On the measurement of frequency and of its sample variance with high-resolution counters

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***Abstract*— A frequency counter measures the input frequency ν averaged over a suitable time τ, versus the reference clock. Beside clock interpolation, modern counters improve the resolution by averaging multiple measurements highly overlapped. In the presence of white noise, the overlapping technique improves the square uncertainty from $\sigma_\nu^2 \propto 1/\tau^2$ to $\sigma_\nu^2 \propto 1/\tau^3$. This is important because the input trigger integrates white noise over the full instrument bandwidth, which is usually of at least 100 MHz. Due to insufficient technical information, the general user is inclined to make the implicit assumption that the counter takes the bare mean. After explaining the overlapped-average mechanism, we prove that feeding a file of contiguous data into the formula of the two-sample (Allan) variance $\sigma_y^2(\tau) = \mathbb{E}\{\frac{1}{2}(\overline{y}_{k+1} - \overline{y}_k)^2\}$ gives the *modified* Allan variance mod $\sigma_y^2(\tau)$. This conclusion is based on the mathematical reverse-engineering of the formulae found in technical specifications. More details are available on the web site arxiv.org, document arXiv:physics/0411227 [1]. Our purpose is to warn the experimentalists against possible mistakes, and to encourage the manufacturers to explain what the instruments really do.**

LIST OF MAIN SYMBOLS

$\overline{}$	as in $\overline{\nu}$, time average (over the duration τ)
$\mathbb{E}$	statistical expectation
T	period, $T = 1/\nu$
$v(t)$	signal (voltage), time domain
w	weight function
x	phase time, i.e., phase noise converted into time
y	fractional frequency fluctuation, $y = \dot{x}$
ν	frequency
ν_{00}	nominal frequency (ν_0 in the general literature)
$\sigma_y^2(\tau)$	variance, Allan variance, modified Allan variance
τ	measurement time

The notation used in this article is that of general literature on phase noise and frequency stability. The reader can find an introduction and an extensive digression in Reference [2].

I. CLASSICAL RECIPROCAL COUNTERS

Figure 1 shows the basic scheme of a reciprocal frequency counter. The binary counter counts the number M of clock pulses that fit in N periods of the input signal. The counter measures the period $\overline{T} = \frac{1}{\nu_c}\frac{M}{N}$ averaged on τ, and displays the frequency $\overline{\nu} = \frac{N}{M}\nu_c$. Interchanging the role of ν and ν_c, the counter—no longer reciprocal—measures (and displays) the average frequency $\overline{\nu}$. The reciprocal scheme has the advantage of higher resolution for the following reasons.

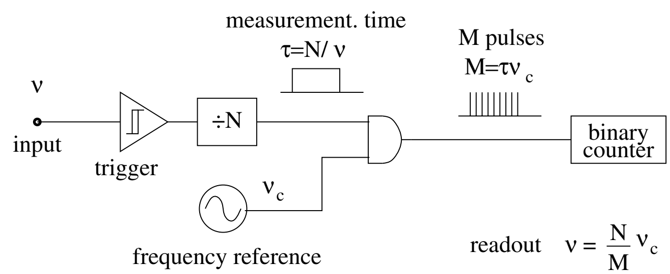

Fig. 1. Basic reciprocal frequency counter.

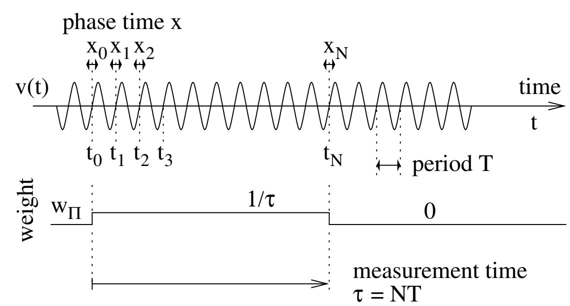

Fig. 2. Rectangular averaging mechanism in simple frequency counters.

1) The clock frequency can be close to the maximum toggling frequency of the technology employed. This choice maximizes the number M of pulses counted in a given time τ, and in turn offers the lowest quantization uncertainty.
2) Interpolation techniques enable the measurement of a fraction of a clock pulse (M is a rational number instead of an integer). The interpolator works well at a clock fixed frequency, not at the arbitrary input frequency. In single-event measurement, the interpolation resolution can be of 10 ps (2–2.5 mm of wavefront propagation in a coaxial cable). An extensive digression on the interpolation techniques is available in [3].

In the classical reciprocal counter, the uniform average over the time interval τ is used as the estimator of the frequency ν. The expectation of ν is

$$\mathbb{E}\{\nu\} = \int_{-\infty}^{+\infty} \nu(t) w_\Pi(t)\, dt \qquad \Pi \text{ estimator} \qquad (1)$$

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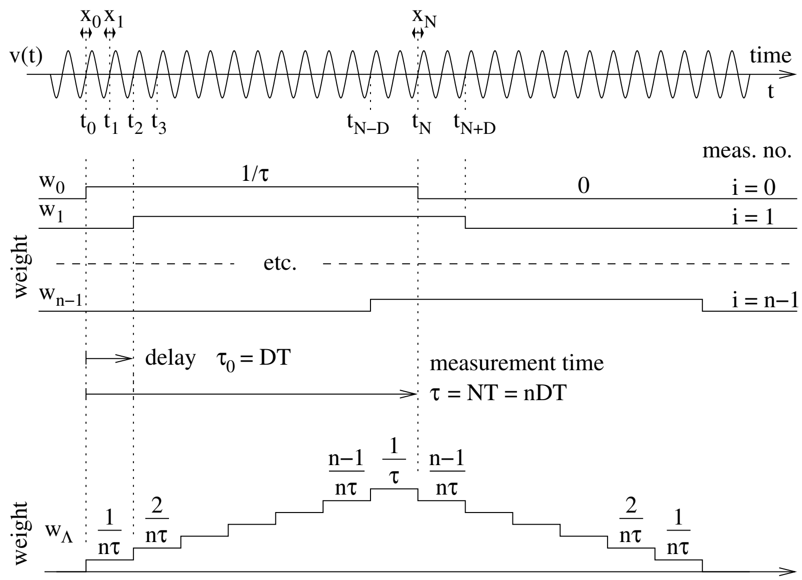

Fig. 3. Triangular averaging mechanism, implemented in some high-resolution frequency counters.

$$w_\Pi(t) = \begin{cases} \frac{1}{\tau} & 0 < t < \tau \\ 0 & \text{elsewhere} \end{cases} \tag{2}$$

With reference to Fig. 2, the measurement of τ is affected by the error $x_0 - x_N$ that results from the trigger noise and from the clock interpolator. The reference clock is assumed ideal. The timing errors x_0 and x_N are independent. In fact, x_0 and x_N are due to the interpolator noise, and to the noise of the input trigger. The interpolator is restarted every time it is used. The trigger noise spans from dc to the trigger bandwidth B, which is at least the maximum operating frequency of the counter. Due to the large input bandwidth (usually in excess of 100 MHz), white noise is dominant. The autocorrelation function of the trigger noise is a sharp pulse of duration $T_R \approx 1/B$. Denoting with σ_x^2 the variance of x, the variance of τ is $2\sigma_x^2$. Consequently, the *classical variance* of the fractional frequency fluctuation is

$$\sigma_y^2(\tau) = \frac{2\sigma_x^2}{\tau^2} \ . \tag{3}$$

The counter output is a stream estimates, one every τ seconds. As the measurement process takes τ, i.e., the duration of the weight function w_Π, the estimates are independent.

II. Enhanced-resolution reciprocal counters

Looking at Fig. 2, there is a lot of unexploited information in the zero-crossings between t_0 and t_n. More sophisticated counters (Fig. 3) measure the frequency by taking a series of n measures $\overline{\nu}_i = N/\tau_i$ delayed by $i\tau_0 = iDT$, where $\tau_i = t_{N+iD} - t_{iD}$, $i \in \{0, \cdots, n-1\}$ is the time interval measured from the (iD)-th to the $(N+iD)$-th zero crossings. The expectation of ν is evaluated as the average

$$\mathbb{E}\{\nu\} = \frac{1}{n}\sum_{i=0}^{n-1} \overline{\nu}_i \qquad \text{where } \overline{\nu}_i = N/\tau_i \ . \tag{4}$$

The above can be written as an integral similar to Eq. (1), but for the weight function w_Π replaced with w_Λ

$$\mathbb{E}\{\nu\} = \int_{-\infty}^{+\infty} \nu(t) w_\Lambda(t)\, dt \qquad \Lambda \text{ estimator} \ . \tag{5}$$

For $\tau_0 \ll \tau$, w_Λ approaches the triangular-shape function

$$w_\Lambda(t) = \begin{cases} \frac{t}{\tau} & 0 < t < \tau \\ 2 - \frac{t}{\tau} & \tau < t < 2\tau \\ 0 & \text{elsewhere} \end{cases} \tag{6}$$

Nonetheless the integral (5) can only be evaluated as the sum (4) because the time measurements take place at the zero crossings. The measures $\overline{\nu}_i$ are independent because the timing errors x_k, $k \in \{0, \cdots, n-1\}$ are uncorrelated. In fact, the interpolator is restarted every time is used, while the delay τ_0 is long as compared to the duration $T_R \approx 1/B$ of the autocorrelation function of the input white noise. The delay τ_0 is lower-bounded by the period T_{00} of the input signal and by the conversion time of the interpolator. The latter may take a few microseconds, which is significantly longer than $1/B$. The *classical variance* is

$$\sigma_y^2(\tau) = \frac{1}{n}\frac{2\sigma_x^2}{\tau^2} \ . \tag{7}$$

At low input frequency, the delay τ_0 is equal to T_{00}, i.e., one period. Thus $D = 1$, $\tau_0 = T_{00}$, and $n = N = \nu_{00}\tau$. Hence Eq. (7) is rewritten as

$$\sigma_y^2(\tau) = \frac{1}{\nu_{00}}\frac{2\sigma_x^2}{\tau^3} \ . \tag{8}$$

At high input frequency, the minimum delay τ_0 is set by the conversion time $1/\nu_I$ of the interpolator, which limits the measurement rate to ν_I measures per second. The number of overlapped measures is $n = \nu_I \tau \leq \nu_{00}\tau$, thus Eq. (7) becomes

$$\sigma_y^2(\tau) = \frac{1}{\nu_I}\frac{2\sigma_x^2}{\tau^3} \; . \qquad (9)$$

The counter output is a stream of estimates, one every τ seconds, while the measurement process takes 2τ. This means that *contiguous measures are overlapped by* τ.

III. How to identify the estimator type

It is to made clear that the enhanced resolution of the Λ-type estimator can only be achieved with multiple measurements, and that the measurement of a single event, like a start-stop time interval, can not be improved in this way.

Searching through the technical information provided by the manufacturers, one observes that the estimation problem is generally not addressed. While in old frequency counters (Π-type estimator), the measurement mechanism is sometimes explained with a figure similar to Fig. 2, the explanation of the overlapped measurements in the Λ-type estimator is not found. As the counter provides an output value every τ seconds, the experimentalist is led to believe that the estimation is always of the Π type.

Due to the large input bandwidth, in actual cases white noise is dominant. Thus the classical variance $\sigma_y^2(\tau)$ follows either the law $1/\tau^2$ or the law $1/\tau^3$. The law $1/\tau^2$ [Eq. (3)] is a mathematical property of the Π-type estimator; the law $1/\tau^3$ [either Eq. (8) or Eq. (9)] is a mathematical property of the Π-type estimator. Manufacturers usually provide formulae for the rms error that look like

$$\sigma_y = \frac{1}{\tau}\sqrt{2(\delta t)^2_{\text{trigger}} + 2(\delta t)^2_{\text{interpolator}}} \qquad (10)$$

or

$$\sigma_y = \frac{1}{\tau\sqrt{n}}\sqrt{2(\delta t)^2_{\text{trigger}} + 2(\delta t)^2_{\text{interpolator}}} \qquad n = \begin{cases} \nu_0\tau & \nu_{00} \leq \nu_I \\ \nu_I\tau & \nu_{00} > \nu_I \end{cases} \qquad (11)$$

The actual formulae may differ slightly. For example the standard deviation σ_y may be replaced with the "frequency error" $(\delta\nu)_{\text{rms}} = \nu_{00}\sigma_y$; the uncertainty and the noise of the reference ν_c may be included or not; the factor 2 in the interpolator noise may appear explicitly or not. Nonetheless, in all cases we should be able to identify a power-law of the type $\sigma_y^2 \propto 1/\tau^2$ or of the type $\sigma_y^2 \propto 1/\tau^3$. For example, the uncertainty *Stanford Research Systems SR-620* [4, p. 27] matches Eq. (10), for the internal estimator is of the Π type. Conversely, the uncertainty *Agilent Technologies 53132A* [5, pp. 3-5 to 3-8] matches Eq. (11), for the internal estimator is of the Λ type.

IV. Sample variances

The Allan variance $\sigma_y^2(\tau)$ [6] is the expected variance of two contiguous samples averaged over the time τ

$$\sigma_y^2(\tau) = \mathbb{E}\left\{\frac{1}{2}\Big[\overline{y}_{k+1} - \overline{y}_k\Big]^2\right\} \; . \qquad \text{AVAR} \qquad (12)$$

The above can be rewritten as

$$\sigma_y^2(\tau) = \mathbb{E}\left\{\Big[\int_{-\infty}^{+\infty} y(t)\, w_A(t)\, dt\Big]^2\right\} \qquad (13)$$

$$w_A = \begin{cases} -\frac{1}{\sqrt{2}\tau} & 0 < t < \tau \\ \frac{1}{\sqrt{2}\tau} & \tau < t < 2\tau \\ 0 & \text{elsewhere} \end{cases} \qquad (14)$$

The modified Allan variance $\operatorname{mod}\sigma_y^2(\tau)$ [7], [8], [9] is

$$\operatorname{mod}\sigma_y^2(\tau) = \mathbb{E}\left\{\frac{1}{2}\left[\frac{1}{n}\sum_{i=0}^{n-1}\left(\frac{1}{\tau}\int_{(i+n)\tau_0}^{(i+2n)\tau_0} y(t)\,dt + \right.\right.\right. \\ \left.\left.\left. - \frac{1}{\tau}\int_{i\tau_0}^{(i+n)\tau_0} y(t)\,dt\right)\right]^2\right\} \qquad \text{MVAR} \qquad (15)$$

with $\tau = n\tau_0$. This variance was originally introduced in the domain of optics [7] because it divides white phase noise from flicker phase noise, which the AVAR does not. This is often useful in fast measurements. MVAR is also related to the sampling theorem and to the aliasing phenomenon [10], [11] because the trigger samples the input process at a rate $1/\tau_0$. For $\tau_0 \ll \tau$, or equivalently for $n \gg 1$, it holds that

$$\operatorname{mod}\sigma_y^2(\tau) = \mathbb{E}\left\{\Big[\int_{-\infty}^{+\infty} y(t)\, w_M(t)\, dt\Big]^2\right\} \qquad (16)$$

$$w_M = \begin{cases} -\frac{1}{\sqrt{2}\tau^2}t & 0 < t < \tau \\ \frac{1}{\sqrt{2}\tau^2}(2t-3) & \tau < t < 2\tau \\ -\frac{1}{\sqrt{2}\tau^2}(t-3) & 2\tau < t < 3\tau \\ 0 & \text{elsewhere} \end{cases} \qquad (17)$$

V. Interpretation of the counter data stream

Let us first remark that

$$w_A(t) = \frac{1}{\sqrt{2}}\Big[w_\Pi(t-\tau) - w_\Pi(t)\Big] \qquad (18)$$

$$w_M(t) = \frac{1}{\sqrt{2}}\Big[w_\Lambda(t-\tau) - w_\Lambda(t)\Big] \qquad (19)$$

This is easy to prove analytically by comparing Eq. (2) to Eq. (14), and Eq. (6) to Eq. (17). A graphical proof is given in Fig. 4. Secondly, let us point out that $\sigma_y^2(\tau)$ [Eq. (12)] and $\operatorname{mod}\sigma_y^2(\tau)$ [Eq. (16)] are formally identical but for the weight function, which is $w_A(t)$ or $w_M(t)$. Thirdly, let us note that $w_A(t)$ [Eq. (18)] and $w_M(t)$ [Eq. (19)] are formally identical but for the weight function, which is $w_\Pi(t)$ or $w_\Lambda(t)$. Joining the above three facts, it follows that

> *if we feed the data stream $\overline{y}_k$ from a Λ-type counter in an algorithm intended to evaluate the Allan variance $\sigma_y^2(\tau)$ [Eq. (12)], the algorithm calculates exactly the modified Allan variance $\operatorname{mod}\sigma_y^2(\tau)$ [Eq. (16)].*

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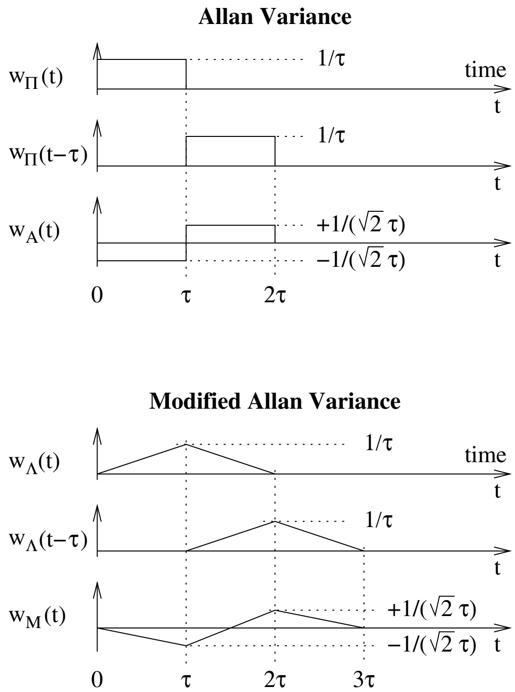

Fig. 4. Relationships between the weight functions.

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