A Numerical Comparison of the Unbiased FIR and Kalman Filters in Applications to GPS-based Timekeeping

Yuriy S. Shmaliy, Oscar Ibarra-Manzano, and Luis Arceo-Miquel
FIMEE, Guianajuato University
Salamanca, Gto, 36730, Mexico
Email: shmaliy@salamanca.ugto.mx

Abstract— **The 3-state unbiased finite impulse response (FIR) filter and the 3-state Kalman filters are investigated for the time interval error (TIE) K-degree polynomial model of a local crystal clock in GPS-based timekeeping in presence of the sawtooth noise induced by the receiver. We show that both algorithms produce consistent estimates for the reference (rubidium) measurements. We also demonstrate that the unbiased FIR algorithm produces a lower error than the standard Kalman filter in presence of the sawtooth noise.**

I. Introduction

Fast and accurate estimation and adjustment of a local clock performance, making possible for a variety of modern digital systems to operate in common time with minimum "slips", is of importance for the Global Positioning System (GPS)-based timekeeping [1], [2]. To obtain filtering in an optimum way, the time interval error (TIE) model of a local clock must be known for the filter memory. In the discrete time, such a model [3] may be written as

$$x_1(n) = x_1(0) + x_2(0)\tau n + \frac{x_3(0)}{2}\tau^2 n^2 + w_1(n,\tau)\,, \quad (1)$$

where $n = 0, 1, ...$; $\tau = t_n - t_{n-1}$ is a time step multiple to the 1 s; t_n is a discrete time; $x_1(0)$ is an initial time error; $x_2(0)$ is an initial fractional frequency offset of a local clock from the reference frequency; $x_3(0)$ is an initial linear fractional frequency drift rate; and $w_1(n,\tau)$ is a random component caused by the oscillator noise and environment.

In GPS-based measurements, the TIE model is observed via the mixture $\xi_1(n) = x_1(n) + v_1(n)$, in which $v_1(n)$ is a noisy component induced at the receiver (noise of a measurement set is usually small). In modern receivers [4], a random variable $v_1(n)$ is uniformly distributed owing to the sawtooth noise caused by a principle of the 1 PPS (one pulse per second) signal formation.

To estimate the states of the clocks, we have studied several filtering algorithms [5]–[12], among which, an unbiased moving average filter for the linear clock model was proposed in [11]. An unbiased approach was then generalized in the finite impulse response (FIR) unbiased filtering algorithms [12] for the clock model of the K-degree.

In this paper, we investigate the 3-state unbiased FIR filtering algorithm for the GPS-based measurements of the TIE model of a local crystal clock in presence of the sawtooth noise induced by the receiver. We also apply the 3-state standard Kalman filter and compare the results obtained with two these algorithms.

II. Three-State Unbiased FIR Filtering Algorithm

Here we present the TIE clock model and the unbiased FIR filtering algorithm as they are described in [12].

A. TIE clock model

Most commonly, the TIE polynomial model projects ahead on a horizon of N points from the start point $n = 0$ with the K-degree Taylor polynomial

$$\begin{aligned} x_1(n) &= \sum_{p=0}^{K} x_{p+1}\frac{\tau^p n^p}{p!} + w_1(n,\tau) \\ &= x_1 + x_2\tau n + \frac{x_3}{2}\tau^2 n^2 + \frac{x_4}{6}\tau^3 n^3 ... + w_1(n,\tau)\,, \end{aligned} \quad (2)$$

where $x_{l+1} \equiv x_{l+1}(0)$, $l \in [0, K]$, are initial states of the clock and $w_1(n,\tau)$ is a noise with known properties. By extending the time derivatives of the TIE model to the Taylor series, the signal and observation equations become, respectively,

$$\lambda(n) = \mathbf{A}(n)\lambda(0) + \mathbf{w}(n,\tau)\,, \quad (3)$$

$$\xi(n) = \mathbf{C}\lambda(n) + \mathbf{v}(n)\,, \quad (4)$$

where $\lambda(n) = [x_1(n)x_2(n)...x_{K+1}(n)]^T$ is a vector $[(K+1)\times 1]$ of the clock states and a time-varying transition matrix $[(K+1)\times(K+1)]$ is

$$\mathbf{A}(n) = \begin{bmatrix} 1 & \tau n & \tau^2 n^2/2 & ... & (\tau n)^K/K! \\ 0 & 1 & \tau n & ... & (\tau n)^{K-1}/(K-1)! \\ 0 & 0 & 1 & ... & (\tau n)^{K-2}/(K-2)! \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 1 \end{bmatrix}. \quad (5)$$

For $M = K - 1$, the observation vector is $\xi(n) = [\xi_1(n)\xi_2(n)...\xi_M(n)]^T$ and a measurement matrix $\mathbf{C}$ of $[(K+$

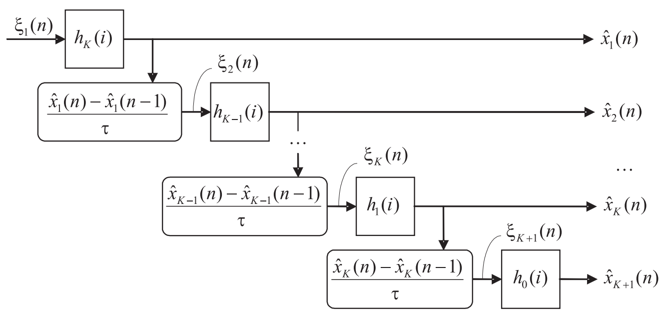

Fig. 1. Structure of the $(K+1)$-State unbiased FIR filtering algorithm for the K-degree TIE polynomial model observable with a single GPS timing receiver.

$1) \times (K+1)]$ is typically unit. The clock noise vector is $\mathbf{w}(n,\tau) = [w_1(n,\tau) w_2(n,\tau) ... w_{K+1}(n,\tau)]^T$ with the components caused by the oscillator noises. Finally, the noise vector $\mathbf{v}(n) = [v_1(n) v_2(n) ... v_M(n)]^T$ contains correlated or uncorrelated components that are not obligatory Gaussian. The GPS noise $\mathbf{v}(n)$ dominates on a horizon N; that is, typically, $\langle w_u^2(n,\tau)\rangle_N << \langle v_l^2(n)\rangle_N$. Therefore, $\mathbf{w}(n,\tau)$ is neglected in the FIR procedure [12]. Note that the TIE noise cannot be discarded in the Kalman algorithm.

B. Three-state unbiased FIR filtering algorithm

The algorithm is illustrated in Fig. 1. The clock first state estimate $\hat{x}_1(n)$ is obtained with $h_K(i)$ at a horizon of N_K points. The observation $\xi_2(n)$ for the second state $x_2(n)$ is then formed by increments of $\hat{x}_1(n)$. Accordingly, $\hat{x}_2(n)$ is achieved with $h_{K-1}(i)$ at a horizon of N_{K-1} points. Inherently, the first accurate value of $\hat{x}_2(n)$ appears at $(N_K + N_{K-1} - 2)$th point starting from $n = 0$. Finally, the last state estimate $\hat{x}_{K+1}(n)$ is calculated with $h_0(i)$ at a horizon of N_0 points, using $\xi_{K+1}(n)$ that is formed in the same manner as $\xi_2(n)$. The first correct value of $\hat{x}_{K+1}(n)$ appears at $(N_K + N_{K-1} + ... + N_0 - K - 1)$th point.

For the quadratic TIE model, $K = 2$, associated with crystal clocks, the 3-state unbiased FIR batch algorithm becomes

$$\hat{x}_1(n) = \sum_{i=0}^{N_2-1} h_2(i)\xi_1(n-i)\,, \tag{6}$$

$$\hat{x}_2(n) = \frac{1}{\tau}\sum_{j=0}^{N_1-1} h_1(j)[\hat{x}_1(n-j) - \hat{x}_1(n-j-1)]\,, \tag{7}$$

$$\hat{x}_3(n) = \frac{1}{\tau N_0}\sum_{r=0}^{N_0-1} [\hat{x}_2(n-r) - \hat{x}_2(n-r-1)]\,, \tag{8}$$

where the unique FIRs $h_2(i)$ and $h_1(i)$ are given by, respectively,

$$h_1(i) = \frac{2(2N-1) - 6i}{N(N+1)}\,, \tag{9}$$

$$h_2(i) = \frac{3(3N^2 - 3N + 2) - 18(2N-1)i + 30i^2}{N(N+1)(N+2)}\,, \tag{10}$$

III. Three-State Kalman Filtering Algorithm

In the state space, the TIE model (1) is given by

$$\begin{bmatrix} x(n) \\ y(n) \\ z(n) \end{bmatrix} = \begin{bmatrix} 1 & \tau & \tau^2/2 \\ 0 & 1 & \tau \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x(n-1) \\ y(n-1) \\ z(n-1) \end{bmatrix} + \begin{bmatrix} w_1(n,\tau) \\ w_2(n,\tau) \\ w_3(n,\tau) \end{bmatrix}, \tag{11a}$$

$$\mathbf{x}(n) = \mathbf{A}\mathbf{x}(n-1) + \mathbf{w}(n,\tau)\,, \tag{11b}$$

and (5) becomes, assuming a single receiver,

$$\xi(n) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(n) \\ y(n) \\ z(n) \end{bmatrix} + v(n)\,, \tag{12a}$$

$$\xi(n) = \mathbf{C}\mathbf{x}(n) + v(n)\,, \tag{12b}$$

The noises $\mathbf{w}(n,\tau)$ and $v(n)$ are mean zero and jointly uncorrelated. The sawtooth noise $v(n)$ has a uniform distribution $p(v) = 1/2v_{max}$ and correlated increments. Its white Gaussian approximation has a variance $V = \sigma_v^2 = \frac{1}{2v_{max}}\int_{-v_{max}}^{v_{max}} v^2 dv = v_{max}^2/3$. The autocorrelation matrix of the white Gaussian noise $\mathbf{w}(n)$ is given by

$$\mathbf{\Psi} = \tau \begin{bmatrix} q_1 + \frac{q_2\tau^2}{3} + \frac{q_3\tau^4}{20} & \frac{q_2\tau}{2} + \frac{q_3\tau^3}{8} & \frac{q_3\tau^2}{6} \\ \frac{q_2\tau}{2} + \frac{q_3\tau^3}{8} & q_2 + \frac{q_3\tau^2}{3} & \frac{q_3\tau}{2} \\ \frac{q_3\tau^2}{6} & \frac{q_3\tau}{2} & q_3 \end{bmatrix}, \tag{13}$$

in which the diffusion coefficients q's, namely q_1, q_2, and q_3, specify the white FM noise (WHFM), white random walk FM noise (WRFM), and white random run FM noise (RRFM), respectively, in the τ-domain power law.

The linear Kalman filtering algorithm reads as follows. Enter the q's, $\mathbf{R}_{n-1}$, and $\hat{\mathbf{x}}_{n-1}$ and then calculate recursively:

$$\tilde{\mathbf{R}}_n = \mathbf{A}\mathbf{R}_{n-1}\mathbf{A}^T + \mathbf{\Psi}\,, \tag{14}$$

$$\mathbf{K}_n = \tilde{\mathbf{R}}_n\mathbf{C}^T(\mathbf{C}\tilde{\mathbf{R}}_n\mathbf{C}^T + V)^{-1}\,, \tag{15}$$

$$\hat{\mathbf{x}}_n = \mathbf{A}\hat{\mathbf{x}}_{n-1} + \mathbf{K}_n(\xi_n - \mathbf{C}\mathbf{A}\hat{\mathbf{x}}_{n-1})\,, \tag{16}$$

$$\mathbf{R}_n = (\mathbf{I} - \mathbf{K}_n\mathbf{C})\tilde{\mathbf{R}}_n\,. \tag{17}$$

Below, we employ the 3-state unbiased FIR algorithm (6)–(8) and the 3-state Kalman algorithm (14)–(17) to estimate the TIE model of an oven crystal clock embedded to the Stanford Frequency Counter SR620. The measurement is done with the GPS timing sensor SynPaQ III and SR620 for $\tau = 1$ s (GPS-measurement). Simultaneously, to get a reference trend, the TIE of the same crystal clock is measured, by SR625, for the rubidium clock (Rb-measurement). Initial time and frequency shifts between two measurements are then eliminated statistically and a transition to $\tau = 10$ s is provided by the data thinning in time. At the early stage, the TIE model was identified to be quadratic, $K = 2$. Then N_l and q's were determined for the FIR and Kalman algorithms, respectively, in the minimum MSE sense.

IV. MEASUREMENTS AND ESTIMATIONS

A. Several Hours Measurements

In this experiment, a short-term measurement of the TIE has been done during several hours (Fig. 2a). The algorithm then was run. The horizons were identified for $\tau = 10s$ to be $N_1 = 155$ or 0.43 hours, $N_2 = 950$ or 2.64 hours, and $N_3 = 860$ or 2.39 hours for the Rb-measurements. Thereafter, we set the values of q's in the Kalman filter to obtain the minimum MSEs for the FIR estimates. Figure 2 and Table I illustrate these studies, showing that the unbiased FIR estimates, $\hat{x}_1(n)$, $\hat{x}_2(n)$, and $\hat{x}_3(n)$, and the relevant Kalman estimates, $\hat{x}(n)$, $\hat{y}(n)$, and $\hat{z}(n)$, respectively, are consistent with, however, some differences.

It follows from Table I that the FIR filter works accurately. Figure 2a shows that $\hat{x}_1(n)$ and $\hat{x}(n)$ track the mean value of the GPS-measurement and that their offsets from the Rb-measurement are coursed mostly by the GPS time uncertainty. In this experiment, a maximum estimate error of about 60 ns was indicated between 8th and 9th hours when a time shift in the 1 PPS signal has occurred.

In follows (Fig. 2b) that $\hat{x}_2(n)$ and $\hat{y}(n)$ fit well the weighted by $1/\tau$ increments of the Rb-measurement. Even so, there are two special ranges (dashed). In the range I, the frequency shift of about 3×10^{-11} has occurred in the span between 7th and 8th hours and no appreciable error is indicated in a range of large time shifts (between 8th and 9th hours in Fig. 2a). We associate it with the frequency shift in SR625. In the range II, the Kalman filter demonstrates a brightly pronounced instability caused likely by the temporary model uncertainty, whereas the FIR estimate is still consistent.

We watch for a bit shifted trends of $\hat{x}_3(n)$ and $\hat{z}(n)$ in Fig. 2c that may be explained by some inconsistency between the q's and N_l. It is also seen that $\hat{z}(n)$ traces much upper $\hat{x}_3(n)$ after about 8.7 hours. We associate it with the Kalman filter instability, like the case of a range II in Fig. 2a.

The experiment was repeated for $\tau = 1$ s. The results are presented in Table II to mention that, on the whole, the picture (Fig. 2) remains the same. The only principle point to notice is that the Allan deviations of all estimates are reduced by a large number of the points. The FIR and Kalman estimates behave here closer to each other, even though the former is still more accurate with its lower error and much lower Allan variance.

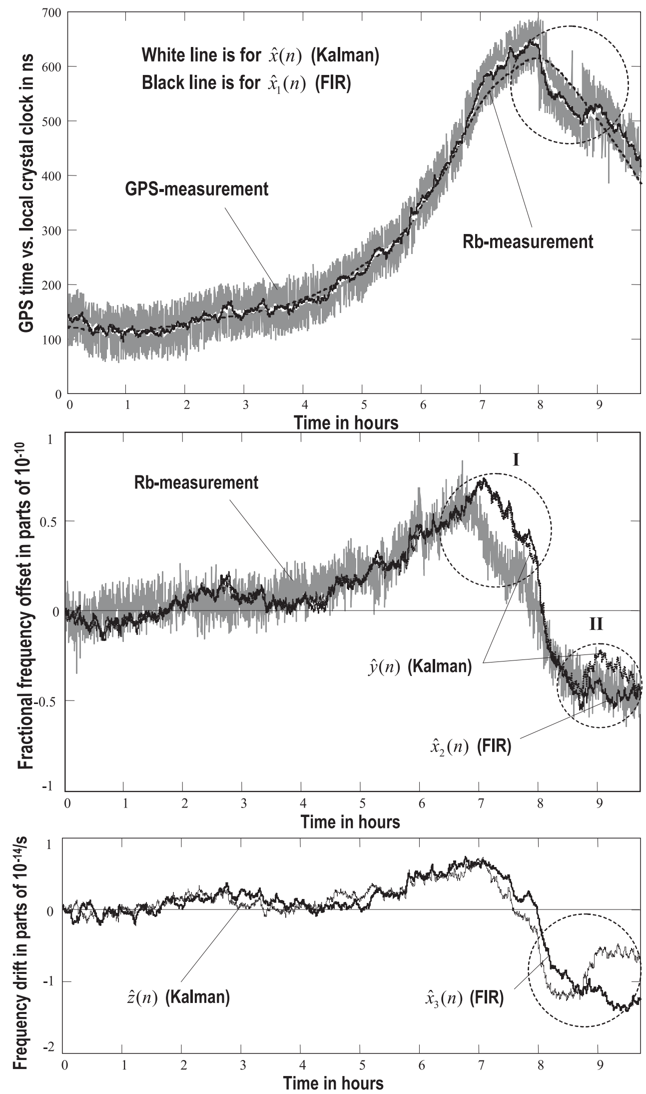

Fig. 2. Short-time measurement and estimation of the crystal clock TIE model with the 3-state unbiased FIR algorithm and the 3-state Kalman filter: (a) TIE, (b) fractional frequency offset, and (c) linear fractional frequency drift rate.

B. Long-Term Measurements

The same crystal clock was later examined during about 2.5 days using only the unbiased FIR filter. The measurements inherently show oscillations caused by day's variations in temperature and, like the previous case, all FIR estimates fit well the Rb-measurement. Employing $\hat{x}_2(n)$, the temperature drift was estimated to be about 2×10^{-10} (14 to 24 °C) and $\hat{x}_3(n)$ calculates the aging rate by $\langle \hat{x}_3(n) \rangle = 0.4 \times 10^{-10}$/day.

TABLE I

AVERAGE ERROR (ERROR) AND ALLAN DEVIATION (σ) OF THE ESTIMATE (EST) FOR 9.7 HOURS AND τ= 10 S : F IS FIR AND K IS KALMAN. ERRORS ARE GIVEN FOR RB-MEASUREMENTS

Est	x, ns		$y, 10^{-12}$		$D, 10^{-16}/s$	
	error	$\sigma_x(10)$	error	$\sigma_y(10)$	error	$\sigma_D(10)$
F	**2.8313**	1.3786	**1.4852**	**0.6399**	**4.1660**	**1.0206**
K	3.1295	**1.3627**	2.5698	0.7025	5.3121	1.3881
K-F	0.2977		1.0846		1.1461	

V. CONCLUSIONS

We investigated an unbiased FIR filter for the GPS-based measurements of the TIE K-degree polynomial model of a

TABLE II

AVERAGE ERROR (ERROR) AND ALLAN DEVIATION (σ) OF THE ESTIMATE (EST) FOR 9.7 HOURS AND τ= 1 S : F IS FIR AND K IS KALMAN. ERRORS ARE GIVEN FOR RB-MEASUREMENTS

Est	x, ns		$y, 10^{-12}$		$D, 10^{-16}/s$	
	error	$\sigma_x(1)$	error	$\sigma_y(1)$	error	$\sigma_D(1)$
F	**2.8127**	**0.1374**	**1.5638**	**0.0641**	**19.147**	**0.1037**
K	2.8965	0.3956	2.4924	0.2131	20.346	0.4315
K-F	0.0838		0.9286		1.1990	

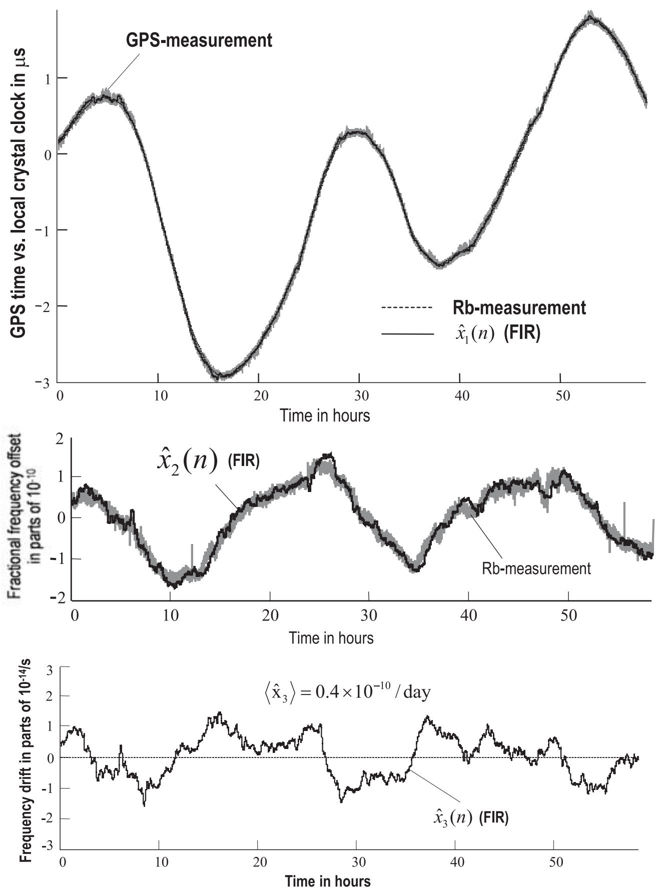

Fig. 3. Long-term measurement and estimation of the crystal clock TIE with the 3-state unbiased FIR algorithm: (a) TIE, (b) fractional frequency offset, and (c) linear fractional frequency drift rate.

local crystal clock. The trade-off between the 3-state unbiased FIR algorithm and the 3-state standard Kalman algorithm has shown their consistency. However, as it was demonstrated experimentally, the FIR filter produces a smaller error and a lower Allan variance for the sawtooth noise.

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