Tracking Nonstationarities In Clock Noises Using the Dynamic Allan Variance

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***Abstract*— In a previous paper [4] we have introduced the concept of dynamic Allan variance, an extension of the classical Allan variance that is commonly used to evaluate the stability of atomic clocks. The Allan variance assumes the stationarity of the (increment of the) clock error signal, a condition that is valid for ideal clocks only. For real clocks one has to pay attention in the evaluation of the clock stability, because even for short time intervals the clock can exhibit a nonstationary behavior. Possible reasons for the lack of stationarity are sudden breakdowns, or, in the long term, clock ageing. Even cyclostationary behaviors can be observed due to daily or seasonal variation of temperature, humidity and other physical quantities that have a direct influence on the clock behavior. The main purpose of the Dynamic Allan variance is to describe the variation in time of the clock stability. In this paper we give a mathematical definition of this quantity. We apply our method to simulated data and to real data coming from a Rubidium clock. The results are very interesting, and they show that the proposed method can track and reveal in a clear and intuitive manner the changes in the behavior of atomic clock data.**

I. Introduction

Atomic clocks and oscillators can vary their behavior with time, due to sudden failures, aging, change in physical quantities that can influence the clock stability, such as temperature and humidity. It is therefore important to introduce a representation that can take into account the time-varying nature of the stability of an oscillator. We have recently proposed a new quantity, the dynamic Allan variance (DAVAR), that is a representation of the instantaneous stability of an atomic clock [4]. In that paper we gave a definition of the dynamic Allan variance and then a practical way to estimate it from experimental data. We have found out that the definition and the estimator are actually different in nature! That is, they correspond to different definitions of dynamic Allan variance. This has suggested the idea that there is no unique definition of instantaneous stability, but that there are instead a variety of different ways of defining such concept. This multiplicity of definitions is already well known in time-frequency analysis [2], a field that characterizes signals with time-varying frequencies.

A discussion of the two definitions of dynamic Allan variance will be presented in a forthcoming paper. In this paper we concentrate on the most intuitive definition of dynamic Allan variance, which basically consists in sliding the Allan variance on the random process that one wants to analyze. In Sect. II we review the concept and notation for the Allan variance. In Sect. III we define the sliding DAVAR and we also give a practical formulation of a possible estimator. The results are very interesting, and they are presented in Sect. IV for simulated data and in Sect. V for experimental data.

II. The Allan variance

Consider the common model for a signal generated by a reference oscillator[1]

$$\mathbf{u}(t) = (U_0 + \epsilon(t))\sin(2\pi\nu_0 t + \phi(t)) \tag{1}$$

An ideal oscillator would have $\epsilon(t) \equiv \phi(t) \equiv 0$, but in reality what happens is that both the nominal amplitude U_0 and frequency ν_0 are affected by random fluctuations, $\epsilon(t)$ and $\phi(t)$. It becomes hence fundamental to characterize the stability of the reference signal, and especially its deviation from the desired frequency value ν_0, since the amplitude deviation $\epsilon(t)$ is usually negligible. The Allan variance [1] is the standard tool for the estimation of the stability of an oscillator. Before reviewing its definition, let's remember the definition of the instantaneous frequency of the oscillator

$$\nu(t) = \nu_0 + \frac{1}{2\pi}\frac{d\phi(t)}{dt} \tag{2}$$

The quantity that is usually adopted is the fractional frequency deviation

$$\mathbf{y}(t) = \frac{\nu(t) - \nu_0}{\nu_0} \tag{3}$$

that is a (dimensionless) relative deviation from the nominal frequency ν_0. Another important quantity is the relative phase deviation $\mathbf{x}(t)$, that satisfies

$$\mathbf{y}(t) = \frac{d\mathbf{x}(t)}{dt} \tag{4}$$

Due to their fluctuating nature, the phase and frequency offset are modelled as stochastic processes. The Allan variance is a measure of the variation in time of $\mathbf{y}(t)$ over different

[1] We use bold letters for random quantities.

observation intervals τ The general Allan variance is the time average of the N-sample variance

$$\sigma^2_{\mathbf{y}}(N,T,\tau) = \left\langle \frac{1}{N-1}\sum_{k=1}^{N}\left(\bar{\mathbf{y}}_k - \sum_{l=1}^{N}\bar{\mathbf{y}}_l\right)^2 \right\rangle \quad (5)$$

where N is the number of samples of $\mathbf{y}(t)$, $T = t_k - t_{k-1}$ is the distance between two consecutive measurements $\bar{y}_k,\ \bar{y}_{k-1}$, and τ is the observation interval that generates the k-th measurement

$$\bar{\mathbf{y}}_k = \bar{\mathbf{y}}(t_k) = \frac{1}{\tau}\int_{t_k-\tau}^{t_k}\mathbf{y}(t)dt = \frac{\mathbf{x}(t_k) - \mathbf{x}(t_k-\tau)}{\tau} \quad (6)$$

A special case is the 2-sample Allan variance, without dead-time, that is obtained when $N = 2$ and $T = \tau$

$$\sigma^2_{\mathbf{y}}(\tau) = \frac{1}{2}\left\langle \mathbf{\Delta}(t_{k+1},\tau)^2\right\rangle \quad (7)$$

where

$$\mathbf{\Delta}(t_k,\tau) = \bar{\mathbf{y}}_{t_k+\tau} - \bar{\mathbf{y}}_{t_k} \quad (8)$$

The time average $\langle . \rangle$ of Eq. (5) and Eq. (7) requires an infinite number of samples. This implies that the stationarity of $\mathbf{\Delta}(t_{k+1},\tau)$ is necessary for the averaging - and hence the variance - to converge in general.

Stationarity of a random variable means that its statistics is invariant in time and hence does not depend on the time instant t. In case of clock noises, the stationarity of the quantity $\mathbf{\Delta}(t_{k+1},\tau)$ is ensured for white frequency and random walk frequency noises, but other types of noises or sudden change in the behavior do not necessarily ensure the stationarity of $\mathbf{\Delta}(t_{k+1},\tau)$.

If we consider continuous measurements, analogous continuous quantities are

$$\mathbf{\Delta}(t,\tau) = h_\tau(t) * \mathbf{y}(t) \quad (9)$$

$$= \int_{-\infty}^{+\infty} h_\tau(t-t')\mathbf{y}(t')dt' \quad (10)$$

where the star sign indicates convolution and $h_\tau(t)$ is the Allan window defined as

$$h_\tau(t) = \begin{cases} -\frac{1}{\tau} & 0 \le t < \tau \\ \frac{1}{\tau} & -\tau \le t < 0 \end{cases} \quad (11)$$

III. The dynamic Allan variance

The most natural way to define the instantaneous stability of a random process $\mathbf{x}(t)$ (representing the clock phase or frequency deviation, or in general any stochastic process) is to *slide* the Allan variance on the data. That is, to do the following things:

1) Obtain the signal $\mathbf{x}_T(t)$ by truncating the original signal $\mathbf{x}(t)$ in an interval of duration T.
2) Estimate the Allan variance $\hat{\sigma}^2_y(t,\tau)$ of $\mathbf{x}_T(t)$.
3) Repeat from step 1 by truncating a different piece of signal $\mathbf{x}(t)$.

The quantity $\hat{\sigma}_y(t,\tau)$ can be interpreted as the instantaneous stability of the process $\mathbf{x}(t)$. Its meaning is the average stability of $\mathbf{x}(t)$ in the interval T. When we slide the interval and we select another part of the signal, if $\mathbf{x}(t)$ is nonstationary we will find in general a different stability. The collection of the deviations $\hat{\sigma}_y(t,\tau)$ is a representation of the instantaneous stability of the clock. Our definition of the dynamic Allan variance follows this simple concept. In particular, we will now derive the dynamic Allan variance for the continuous time case, and we will then examine a possible estimator of it.

A. Definition

We start with the random process $\mathbf{y}(t')$, that is the frequency deviation. We truncate the frequency deviation on the interval $t - T/2 \le t' \le t + T/2$

$$\mathbf{y}_T(t') = \mathbf{y}(t')P_T(t-t') \quad (12)$$

where $P_T(t)$ is the rectangular window defined as

$$P_T(t) = \begin{cases} 1, & |t| \le T/2 \\ 0, & \text{elsewhere} \end{cases} \quad (13)$$

We now build the increment process $\mathbf{\Delta}(t,t',\tau)$ by convolving $\mathbf{y}_T(t')$ with the Allan window $h_\tau(t)$

$$\mathbf{\Delta}(t,t',\tau) = \int h_\tau(t'-t'')\mathbf{y}_T(t'')dt''$$

To avoid problems with the boundary of the window, we set

$$t-(T/2-\tau) \le t' \le t+(T/2-\tau) \quad (14)$$

$$0 < \tau \le \tau_M \quad (15)$$

The value τ_M can be chosen for example as

$$\tau_M = \lfloor T/3 \rfloor \quad (16)$$

where $\lfloor\rfloor$ denotes the integer part rounded towards $-\infty$. We then average in time the squared increment, to obtain an estimate of the sliding dynamic Allan variance

$$\sigma^2_y(t,\tau) = \frac{1}{2}\left\langle \mathbf{\Delta}^2(t,t',\tau)\right\rangle \quad (17)$$

$$= \frac{1}{2T}\int_{t-T/2+\tau}^{t+T/2-\tau}\mathbf{\Delta}^2(t,t',\tau)dt' \quad (18)$$

This quantity is still a stochastic process. Therefore we define the sliding dynamic Allan variance, or DAVAR, to be the expectation value of this quantity

$$\sigma^2_y(t,\tau) = E[\sigma^2_y(t,\tau)] \quad (19)$$

$$= \frac{1}{2}E\left[\left\langle \mathbf{\Delta}^2(t,t',\tau)\right\rangle\right] \quad (20)$$

The sliding Allan variance is now defined as a deterministic quantity for any t and τ values within the chosen range.

We can rewrite the DAVAR as a function of the phase offset $\mathbf{x}(t)$. Because of Eq. (4), the increment from Eq. (6) can be written as

$$\mathbf{\Delta}(t,t',\tau) = \frac{1}{\tau}\int_{t'}^{t'+\tau}\mathbf{y}_T(t'')dt'' - \frac{1}{\tau}\int_{t'-\tau}^{t'}\mathbf{y}_T(t'')dt'' \quad (21)$$

$$= \frac{1}{\tau}\left[\mathbf{x}_T(t'+\tau) - 2\mathbf{x}_T(t') + \mathbf{x}_T(t'-\tau)\right] \quad (22)$$

and hence

$$\sigma_y^2(t,\tau) = \frac{1}{2\tau^2 T}\times \int_{t-T/2+\tau}^{t+T/2-\tau} E\left[(\mathbf{x}_T(t'+\tau) - 2\mathbf{x}_T(t') + \mathbf{x}_T(t'-\tau))^2\right] dt' \quad (23)$$

with

$$0 < \tau \leq \tau_M \quad (24)$$

which is the desired form.

B. Estimator

The dynamic Allan variance has been defined in Eq. (23) for the continuous time case. One may ask whether it is possible to estimate these quantities from experimental data. The way the dynamic Allan variance has been defined leads directly to the formulation of a continuous time estimator. The estimator is

$$\hat{\sigma}_y^2(t,\tau) = \frac{1}{2\tau^2 T}\times \int_{t-T/2+\tau}^{t+T/2-\tau} (\mathbf{x}_T(t'+\tau) - 2\mathbf{x}_T(t') + \mathbf{x}_T(t'-\tau))^2 dt' \quad (25)$$

We immediately see that

$$\sigma_y^2(t,\tau) = E\left[\hat{\sigma}_y^2(t,\tau)\right] \quad (26)$$

and hence the estimator is unbiased. With a single data sequence we can estimate only one sample of dynamic Allan variance $\hat{\sigma}_y^2(t,\tau)$ and this single sample in the estimate of the theoretical DAVAR.

IV. Analysis of simulated data

In order to check the validity of our method, we have applied the dynamic Allan variance to a set of simulated nonstationary data. In particular we have considered three basic cases:

1) White noise with a "jump". A random phase process $\mathbf{x}(t)$ is built by using the following model

$$\mathbf{x}(t) = \sigma(t)\mathbf{w}(t) \quad (27)$$

where $\mathbf{w}(t)$ is a white noise with unit variance and $\sigma(t)$ is a function of time that takes two values

$$\sigma(t) = \begin{cases} \sigma_1, & t \leq t_0 \\ \sigma_2, & t > t_0 \end{cases} \quad (28)$$

2) White noise with a "bump". A random phase $\mathbf{x}(t)$ is generated using the same model of point 1 above, where the time-varying variance $\sigma(t)$ is now given by

$$\sigma(t) = \begin{cases} \sigma_1, & t \leq t_1 \\ \sigma_2, & t_1 < t \leq t_2 \\ \sigma_1, & t > t_2 \end{cases} \quad (29)$$

3) Two noises in sequence. The random phase $\mathbf{x}(t)$ is given by the sequence of data obtained from a white noise (white PM noise) for the first half and then a Wiener process corresponding to a white FM noise in the second half.

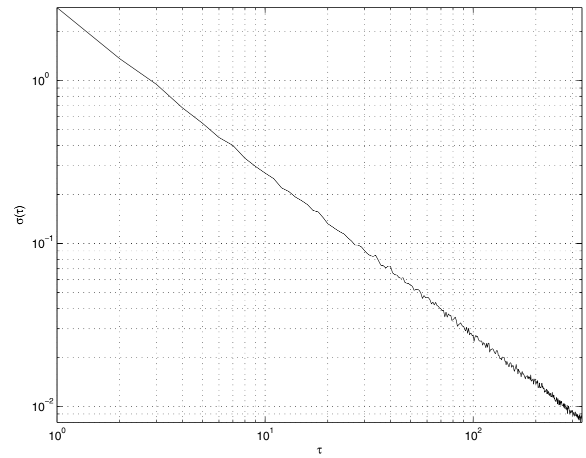

Fig. 1. The Allan deviation of the random white phase $\mathbf{x}(t)$ as defined in Eq. (27) and Eq. (28).

For each case we show the Allan variance, the definition of the DAVAR from Eq. (23), numerically estimated over a large number of realizations, and its estimated value from Eq. (25), estimated over a single data sample. The results are very interesting. For the first two cases, the DAVAR in Fig. 2 and Fig. 5 is able to track the change in the variance in a very clear way. We point out that the steep changes in the DAVAR values happen at time instants that are also related to the length T of the analysis window. In particular, the shortest the window, the sooner the variation will be tracked. Unfortunately one cannot reduce the window length T at will, because with a short window, the amount of data is reduced and therefore the quality of the estimate is very poor giving a large uncertainty of the DAVAR estimate. The DAVAR estimates, shown in Fig. 3 and Fig. 6 respectively, demonstrate that the quality of the estimates is inversely proportional to T. There is hence a *trade-off* between the window length T and the quality of the estimate. It is interesting to point out that the *classical* Allan variance, shown in Fig. 1 and Fig. 4, cannot track the nonstationarity in the random signal and would erroneously point out an average stationary behavior for the whole period.

For the third case, the DAVAR is again able to highlight the change in slope of the random phase $\mathbf{x}(t)$, as shown in Fig. 8. The estimator is represented in Fig. 9. Again the Allan variance, shown in Fig. 7, does not point out the change from white noise to the Wiener process affecting $\mathbf{x}(t)$.

V. Analysis of experimental data

We have studied measures from a Rubidium clock. We applied an algorithm (see the Conclusion) that estimates the sliding DAVAR from a single set of data, and the result is shown in Fig. 10 (the dynamic Allan deviation, defined as the square root of the DAVAR, is represented). The DAVAR alerts that the signal is nonstationary. In particular some long term noises seem to appear twice and immediately disappear.

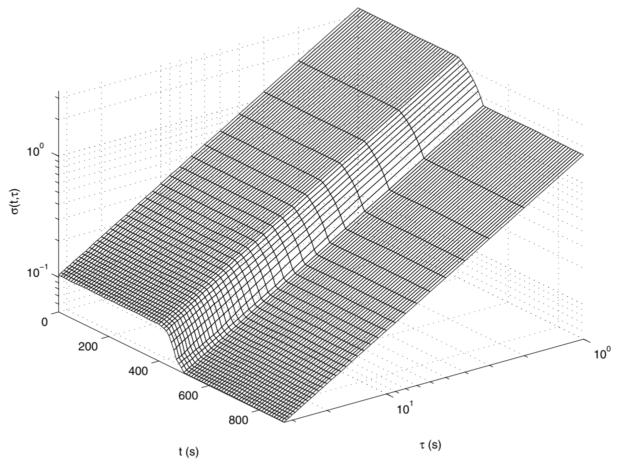

Fig. 2. The dynamic Allan deviation of the random phase $\mathbf{x}(t)$ as defined in Eq. (27) and Eq. (28).

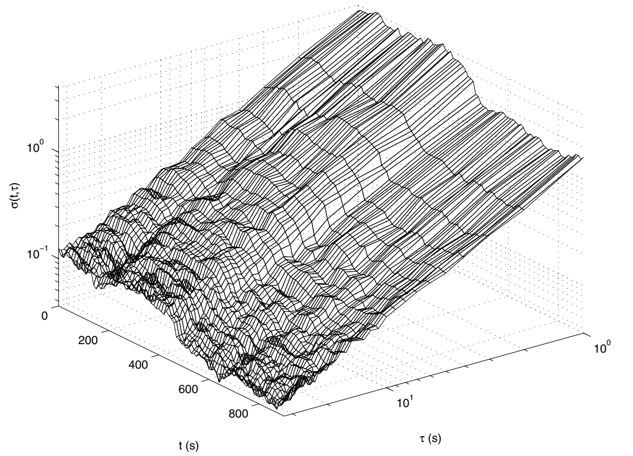

Fig. 3. The estimate of the dynamic Allan deviation of the random phase $\mathbf{x}(t)$ as defined in Eq. (27) and Eq. (28).

Such a behavior may be due to a frequency jump which is interpreted as the insurgence of a long term noise. A complete characterization of the DAVAR as a tool to identify and classify nonstationary behaviors is currently under investigation. A possible application of the DAVAR is also reported in [3].

VI. Conclusion

We have analyzed the definition of the dynamic Allan variance, a tool useful to track nonstationarities in clock behavior, and we have proposed a sound mathematical definition for both the theoretical DAVAR and its estimator.

Many variations in the dynamic Allan variance estimate are due to random fluctuations and are not a direct proof of nonstationarity. We are currently working at defining surfaces of confidence that can be used in association with the DAVAR surface in order to establish the presence of anomalous changes.

To encourage the use and test of the proposed DAVAR we have prepared a free Matlab code that can be downloaded

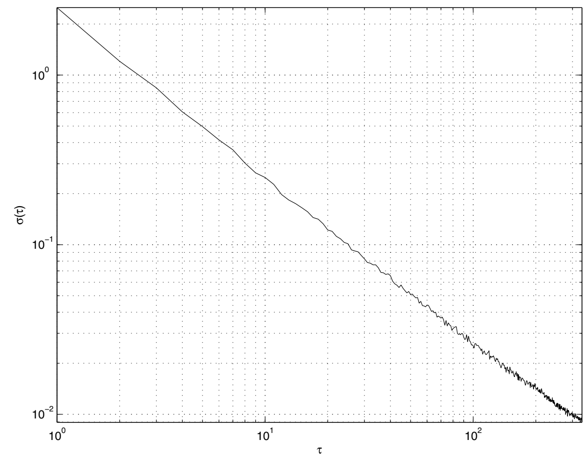

Fig. 4. The Allan deviation of the random phase $\mathbf{x}(t)$ as defined in Eq. (27) and Eq. (29).

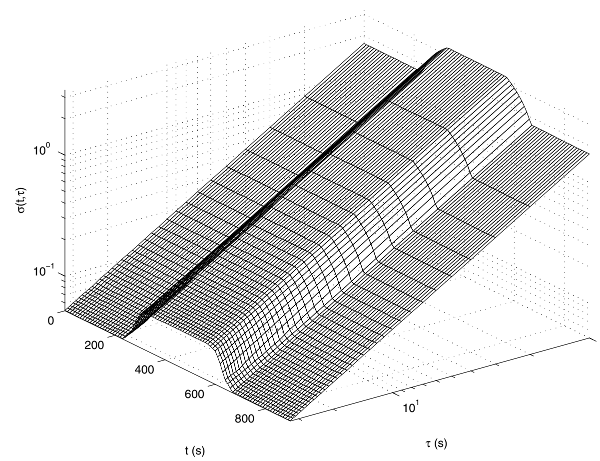

Fig. 5. The dynamic Allan deviation of the random phase $\mathbf{x}(t)$ as defined in Eq. (27) and Eq. (29).

from `www.ien.it/tf/ts/clock_behavior.shtml`. Any comment from users is welcome.

References

[1] D. W. Allan, "Statistics of atomic frequency standards," *Proc. IEEE*, vol. 54, pp. 221-230, 1966.
[2] L. Cohen, *Time-Frequency Analysis*, Prentice-Hall, 1995.
[3] D. Orgiazzi, P. Tavella, G. Cerretto, P. Collins and F. Lahaye, "First Evaluation of a Rapid Time Transfer within the IGS Global Real-Time Network," to appear in *Proc of 2005 Joint IEEE International Frequency Control Symposium and Precise Time and Time Interval (PTTI) Systems and Applications Meeting*, 29-31 August 2005, Vancouver, BC, Canada
[4] L. Galleani and P. Tavella, "The characterization of clock behavior with the dynamic Allan variance," *IEEE FCS-EFTF 2003*, 5-8 May 2003, Tampa, Florida, US.

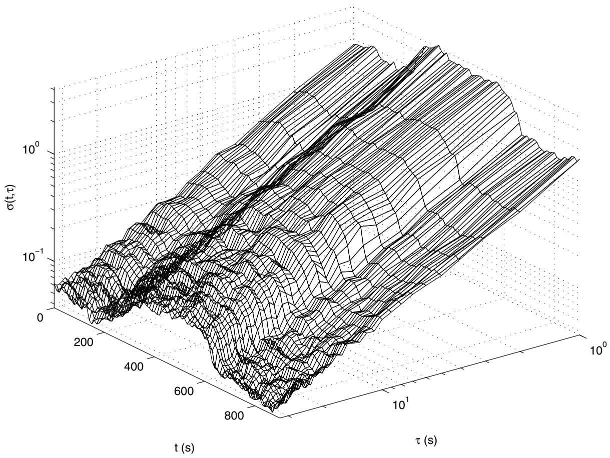

Fig. 6. The estimate of the dynamic Allan deviation of the random phase $\mathbf{x}(t)$ as defined in Eq. (27) and Eq. (28).

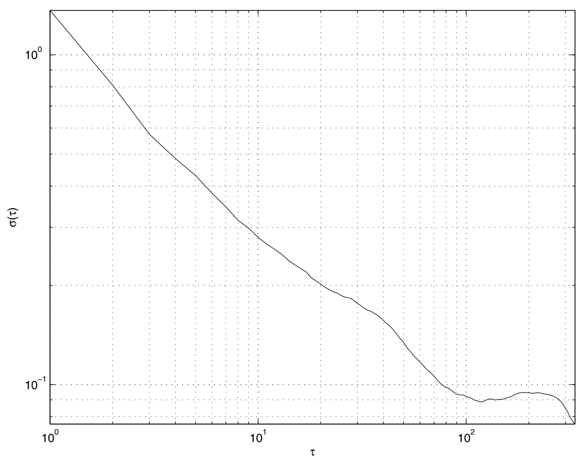

Fig. 7. The Allan devaition of a random phase $\mathbf{x}(t)$ that is made by the sequence of a white noise (white PM noise) and a Wiener process (white FM noise).

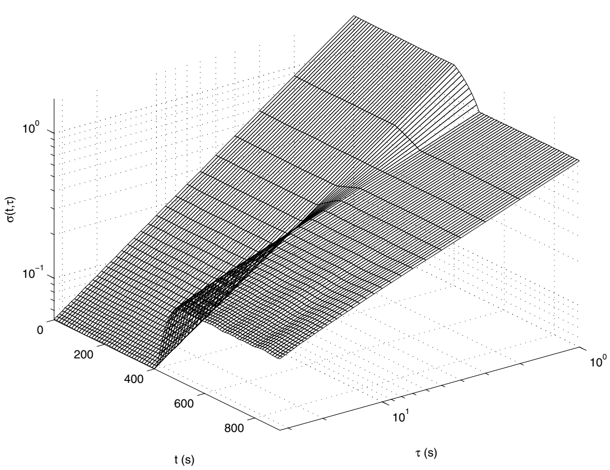

Fig. 8. The dynamic Allan deviation of a random phase $\mathbf{x}(t)$ that is made by the sequence of a white noise (white PM noise) and a Wiener process (white FM noise).

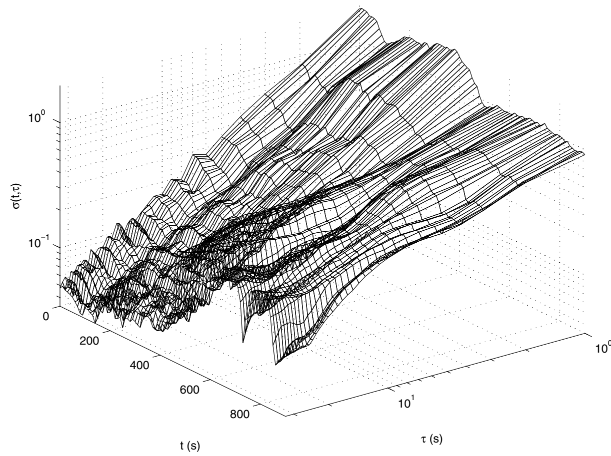

Fig. 9. The estimate of the dynamic Allan deviation of a random phase $\mathbf{x}(t)$ that is made by the sequence of a white noise (white PM noise) and a Wiener process (white FM noise).

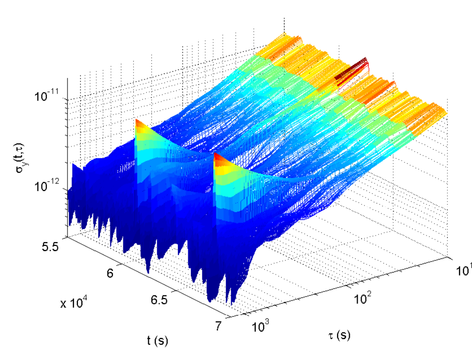

Fig. 10. Dynamic Allan deviation of experimental data coming from a Rubidium clock. Notice the nonstationary behavior of the signal.