HOW TO DEAL WITH FFT SAMPLING INFLUENCES ON ADEV CALCULATIONS

Po-Cheng Chang
National Standard Time & Frequency Lab., TL, Taiwan
12, Lane 551, Min-Tsu Road, Sec. 5, Yang-Mei, Taoyuan, Taiwan 326
Tel: 886 3 424 5179, Fax: 886 3 424 5474, E-mail: *betrand@cht.com.tw*

Abstract

We have done some work to reveal that Fast Fourier Transform (FFT) sampling may induce unreasonable Allan deviation (ADEV) values while the numerical integration is used for the time and frequency (T&F) conversion. These ADEV errors occur because parts of the FFT sampling have no contributions to the ADEV calculation for some τ. For example, when averaging interval τ = 0.004 (s) and Fourier frequency f = 250 (Hz), the term sin ($\pi\tau f$), which plays a key role in the mathematical conversion, is zero. If FFT sampling in a specific frequency range is only at multiples of 250 (Hz), spectral density in this range has no contributions to ADEV calculations for τ = 0.004 × N (s), where N is a positive integer.

Our lab has also found such errors in related commercial software. In order to solve this problem without skipping over effects from certain values of τ, we try to change the original sampling data in several ways, like dividing sampling spaces into narrower ones or shifting the FFT sampling frequency a small amount, etc. The regenerated data using interposition techniquess are then calculated via the T&F conversion. According to our tests, the FFT sampling within logarithmic frequency space exceeds the others at reducing ADEV errors.

As for spur effects, the spectral density with spurs is likely to double or triple ADEV values from the same density with spurs removed in our case, so it is meaningful for laboratories to reduce ac power and other periodic noises in their own environment. The power-law processes can also perform the T&F conversion and identify different noise types in the spectral density. ADEV results calculated from this way are in good agreement with those from the numerical integration.

I. INTRODUCTION

Spectral density is a measure of frequency stability in the frequency domain because of its functional dependence on Fourier frequency. Allan deviation (ADEV), on the other hand, is an example of a time domain measure. In a strict mathematical sense, these two descriptions are connected by the Fourier transform relationship. For very short averaging intervals (τ < 0.5 s), it is not easy to measure the frequency stability of a device using a timing measurement instrument (ex: counter) because of its capability limitations. The existence of a time and frequency (T&F) relationship provides us a useful access to obtain ADEV via its spectral density.

To perform the mathematical conversion, the numerical integration, or trapezoidal integration to be precise, is a direct way to proceed. Generally, Fast Fourier Transform (FFT) sampling for measuring spectral density is different in individual Fourier frequency sections. The higher frequency section is sampled with wider frequency space than the lower one, while in each section sampling is evenly spaced. It is possible for the numerical integration to generate some unreasonable ADEV values due to influences of FFT sampling. For example, when averaging interval $\tau = 0.004$ (s) and Fourier frequency $f = 250$ (Hz), the term $\sin(\pi\tau f)$, which plays a key role in the mathematical conversion, is zero. If FFT sampling in a specific frequency range are only at multiples of 250 (Hz), the spectral density in this range has no contributions to the ADEV calculation for $\tau = 0.004 \times N$ (s), where N is a positive integer. In order to solve the above problem, the FFT sampling data are regenerated within logarithmic frequency space using an interposition technique. The errors of ADEV are then improved obviously using the regenerated data.

In addition, the power-law processes, each of which varies as an integer power of Fourier frequency with corresponding coefficient h_α, are frequently used for describing spectral density. By locating each particular noise process in its dominant range of Fourier frequency with standard regression techniques, the coefficient h_α could be properly determined. ADEV with different τ could be obtained easily using the Cutler's formula with these coefficients.

It is almost inevitable that some spurs, which are mainly from ac power and other periodic noises, are observed in spectral density of a measurement. Those spurs can't be described well by the power-law processes, so only the numerical integration is adopted while their influences on the ADEV calculation are estimated. ADEV from the spectral density with spurs is compared with one from the above spectral density with spurs removed. Finally, we find ADEV results using both the power-law processes and the numerical integration from the same spectral density (spurs removed) are in good agreement with each other. That means the evaluation of coefficients h_α is acceptable.

II. CONVERSION BETWEEN T&F DOMAIN

The Fourier transform relation between time and frequency domain is as follows **[1-3]**:

$$\sigma_y^{\,2}(\tau) = 2\int_0^{f_h} S_y(f)\frac{\sin^4(\pi\tau f)}{(\pi\tau f)^2}df \tag{1}$$

$S_y(f)$ is the spectral density of normalized frequency fluctuations, f_h is the high frequency cutoff of a low pass filter and $\sigma_y(\tau)$ is ADEV. $S_y(f)$ can also be represented by the addition of all the power-law processes with corresponding coefficients h_α ($\alpha = -2,-1,0,+1,+2$):

$$S_y(f) = \begin{cases} \sum_{\alpha=-2}^{+2} h_\alpha f^\alpha & for \quad 0 < f < f_h \\ 0 & for \quad f > f_h \end{cases} \tag{2}$$

For $2\pi f_h\tau >> 1$, combine formula (1) and (2) to get Cutler's formula:

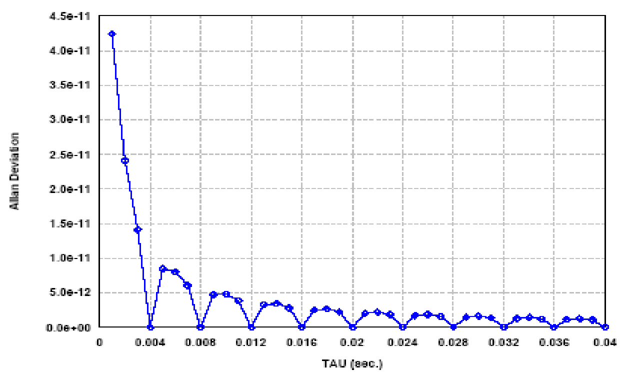
$$\sigma_y^{\,2}(\tau) = h_{-2}\frac{(2\pi)^2}{6}\tau + h_{-1}2\ln 2 + \frac{h_0}{2\tau} + h_1\frac{1.038 + 3\ln(2\pi f_h\tau)}{(2\pi)^2\tau^2} + h_2\frac{3f_h}{(2\pi)^2\tau^2} \tag{3}$$

ADEV can be calculated directly from formula (1) using the numerical integration or from formulas (2) and (3) using regression techniques.

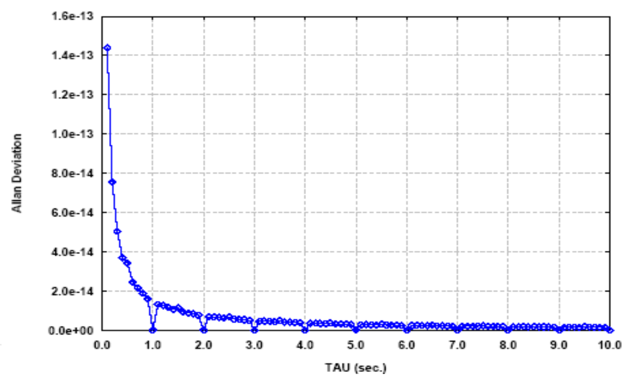
III. HOW TO DEAL WITH FFT SAMPLING INFLUENCES

A. FFT SAMPLING INFLUENCES

FFT sampling may cause errors in ADEV calculations when the numerical integration is adopted for the T&F conversion. For example, if a spectral density in the Fourier frequency range 12500 ~ 99750 (Hz) with a sampling space 250 (Hz) is converted, ADEV is zero when τ equals multiples of 0.004 (s), which is shown in Figure 1a. This is no doubt because the sampling frequencies all make the term sin ($\pi\tau f$) in (1) equal to zero. We have discussed the phenomenon in the Introduction of this paper. Figure 1b is another example for FFT sampling influences. If spectral density in Fourier frequency range 1 ~ 1000 (Hz) with sampling space 1 (Hz) is calculated, ADEV is zero when τ equals multiples of 1 (s).



(a)



(b)

Figure 1. FFT sampling influences on ADEV calculations using the numerical integration for spectral density in the Fourier frequency range (a) 12500 ~ 99750 (Hz) with a sampling space of 250 (Hz), and (b) 1 ~ 1000 (Hz) with sampling space of 1 (Hz).

Such errors are also found in commercial software for the T&F conversion. Before running the software, a sampling space of 1220.7 (Hz) in a certain frequency section of the experimental data is observed. The inverse of this frequency space is about 819.2 (ms), so 819.2 (ms), together with two neighboring values 819.0 (ms) and 820 (ms) are selected for testing the software. The corresponding ADEV are 2.44×10^{-16}, 1.76×10^{-15}, and 1.76×10^{-15} respectively. It can be seen that the ADEV with an averaging interval of 819.2 (ms) is much smaller than that with adjacent averaging intervals, as shown in Figure 2. The reason for this is FFT sampling. In other word, ADEV errors occur because parts of the FFT sampling have no contributions to the ADEV calculation when some values of τ are adopted.

B. METHODS FOR IMPROVEMENTS

Figure 3a shows calculated ADEV from spectral density consisting of several Fourier frequency sections. For some τ, ADEV values obviously fall below the main curve. In order to solve this problem, we have to reduce the possibility of sin ($\pi\tau f$) = 0. For this purpose, let τ equal some logarithmic values and look at the way in which ADEV behaves. Figure 3b shows that the falling ADEV is improved with these selected τ values. Spur influences are not included in the above calculation.

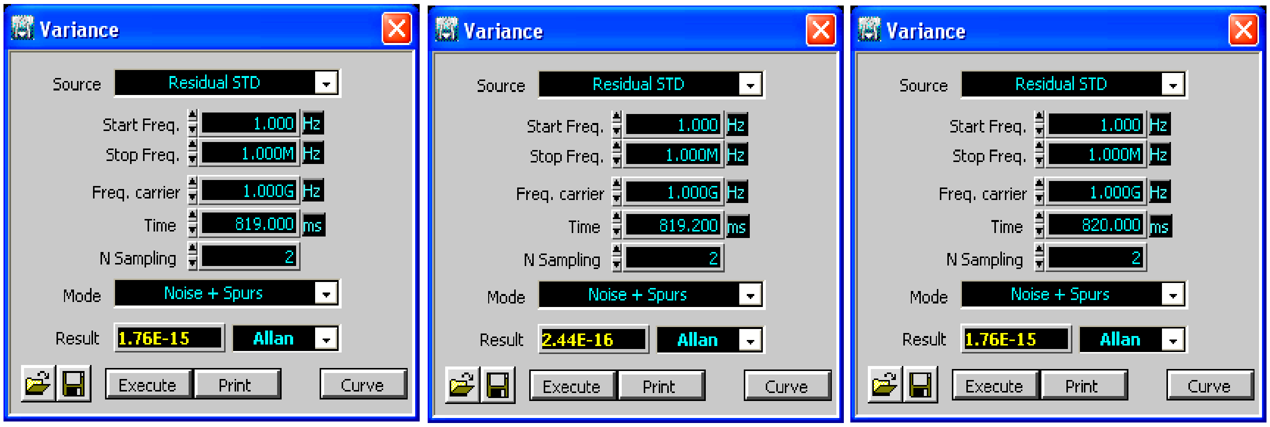
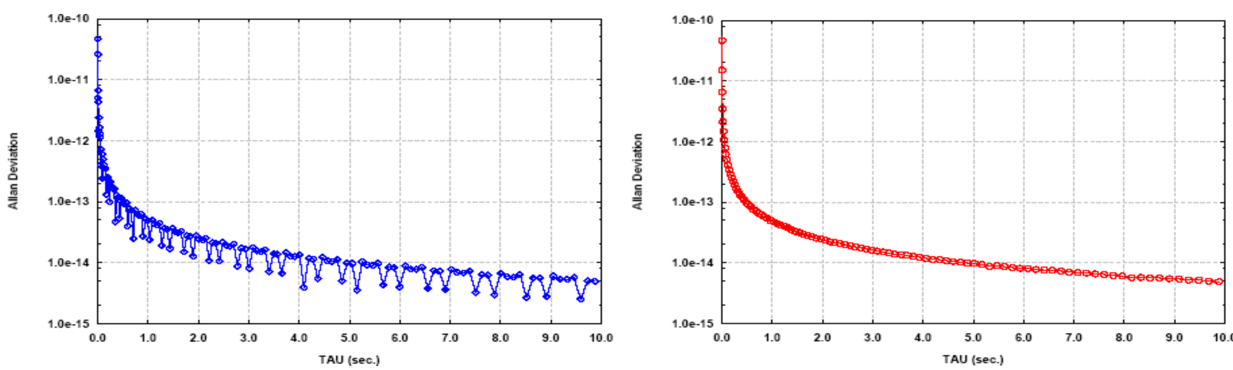


Figure 2. Errors occur in commercial software. The ADEV with an averaging interval of 819.2 (ms) is much smaller than that with adjacent averaging intervals.



(a) **(b)**

Figure 3. (a) ADEV fall obviously below the main curve for some τ using the numerical integration. (b) The falling ADEV are improved with τ of logarithmic values.

This method seems to work well, but it tactfully skips averaging intervals equal to certain values for which ADEV are likely to be underestimated. Furthermore, a logarithmic τ is seldom used in the general case. If τ equal to a non-negative integer with a finite decimal is preferred, changing the original FFT sampling data may be another promising way. After using an interposition technique, we have three kinds of sampling data. They are respectively regenerated with subdivided sampling spaces, sampling points shifted a small amount, and a logarithmic sampling space. The first two have no obvious improvements on the falling ADEV, while the last one works well after our tests. In the following sections, if the numerical integration is used for ADEV calculation, related FFT sampling data are all regenerated using logarithmic sampling space.

IV. ANALYSIS OF CALCULATION RESULTS

Spectral density of a phase noise measurement is illustrated in Figure 4. It's from a noise floor test of our laboratory's measurement system with a Fourier frequency range of 0.12 ~ 99750 (Hz). The measure $L(f)$ on the y-axis is the prevailing expression of phase noise among manufacturers and users of frequency standards. Its relation to $S_y(f)$ can be expressed as **[4]**:

$$L(f) = \frac{1}{2}\left(\frac{v_0^{\;2}}{f^2} S_y(f)\right) \tag{4}$$

where v_0 is the carrier frequency. The x-axis stands for Fourier frequency. $L(f)$ is usually reported in a dB format:

$$\frac{dB_C}{H_Z} = 10\log(L(f)) \tag{5}$$

The blue line is spectral density with spurs, while the red line is that with spurs removed. Those spurs are mainly from ac power and other periodic noises, which usually sneak into measurement results. The spur effects on the ADEV calculation will be evaluated later on. Besides, the power-law processes are another way for performing the T&F conversion. Calculation results from both the numerical integration and the power-law processes will also be compared.

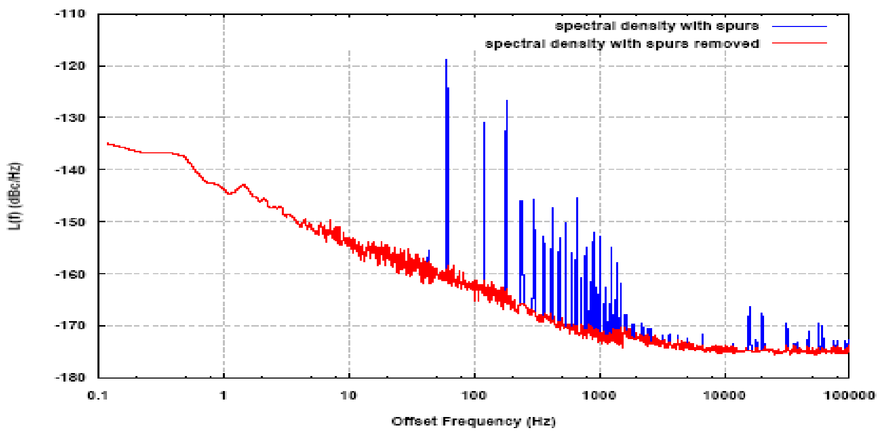
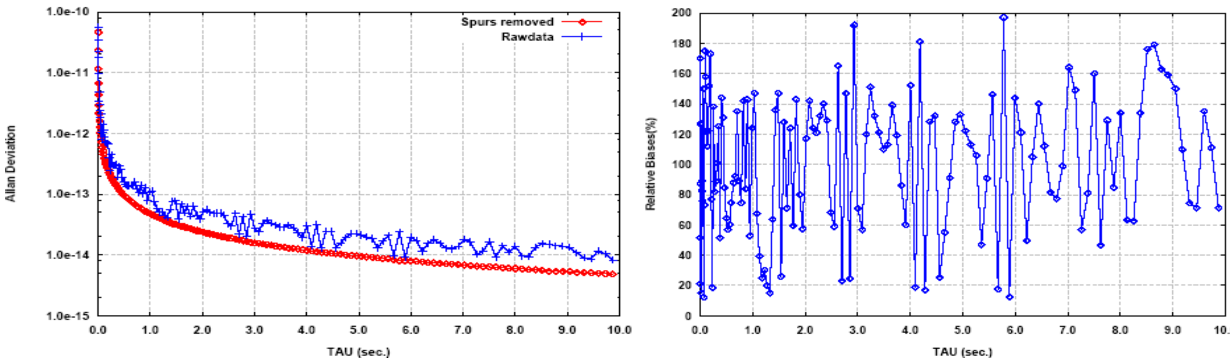


Figure 4. Spectral density of a phase noise measurement. The spurs are mainly from ac power and other periodic noises.

A. Effects of Spurs

Since spurs in spectral density are beyond the model of power-law processes, numerical integration is the only way applicable for the mathematical conversion. Both spectral densities in Figure 4 are converted with τ ranging from 0.001 to 10 (s) and its increment equal to 0.001 (s). We can see in Figure 5a that the blue line (spurs-included ADEV) varies up and down irregularly above the red one (spurs-removed ADEV), depending on τ. That means the spurs have nonnegligible influences on the ADEV calculations. Figure 5b shows relative biases of the results, and some biases may reach 200%. In other words, the spectral density with spurs may triple ADEV values from the one with spurs removed in this case.

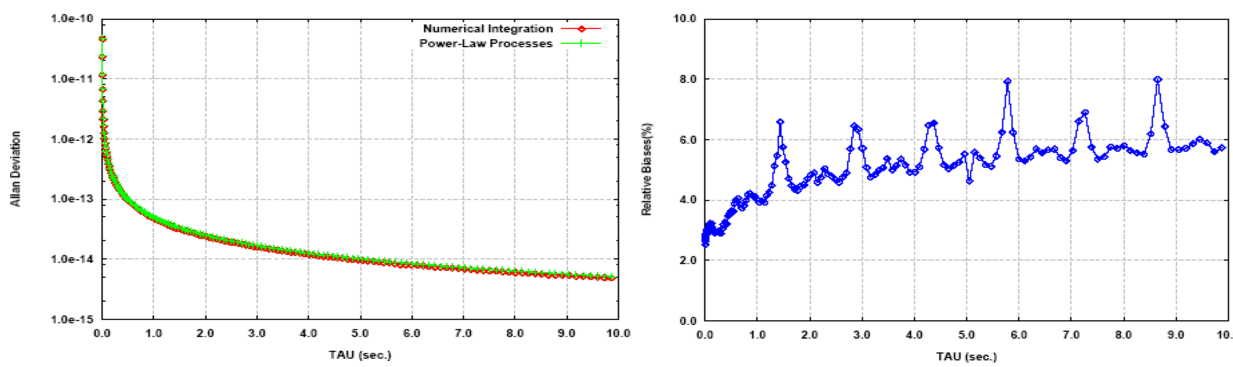


(a) **(b)**

Figure 5. Spurs obviously affect the ADEV calculations. (a) The blue line (spurs-included ADEV) varies up and down irregularly above the red line (spurs-removed one), depending on τ. (b) Relative biases of the ADEV results may reach 200%.

B. Comparisons Between Conversion Methods

According to the power-law processes, the five noise types are Random Walk FM, Flicker FM, White FM, Flicker PM, and White PM, with α equal to -2, -1, 0, +1, and +2 respectively. In Figure 4, we observe the spurs-removed spectral density and find that when Fourier frequency increases by one decade, $L(f)$ also goes down by one decade in the range of 0.12 ~ 1000 (Hz). This indicates that the dominant noise process here is Flicker PM. In the range of 10 ~ 99.75 (kHz), $L(f)$ is almost the same, so the dominant noise process should be White PM. With standard regression techniques, the coefficients $h_{+1} = 4.68 \times 10^{-28}$ and $h_{+2} = 2.61 \times 10^{-31}$ could be obtained. From Cutler's formula, the generated ADEV could be compared with the one using the numerical integration. Figure 6a shows that ADEV results from the two methods match each other quite well when τ is from 0.001 to 10 (s) with an increment of 0.001 (s). Furthermore, their relative biases are all below 10%, as shown in Figure 6b.



(a) **(b)**

Figure 6. (a) ADEV from the numerical integration and the power-law processes match each other quite well. (b) Relative biases of the ADEV results are all below 10%.

V. CONCLUSIONS

Underestimated ADEV may occur when numerical integration is used for the T&F conversion. This is because parts of the FFT sampling have no contributions to the ADEV calculation for some τ. The phenomenon is illustrated by several examples, including results from certain commercial software. In order to solve this problem, we have to reduce the possibility that $\sin(\pi\tau f) = 0$. After testing a number of possible ways, FFT sampling with logarithmic frequency space exceeds the others at improving the ADEV errors while τ has values of a non-negative integer with a finite decimal.

As for spur effects, the spectral density with spurs is likely to double or triple the ADEV from the density with spurs removed, so it is important and meaningful for laboratories to reduce ac power and other periodic noises in the environment. The power-law processes can also perform the T&F conversion with the advantage of identifying different noise types in the spectral density. In Figure 4, two noise types including Flicker PM and White PM are identified. Finally, we compare the generated ADEV from the numerical integration and the power-law processes, and results show that they can match each other quite well.

REFERENCES

[1] S. R. Stein, 1985, *"Frequency and Time – Their Measurement and Characterization,"* **Precision Frequency Control** (Academic Press, New York), **Vol. 2,** chap. 12, pp. 203-204.

[2] *"Characterization of Frequency and Phase Noise,"* Report 580 of the International Radio Consultative Committee (C.C.I.R), pp. 142-150, 1986.

[3] D Allan, H. Hellwig, P. Kartaschoff, J. Vanier, J. Vig, G. M. R. Winkler, and N. F. Yannoni, 1988, *"Standard Terminology for Fundamental Frequency and Time Metrology,"* in Proceedings of the 42^{nd} Annual Symposium on Frequency Control, 1-3 June 1988, Baltimore, Maryland, USA (IEEE Publication 88CH2588-2), pp. 419-425.

[4] Femtosecond Systems, Inc., FSS1000E Phase Noise Detector FSS1011A Delay Line Operation Manual, chap. 2.