Discrete Simulation of Power Law Noise

Neil Ashby[1,2]

[1]University of Colorado, Boulder, CO 80309-0390 USA
[2]National Institute of Standards and Technology, Boulder, CO 80305 USA
ashby@boulder.nist.gov

Abstract—A method for simulating power law noise in clocks and oscillators is presented based on modification of the spectrum of white phase noise, then Fourier transforming to the time domain. Symmetric real matrices are introduced whose traces–the sums of their eigenvalues–are equal to the Allan variances, in overlapping or non-overlapping forms, as well as for the corresponding forms of the modified Allan variance. Diagonalization of these matrices leads to expressions for the probability distributions for observing a variance at an arbitrary value of the sampling or averaging interval, and hence for estimating confidence in the measurements. Comparison with flicker-like noise from a spectrum analyzer is discussed.

Key words: simulation, Allan variance, power law noise

I. INTRODUCTION

The characterization of clock performance by means of average measures such as Allan variance, Hadamard variance, Theo, and modified forms of such variances, is widely applied within the time and frequency community as well as by most clock and oscillator fabricators. Such variances are measured by comparing the times t_k on a device under test, with the times at regular intervals $k\tau_0$ on a reference. Imperfections in performance of the clock under test are studied by analyzing noise in the time deviation sequence $x_k = t_k - k\tau_0$ or the fractional frequency difference during the sampling interval $\tau = s\tau_0$:

$$\Delta_{k,s}^{(1)} = \frac{x_{k+s} - x_k}{s\tau_0}. \tag{1}$$

The frequency spectrum of fractional frequency differences can usually be adequately characterized by linear superposition of a small set of types of power law noise. The frequency spectrum of the fractional frequency differences of a particular noise type is given by a one-sided spectral density [1,2]

$$S_y(f) = h_\alpha f^\alpha, \quad f > 0. \tag{2}$$

The units of spectral density are Hz^{-1}. For the common power-law noise types, α varies in integral steps from +2 down to -2 corresponding respectively to white phase modulation, flicker phase modulation, white frequency modulation, flicker frequency modulation, and random walk of frequency.

Simulation of clock noise can be extremely useful in testing software algorithms that use various windowing functions and Fourier transform algorithms to extract spectral density and stability information from measured time deviations, and especially in predicting the probability for observing a particular value of some clock stability variance. This paper develops a simple simulation method for a time difference sequence that guarantees the spectral density will have some chosen average power law dependence. Expressions for the common variances and their modified forms can be derived that agree with expressions found in the literature, except in some cases for the modified Allan variance. This approach also leads to predictions of probabilities for observing a variance of a particular type at particular values of the averaging time. A broad class of probability functions naturally arises. These only rarely correspond to chi-squared distributions.

This paper is organized as follows. Section II introduces the basic simulation method. Section III applies the method to the overlapping Allan variance. A symmetric real matrix is defined such that the Allan variance is equal to the sum of the eigenvalues of the matrix. Section IV shows how diagonalization of the averaged squared second-difference operator, applied to the simulated time series, leads to expressions for the probability of observing a value of the variance for some chosen value of the sampling or averaging time. In Section V the probability for observing the spectral density as a function of frequency is compared with measurements of flicker-like noise. Probability distributions for observing the Allan variance, in terms of eigenvalues, are discussed in Section VI. Section VII applies the approach to measurements with dead time and Section VIII summarizes the results.

II. DISCRETE TIME SERIES

We imagine the noise amplitudes at Fourier frequencies f_m are generated by a set of N normally distributed random complex numbers $w_n = u_n + iv_n$, each component having mean zero and variance σ, that would by themselves generate a simulated spectrum for white phase noise. These random numbers are divided by a function of the frequency, $|f^{1-\alpha/2}|$, producing a spectral density that has the desired frequency characteristics. For ordinary power law noise, the exponent is a multiple of ½, but it could be anything. The frequency noise is then transformed to the time domain, producing a time series with the statistical properties of the selected power law noise. The Allan variance, Modified Allan variance, Hadamard Variance, variances with dead time, and other quantities of interest can be calculated using either the frequency noise or the time series.

In this paper we discuss applications to calculation of various versions of the ordinary overlapping Allan variance. Of considerable interest are results for the probability of observing a value of the Allan variance for particular values of the sampling time τ and time series length N. The derivations in this paper are theoretical predictions. A natural frequency cutoff occurs at $f_h = 1/(2\tau_0)$, where τ_0 is the time between successive time deviation measurements. This number is not necessarily related in an obvious way to some hardware bandwidth. The measurements are assumed to be made at times $k\tau_0$, and the time errors or residuals relative to the reference clock are denoted by X_k. The averaging or sampling time is denoted by $\tau = s\tau_0$, where s is an integer. The total length of time of the measurement series is $T = N\tau_0$. The possible frequencies that occur in the Fourier transform of the time residuals are

$$f_m = \frac{m}{N\tau_0}; \qquad -\frac{N}{2} + 1 \le m \le \frac{N}{2}. \tag{3}$$

Noise Sequences. In order that a set of noise amplitudes in the frequency domain represent a real series in the time domain, the amplitudes must satisfy the reality condition

$$w_{-m} = u_{-m} + iv_{-m} = (w_m)^* = u_m - iv_m. \tag{4}$$

N random numbers are placed in $N/2$ real and $N/2$ imaginary parts of the positive and negative frequency spectrum. Since the frequencies $\pm 1/(2\tau_0)$ represent essentially the same contribution, we can omit the variable $v_{N/2}$. We shall assume the variance of the noise amplitudes satisfies

$$\langle (w_m)^* w_n \rangle = \langle u^2 + v^2 \rangle = 2\sigma^2 \delta_{mn}, \qquad m \ne 0, \frac{N}{2}. \tag{5}$$

The random variables corresponding to unequal values of the frequency index are assumed to be uncorrelated. In order to avoid division by zero, we shall always drop the Fourier amplitude corresponding to zero frequency. This only means that the average of the time residuals in the time series will be zero, and has no effect on any variance that involves time differences.

We perform a discrete Fourier transform of the frequency noise and obtain the amplitude of the k^{th} member of the simulated time series:

$$X_k = \sqrt{\frac{h_\alpha}{16\pi^2\sigma^2(N\tau_0)}}\sum_m \frac{e^{-2\pi imk/N}}{|f_m|^{1-\alpha/2}} w_m. \tag{6}$$

The multiplicative constant has been carefully chosen to give the correct spectral density; for white phase noise $\alpha = 2$ and the time series would give rise to a constant average spectral density; for other noise types Eq. (6) leads to the correct power law noise for spectral density. To show this, the average two-sided spectral density of the time residuals is obtained from a single term in Eq. (6):

$$s_x(f_m) = \frac{h_\alpha}{16\pi^2\sigma^2(N\tau_0)f_m^{2-\alpha}}\langle\frac{|w_m|^2}{\Delta f}\rangle = \frac{h_\alpha}{8\pi^2 f_m^{2-\alpha}} \tag{7}$$

where $\Delta f = 1/(N\tau_0)$ is the spacing between successive allowed frequencies. The average two-sided spectral density of fractional frequency fluctuations is given by the well-known relation [1]

$$s_y(f) = (2\pi f)^2 s_x(f), \tag{8}$$

and in agreement with convention, Eq. (2), the one-sided spectral density is

$$S_y(f) = \begin{cases} 0, & f < 0 \\ 2s_y(f) = h_\alpha f^\alpha, & f > 0. \end{cases} \tag{9}$$

Thus the simulation will always give rise to the conventional form for the one-sided spectral density [3]. In the next section we discuss the form of the simulated overlapping Allan variance.

III. OVERLAPPING ALLAN VARIANCE

Consider the second-difference operator defined by

$$\Delta_{j,s}^{(2)} = \sqrt{\frac{1}{2\tau^2}}\left(X_{j+2s} - 2X_{j+s} + X_j\right). \tag{10}$$

Here the averaging time is $\tau = s\tau_0$. The fully overlapping Allan variance is formed by averaging the square of this quantity over all possible values of j from 1 to $N - 2s$. Thus

$$\sigma_y^2(\tau) = \langle\frac{1}{N-2s}\sum_{j=1}^{N-2s}(\Delta_{j,s}^{(2)})^2\rangle. \tag{11}$$

In the context of the present simulation, "averaging" means taking an average over all possible values of the random variables w_m. In terms of the time series, Eq. (6), the second difference can be reduced using elementary trigonometric identities:

$$\Delta_{j,2}^{(2)} = -\sqrt{\frac{h_\alpha}{2\pi^2\tau^2\sigma^2(N\tau_0)}}\sum_m \frac{w_m}{|f_m|^{1-\alpha/2}} e^{-\frac{2\pi im(j+s)}{N}}\left(\sin(\frac{\pi ms}{N})\right)^2. \tag{12}$$

By construction this second difference is a real quantity. So we can form the square in Eq. (11) by multiplying the second difference by its complex conjugate:

$$\sigma_y^2(\tau) = \frac{h_\alpha}{2\pi^2\tau^2\sigma^2(N\tau_0)(N-2s)} \sum_{m,n,j} \langle \frac{w_m (w_n)^*}{\sigma^2} \rangle \frac{\left(\sin\left(\frac{\pi m s}{N}\right) \sin\left(\frac{\pi n s}{N}\right)\right)^2}{|f_m f_n|^{1-\frac{\alpha}{2}}} e^{-\frac{2\pi i (m-n)(j+s)}{N}}. \quad (13)$$

The average of the product of random variables only contributes when $m = n$, so (we don't worry here about the special case of maximum frequency) the average Allan variance reduces to

$$\sigma_y^2(\tau) = \frac{h_\alpha}{\pi^2\tau^2(N\tau_0)} \sum_m \frac{\left(\sin\left(\frac{\pi m s}{N}\right)\right)^4}{|f_m|^{2-\alpha}}, \quad (14)$$

since every term in the sum over j contributes the same amount. The zero frequency term is excluded from the sum. For convenience we introduce the abbreviation

$$K = \frac{2h_\alpha}{\pi^2\tau^2(N\tau_0)}. \quad (15)$$

If we sum over positive frequencies only, a factor of 2 comes in except at the maximum frequency. Then if the frequencies are spaced densely enough to pass from the sum to an integral, the frequency spacing $df = 1/(N\tau_0)$ is already present in the sum and we obtain the well-known result

$$\sigma_y^2(\tau) = 2 \int_0^{f_h} \frac{S_y(f) df}{\pi^2\tau^2 f^2} (\sin(\pi f \tau))^4. \quad (16)$$

Similar arguments lead to known expressions for the non-overlapping and modified versions of the Allan variance as well the Hadamard variance. These cases will be discussed elsewhere.

IV. PROBABILITIES

In the present section we shall develop expressions for the probability of observing a particular value A_o for the overlapping Allan variance in a single measurement, or in a single simulation run. A_o is a random variable representing a possible value of the overlapping variance. We use a subscript "o" to denote the completely overlapping case. To save writing, we introduce the following abbreviations:

$$F_m^j = \frac{\left(\sin\left(\frac{\pi m s}{N}\right)\right)^2}{\left|f_m^{1-\frac{\alpha}{2}}\right|} \cos\left(\frac{2\pi m(j+s)}{N}\right) \; ; \qquad G_m^j = \frac{\left(\sin\left(\frac{\pi m s}{N}\right)\right)^2}{\left|f_m^{1-\frac{\alpha}{2}}\right|} \sin\left(\frac{2\pi m(j+s)}{N}\right). \quad (17)$$

The dependence on s, α, and N is suppressed, but is to be understood. We write the second difference in terms of a sum over positive frequencies only, keeping in mind that the most positive and the most negative frequencies only contribute a single term since $\sin(\pi(j+s)) = 0$. The imaginary contributions cancel, and we obtain

$$\Delta_{j,s}^{(2)} = -\sqrt{K} \sum_{m>0} \left(F_m^j \frac{u_m}{\sigma} + G_m^j \frac{v_m}{\sigma} \right). \quad (18)$$

The Allan variance is then given after averaging by the following equation

$$\sigma_y^2(\tau) = \frac{K}{N-2s} \sum_j \sum_{m>0} \left(\left(F_m^j\right)^2 + \left(G_m^j\right)^2 \right). \quad (19)$$

To compute the probability that a particular value A_o is observed for the Allan variance, given all the possible values that the random that the random variables $u_1, v_{1,\ldots} u_{N/2}$ can have, we form the integral [6]

$$P(A_o) = \int \delta\left(A_o - \frac{1}{N-2s}\sum_j \left(\Delta_{j.s}^{(2)}\right)^2\right) \prod_{m>0}\left(e^{-\frac{u_m^2+v_m^2}{2\sigma^2}}\frac{du_m dv_m}{2\pi\sigma^2}\right). \quad (20)$$

The delta function constrains the second difference to the specific value A_o while the random variables $u_1, v_1, \ldots u_m, v_m, \ldots u_{N/2}$ range over their (normally distributed) values. There is no integral for $v_{N/2}$. Inspecting this probability and Eq. (17) for the second difference indicates that we could dispense with the factors of σ^{-1} and work with normally distributed random variables having variances unity. Henceforth we could set $\sigma = 1$.

The exponent involving the random variables is a quadratic form that can be written in matrix form by introducing the $N-1$ dimension column vector U defined by its transpose

$$U^T = [u_1\ v_1 \ldots u_m\ v_m\ \ldots u_{N/2}]. \quad (21)$$

Then

$$\tfrac{1}{2}\sum_{m>0}(u_m^2 + v_m^2) = \tfrac{1}{2}U^T U = \tfrac{1}{2}U^T \mathbf{1} U, \quad (22)$$

where **1** represents the unit matrix. The delta-function in Eq. (20) can be written in exponential form by introducing one of its well-known representations [4], an integral over angular frequencies ω:

$$P(A_o) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{i\omega\left(A_o - \frac{1}{N-2s}\sum_j\left(\Delta_{j.s}^{(2)}\right)^2\right)}\prod_{m>0}\left(e^{-\frac{u_m^2+v_m^2}{2\sigma^2}}\frac{du_m dv_m}{2\pi\sigma^2}\right). \quad (23)$$

The contour of integration goes along the real axis in the complex ω plane.

The squared second difference is a complicated quadratic form in the random variables $u_1\ v_1 \ldots u_m\ v_m\ \ldots u_{N/2}$. If this quadratic form could be diagonalized without materially changing the other quadratic terms in the exponent, then the integrals could be performed in spite of the imaginary factor i in the exponent. To accomplish this we introduce a column vector C^j that depends on j, m, s, N and whose transpose is

$$\left(C^j\right)^T = [F_1^j, G_1^j, \ldots F_m^j, G_m^j, \ldots G_{N/2-1}^j, F_{N/2}^j]. \quad (24)$$

Dependence on the averaging time parameter s is not written explicitly but is understood. The column vector contains all the dependence of the second difference on frequency and on the particular power law noise, but does not depend on the random variables. We use indices m, n as matrix (frequency) indices. The second difference operator can then be written very compactly as a matrix product:

$$\Delta_{j.s}^{(2)} = -\sqrt{K}\left(C^j\right)^T U = -\sqrt{K}U^T C^j, \quad (25)$$

and then

$$\frac{1}{N-2s}\sum_j\left(\Delta_{j.s}^{(2)}\right)^2 = U^T\left(\frac{K}{N-2s}\sum_j C^j\left(C^j\right)^T\right)U. \quad (26)$$

The matrix in parentheses,

$$H_o = \left(\frac{K}{N-2s}\sum_j C^j\left(C^j\right)^T\right), \quad (27)$$

is real and symmetric, and does not depend on the random variables. A real symmetric matrix can be diagonalized by an orthogonal transformation, which we denote by O [5,6]. We shall not need to determine this orthogonal transformation explicitly, but it could be found by first finding the eigenvalues ε and eigenvectors ψ_ε by solving the equation $H_o\psi_\varepsilon = \varepsilon\psi_\varepsilon$. The transformation O is a matrix of dimension $(N-1)\times(N-1)$ consisting of the components of the normalized eigenvectors placed in columns. Then $H_oO = OE$, where E is a diagonal matrix with entries equal to the eigenvalues of the matrix H_o. Then since the inverse of an orthogonal matrix is just the transpose of the matrix itself,

$$O^T H_o O = E. \tag{28}$$

The matrix is thus diagonalized, at the cost of introducing a linear transformation of the random variables:

$$\frac{1}{N-2s}\sum_j \left(\Delta_{j.s}^{(2)}\right)^2 = U^T H_o U = U^T O O^T H_o O O^T U = U^T O E O^T U. \tag{29}$$

We introduce $N-1$ new random variables by means of the transformation

$$V = O^T U. \tag{30}$$

Then the term in the exponent of Eq. (23) representing the Gaussian distributions is

$$-\frac{1}{2}U^T\mathbf{1}U = -\frac{1}{2}U^T O\mathbf{1}O^T U = -\frac{1}{2}V^T\mathbf{1}V = -\frac{1}{2}\sum_{n=1}^{N-1} V_n^2. \tag{31}$$

The normal distributions of the random variables remain basically unchanged. Furthermore, the determinant of an orthogonal matrix is ± 1, so changes in the volume element of integration over the space of random variables remains unchanged in form:

$$dV_1 dV_{2\ldots} dV_{N-1} = |\det(O)| dU_1 dU_{2\ldots} dU_{N-1} = dU_1 dU_{2\ldots} dU_{N-1}. \tag{32}$$

The probability is therefore

$$P(A_o) = \int \frac{d\omega}{2\pi} e^{i\omega\left(A_o - \sum_k \varepsilon_k V_k^2\right)} \prod_i \left(e^{-\frac{V_i^2}{2}} \frac{dV_i}{\sqrt{2\pi}}\right). \tag{33}$$

An eigenvalue of zero will not contribute in any way to this probability since the random variable corresponding to a zero eigenvalue just integrates out.

Suppose the eigenvalue ε_i is repeated μ_i times. Integration over the random variables then gives a useful form for the probability:

$$P(A_o) = \int \frac{d\omega}{2\pi} \frac{e^{i\omega A_o}}{\prod_i (1+2i\varepsilon_i\omega)^{\mu_i/2}}. \tag{34}$$

Finally the contour integral may be deformed and closed in the upper half complex plane where it encloses the singularities of the integrand. Evaluating the integral gives the probability, from which confidence intervals of interest may be calculated. We shall illustrate this in succeeding sections. A similar reduction of probabilities can be developed for other interesting variances.

Properties of the eigenvalues. We state here without proof a number of properties of the eigenvalues. It may be shown that all eigenvalues are either positive or zero. The zero eigenvalues can be completely eliminated by defining a matrix similar to that in Eq. (27) that is reduced in size. In general the dimension of the matrix H_o is $N-1$ but only $N-2s$ eigenvalues are non-zero. The number of significant eigen-

values is in fact equal to the number of terms in the sum over j in Eq. (11). Numerical calculation of many cases seems to indicate that except for white PM, the non-zero eigenvalues are all distinct. When repeated eigenvalues occur an even number of times, the integrand of Eq. (34) has poles and the integral can easily be evaluated. In some of these cases chi-squared probability distributions can arise. In all other cases repeated eigenvalues have never been found so far. The largest eigenvalues correspond to branch points of the contour integral that are closest to the real axis. When the eigenvalues are all distinct, after closing the contour in the upper half ω-plane the contour can be reduced to integrals between alternating pairs of adjacent singularities; this is a consequence of the square roots that occur in the denominator of Eq. (34).

An interesting result arises when one computes the trace of the matrix H_o. Since the trace is not changed by an orthogonal transformation,

$$Trace(O^T H_o O) = Trace(H_o) = \sum_i \varepsilon_i. \tag{35}$$

The sum of the diagonal elements of H_o equals the sum of the eigenvalues of H_o. If we then explicitly evaluate the sum of diagonal elements we find

$$\sum_i \varepsilon_i = \frac{K}{N-2s}\sum_j Trace(C^j(C^j)^T) = \frac{K}{N-2s}\sum_j \sum_{m>0}\left(\left(F_m^j\right)^2 + \left(G_m^j\right)^2\right) = \sigma_y^2(\tau) \tag{36}$$

from Eq. (19). Thus the overlapping Allan variance is equal to the sum of the eigenvalues of the matrix H_o. Similar results can be established for the other types of variances.

V. SPECTRAL DENSITY: AN APPLICATION TO FLICKER-LIKE NOISE

In this section we apply these methods to the calculation of the probability distribution for a power-law spectral density and apply it to actual measurements. From Eqs. (7-9), before averaging, a single-sided spectral density at a particular frequency is simulated as

$$S_y(f_m) = \frac{h_\alpha f_m^\alpha}{2\sigma^2}(u_m^2 + v_m^2). \tag{37}$$

Following the procedure of Eq. (23), the probability of finding a value S will be

$$P(\mathrm{S}) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{i\omega\left(\mathrm{S}-\frac{h_\alpha f_m^\alpha}{2\sigma^2}(u_m^2+v_m^2)\right)}e^{-\frac{u_m^2+v_m^2}{2\sigma^2}}\frac{du_m dv_m}{2\pi\sigma^2}. \tag{38}$$

The integrals over the random variables are straightforward; no additional diagonalization is necessary. Each integral gives the same factor so the probability is

$$P(\mathrm{S}) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left(\frac{e^{i\omega S}}{1+i\omega h_\alpha f_m^\alpha}\right) = \frac{e^{-S/(h_\alpha f_m^\alpha)}}{h_\alpha f_m^\alpha}. \tag{39}$$

For comparison of this probability distribution with measurements, Archita Hati and Craig Nelson of NIST provided 1024 independent measurements of amplifier noise as a function of frequency offset from the carrier; one such run is illustrated in Figure 1. Usually many such runs are averaged and the average spectral density is presented as the measurement result. In Figure 1 the frequency spacing between points is 0.61875 Hz. The conversion from time residuals to spectral density is not known, but by selecting a small range of frequencies and collecting data from many such independent runs, the number of times a spectral density occurs within the range of a particular bin can be reduced to a probability and compared with Eq. (39).

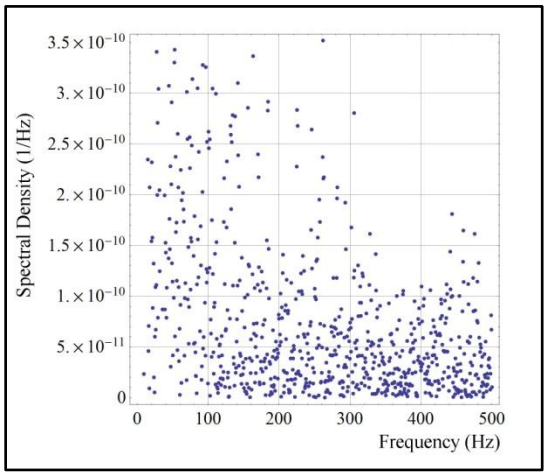

Figure 1. Flicker-like spectral density data from an amplifier.

When these data are plotted on a log-log scale it is found that the spectral density is proportional to f^{α} where $\alpha = -0.83$, so the noise process is not exactly flicker noise, but it is similar to flicker noise. The probability function, Eq. (39) is then determined except for one overall constant, h_{α}, which is the unknown conversion between time residuals and spectral density (the density was provided as rms voltage per root Hz). Figure 2 shows plots of the theoretical number of occurrences, proportional to Eq. (39), at two frequencies, together with a histogram of the number of occurrences constructed from the data. The values of spectral density were collected into bins of width 4.415×10^{-12} Volts^2/Hz for plotting as a histogram.

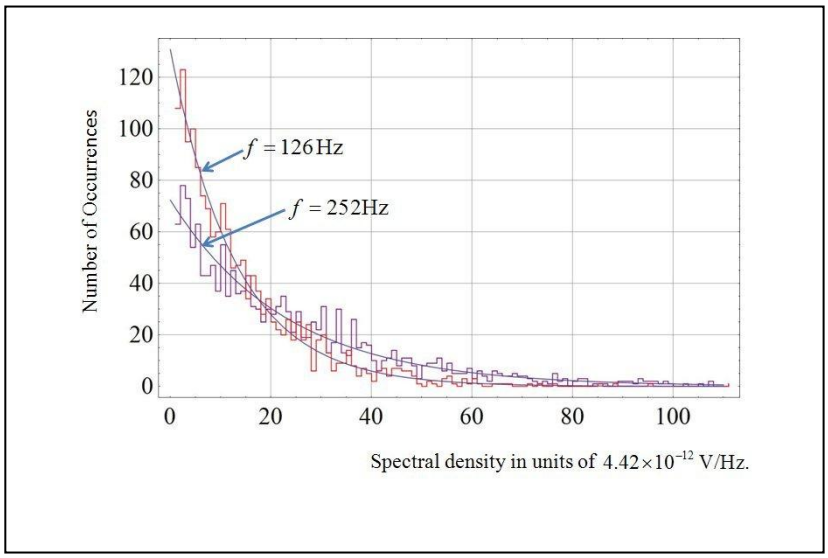

Figure 2. Histograms of occurrences of spectral densities of flicker-like noise at two frequencies; smooth lines are simulated results, Eq. (39).

The probability function Eq. (39) can easily be used to find the mean, median and 25% and 75% confidence levels of the spectral density distribution and hence to compare with the data. Figure 3

provides these theoretical plots, with no additional adjustment of parameters. The corresponding values obtained from the data are given at several frequencies. The simulated probability distribution, Eq. (39), begins to disagree with the data at frequencies lower than about 70 Hz.

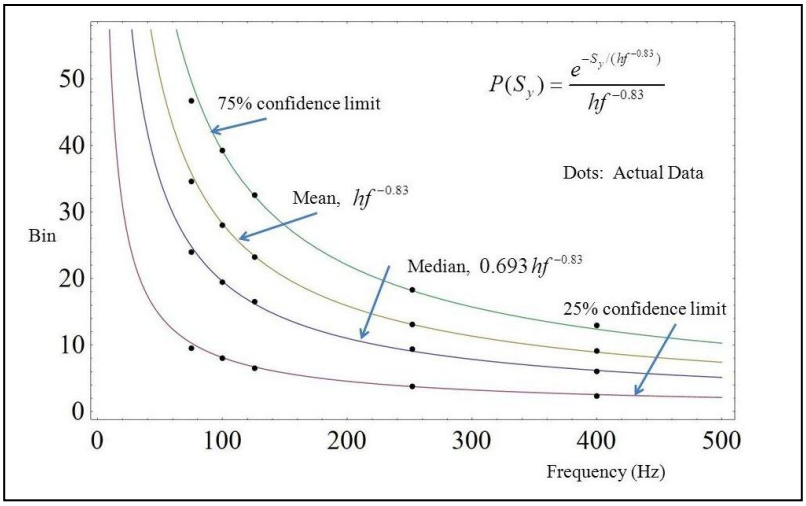

Figure 3. Comparison between probability function, Eq. (39), and experimental data. The dots are computed directly from the data at frequencies 75 Hz, 100 Hz, 126 Hz, 252 Hz,and 400 Hz.

VI. OTHER PROBABILITY DISTRIBUTIONS

If a single eigenvalue occurs once only, then Eq. (34) has a single factor in the denominator

$$P(A_o) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{d\omega e^{i\omega A_o}}{\sqrt{1+2i\omega\varepsilon}} = \frac{1}{\sqrt{2\pi\sigma_y^2(\tau)}} \frac{e^{-A_o/\left(2\sigma_y^2(\tau)\right)}}{\sqrt{A_o}}. \tag{40}$$

For the overlapping variance, when s has its maximum value $N/2-1$, there are two unequal eigenvalues. The probability integral can be performed by closing the contour in the upper half plane. There are two branch points on the imaginary axis, connected by a branch line. The contour integral can be evaluated analytically and gives [7,8]

$$P(A_o) = \frac{1}{2\sqrt{\varepsilon_1\varepsilon_2}} e^{-\frac{A_o}{4}\left(\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}\right)} I_0\left(\frac{A_o}{4}\left(\frac{1}{\varepsilon_2}-\frac{1}{\varepsilon_1}\right)\right), \tag{41}$$

where I_0 is the modified Bessel function of order zero and $\varepsilon_1 > \varepsilon_2$. This probability is automatically normalized. It differs from a chi-squared distribution of order 1 in that it does not have a singularity at $A_o = 0$. Extensive simulations of Allan variance for flicker FM have been carried out; these agree with Eq. (41) showing at least that the theory is internally self-consistent, but comparison with measurements have not yet been made.

Evaluation of the contour integral is generally difficult except in a few cases, such as when eigenvalues are repeated an even number of times, or when there are a very small number of eigenvalues. This indicates that applications are most likely to be useful when there are a very small number of terms that contribute to the Allan variance—e.g., for long averaging times.

VII. OTHER VARIANCES

The mathematical relationships developed in this paper can be applied to many other variances. These include but are not limited to the non-overlapping Allan variance, modified Allan variance, and both modified and un-modified forms of the Hadamard variance [9]. Application can also be made to cases in which there is dead time between measurements of average frequency during the sampling intervals. Suppose for example that the measurements consist of intervals of length $\tau = s\tau_0$ during which an average frequency is measured, separated by dead time intervals of length $D - \tau$ during which no measurements are made. Let the index j label the measurement intervals with $j = 1,2 \ldots N$. An appropriate variance can be defined in terms of the difference between the average frequency in the j^{th} interval and that in the interval labeled by $j + r$:

$$\Delta^{(2)}_{j,r,s} = \frac{1}{\sqrt{2}}\left(\bar{y}_{j+r,s} - \bar{y}_{j,s}\right), \tag{42}$$

where $\bar{y}_{j,s}$ is the average frequency in the interval j of length $s\tau_0$. Then an appropriate variance can be defined as

$$\Psi(\tau, D) = \langle\left(\Delta^{(2)}_{j,r,s}\right)^2\rangle. \tag{43}$$

If the measurements are sufficiently densely spaced that it is possible to pass from a sum to an integral over frequencies, then this can be shown using elementary trigonometric identities to reduce to

$$\Psi(\tau, D) = \frac{2}{D\tau}\int_0^{f_h} df\, \frac{S_y(f)}{(\pi f)^2}(\sin(\pi f r D))^2(\sin(\pi f \tau))^2. \tag{44}$$

When $D = \tau$ and $r = 1$ there is no real dead time and this variance reduces to the ordinary Allan variance.

VIII. SUMMARY AND CONCLUSIONS

In this paper a method of simulating time series for the common power-law noises has been described. The approach is designed to guarantee that the simulation gives rise to conventional forms for power-law dependence of commonly used noise spectral densities, and agrees with well-established relations between spectral densities and Allan variances. Applications reported here have been mostly restricted to the overlapping Allan variance, but can be applied equally well to other variances used to characterize clock stability. It has been shown that the Allan variance is equal to the sum of the eigenvalues of an appropriately defined matrix. Diagonalization of quadratic forms that arise when computing the average variance leads to expressions for the probabilities of observing particular values of the variance for a given sampling time $\tau = s\tau_0$. The probabilities are expressed in terms of integrals that depend on the eigenvalues of matrices formed from squares of the second differences used to define the variance. Generally speaking, the number of eigenvalues equals the number of terms occurring in the sums used to define averages of the second-difference operators. This number gets smaller as the sampling time τ gets larger. The probability distribution $P(A)$ for some variance A is useful in evaluating confidence intervals about the average variance. Probability distributions $P(S)$ for spectral density are easily derived; comparison with measurements for one set of flicker-like noise measurements appear to be mostly in agreement. The eigenvalues are usually distinct; only for white PM have eigenvalues been observed to occur with multiplicities other than unity. Methods for computing the probabilities have been presented in a few useful cases.

Other methods of simulating power law noise have been published; the present approach differs from that of [11] in that here no causality condition is imposed. This approach to simulation will be applied to other variances used to characterize clock stability in future papers.

REFERENCES

[1] "Characterization of Frequency and Phase Noise," Report 580 of the International Radio Consultative Committee (C.C.I.R.), 1986, pp 142-150, Reprinted in NIST Technical Note 1337, 1990, pp TN-162-170.

[2] D. B. Sullivan, D. W. Allan, D. A. Howe, and F. L. Walls, eds., "Characterization of Clocks and Oscillators," NIST Technical Note 1337, March 1990, U. S. Government Printing Office, Washington, D.C.

[3] J. A. Barnes, et al., "Characterization of Frequency Stability," IEEE Trans. Inst. Meas. 1M-20(2), May 1971; Section III. discusses the one-sided spectral density.

[4] M. M. J. Lighthill, Introduction to Fourier Analysis and Generalized Functions, Cambridge University Press (1958).

[5] G. Strang, Linear Algebra and its Applications, 2nd ed., Academic Press (New York) 1980, Section 5.5.

[6] R. Stoll, Linear Algebra and Matrix Theory, McGraw-Hill (New York) 1980.

[7] M. M. J. Lighthill, Introduction to Fourier Analysis and Generalized Functions, Cambridge University Press (1958).

[8] M. Abramowitz and I. Stegun, Handbook of Mathematical Functions, NBS Applied Mathematics Series 55, Tenth Printing, December 1972, p. 374

[9] V. I. Smirnov, "Complex Variables-Special Functions," III.2, in "A Course of Higher Mathematics," Pergamon Press, 1964, Section 60, pp 232-233.

[10] S. A. Hutsell, "Relating the Hadamard variance to MCS Kalman filter clock estimation," Proc. 27th Annual PTTI Systems and Applications Meeting, San Diego, CA. Nov. 29-Dec. 1 1995, pp. 291-301.

[11] N. J. Kasdin and T. Walter, "Discrete Simulation of Power Law Noise," Proc. 1992 IEEE Frequency Control Symposium, pp 274-283.