# Refining the evaluation of uncertainties in [*UTC – UTC (k)*]


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**Abstract— We refine the evaluation of the uncertainty on *[UTC – UTC (k)]*, by taking into account in particular the contribution of the correlations. To easily handle the different computational requirements we use a matrix formulation. Using this matrix formalism we re-analyze the link-based uncertainties and we obtain the same solution as given in [1]. Next we evaluate the site–based uncertainties and compare the results with the link-based uncertainties.**


## I. INTRODUCTION

In this paper we refine the evaluation of the uncertainty in [*UTC – UTC(k)*]. In particular we evaluate the contribution of the correlations in a matrix formalism and consider the effects of two different types of uncertainties:

1. **Link-based uncertainties**, the same as presented in previous paper [1] where we considered the links measuring [*UTC(i) – UTC(j)*] as the main source of uncertainty and we introduced the link uncertainty values as published in BIPM Circular T (Section 6), considering those value as "global link" uncertainty and not ascribing the uncertainty to the different laboratories. The different links are assumed as uncorrelated and the only correlations are due to partially overlapping links.
2. **Site-based uncertainties** wherein the uncertainty of the links is considered as due to different contributions, mainly the time transfer equipment of the laboratories that are being used for different links may correlate the results.

To easily handle the different computational requirements we use a matrix formulation presented in the first part. In the second part we re-analyze the link-based uncertainties using this matrix formulation and we obtain the same solution given in [1]. After that, we evaluate the site–based uncertainties and compare the results with the link-based approach to finally estimate the uncertainty of [*UTC – UTC(k)*].

## II. INTRODUCTION TO MATRIX FORMULATION

We introduce a matrix formulation for dealing with the uncertainties [*UTC – UTC (k)*]. This is based on the general optimal estimation theory and it is possible to find the complete description of these concepts in [2,3].

We start defining the covariance of two random variables $X$ and $Y$ as:

$$Cov(X,Y) = E((X - E(X))(Y - E(Y))) \qquad (1)$$

and, in particular, $Cov(X,Y)=E(XY)$ if the variables $X$ and $Y$ have zero mean. It appears that $Cov(X,X)=Var(X)$. We can have a different situation considering $n$ random variables $X_1, X_2, ..., X_n$. In this case we define the column vector $\boldsymbol{X}$ as $\boldsymbol{X} = (X_1, X_2, ..., X_n)^T$, where superscript $^T$ denotes transposition. In general the components of $\boldsymbol{X}$ may be correlated and have non zero means values. We denote the respective means as $E(X_1), E(X_2), ...., E(X_n)$ and arrange them in the vector $\boldsymbol{E(X)}$.

We can define a matrix that describes the variances and covariances of the $n$ variables, this is the *covariance matrix* defined as:

$$\boldsymbol{\Sigma_X} = \boldsymbol{E}\left((\boldsymbol{X} - \boldsymbol{E}[\boldsymbol{X}])(\boldsymbol{X} - \boldsymbol{E}[\boldsymbol{X}])^{\boldsymbol{T}}\right). \qquad (2)$$

We recognize the variance of the variable $X_i$ in the *(i,i)* diagonal element while the off diagonal terms *(i,j)* contains the covariance between $X_i$ and $X_j$ as defined in (1). We now define a new set of random variables $Y_1, Y_2, ...., Y_n$

that are linearly related to $X_1, X_2, ..., X_n$ via the equation

$$Y = AX + B \tag{3}$$

where $Y$ is the column vector $(Y_1, Y_2, ...., Y_n)^T$, $B$ is a constant vector and $A$ is called design matrix.

We find the mean of the $Y$ by evaluating the expectation values at both sides of the linear transformation $Y = AX + B$ and we obtain $E(Y) = AE(X) + B$. The covariance matrix of $Y$ can be obtained using the definition analogous to (2) and:

$$\begin{aligned} \Sigma_Y &= E\left((Y - E(Y))(Y - E(Y))^T\right) = \\ &= E\left(A(X - E(X))(X - E(X))^T A^T\right) = A\Sigma_X A^T \end{aligned} \tag{4}$$

This transformation of covariance matrices is the base of the theory of optimal estimation [2]. More information about its application in statistical and probability problems may be found in [3]. On the application of optimal estimation theory in metrology and time related problems and in the definition of ensemble time scale the reader is referred to [4, 5, 6] and the references quoted there. We are going to use these relations many times in this work.

## III. LINK-BASED UNCERTAINTY

We start again from the approach presented in [1] to evaluate the uncertainty on *UTC-UTC(j)* and we translate it in the matrix formulation. We considered the definition of the ensemble time EAL as:

$$\begin{cases} \sum_{i=1}^{N} w_i x_i(t) = \sum_{i=1}^{N} w_i h_i'(t) \\ x_i(t) - x_j(t) = x_{i,j}(t) \end{cases}$$

where $N$ is the number of the atomic clocks, $w_i$ the weight of the clocks, $h_i(t)$ is the reading of clock $H_i$ at time $t$, and $h_i'(t)$ is the prediction of the reading of clock $H_i$ to guarantee the continuity of the time scale and $x_i(t) = EAL(t) - h_i(t)$. The solution is:

$$x_j = EAL(t) - h_j(t) = \sum_{i=1}^{N} w_i\left[h_i(t) + h_i'(t)\right] - h_j(t)$$

where the variables are the weights, the measures $x_{i,j}(t) = x_j(t) - x_i(t)$ and the predictions $h_i'(t)$. The predictions and the weights are fixed by appropriate algorithms based on the past clock behaviour, therefore we consider the measures $x_{i,j}$ are the only contributors to the uncertainties in the knowledge of $x_j$. Moreover, *[UTC – EAL]* depends only on pre-determined leap seconds and frequency steers that do not add uncertainty. The uncertainties of [*UTC – UTC (k)]* are therefore close to the uncertainties of [*TAI – UTC (k)]* and [*EAL – UTC (k)]*.

The equation above is valid for any clock contributing to UTC, without loosing generality we consider $h_j$*=UTC(PTB)* to have an explicit example for the following evaluations but the relationship may be applied to any other clock or *UTC(k)* time scale. Therefore this relation becomes:

$$\begin{aligned} x_{UTC(PTB)} &= EAL(t) - UTC(PTB) = \\ &\sum_{i=1}^{N} w_i\left[h_i(t) + h_i'(t)\right] - UTC(PTB) \end{aligned} \tag{4.1}$$

where the $x_{i,j}$ becomes $x_{i,UTC(PTB)}$ and every measures are reported to PTB. All the other clocks can be referred to *EAL* by using the relationship:

$$x_i(t) = x_j(t) + x_{i,j}(t) \tag{4.2}$$

where $x_i$ is the difference between the clock $i$ or *UTC(i)* and *EAL.* In [1] we have obtained the uncertainty of *UTC-UTC (j)* using the law of propagation of uncertainty [7] and we obtained:

$$\begin{aligned} u_{x_j}^2 &= \sum_{i=1}^{N}\left(\frac{\partial x_j}{\partial x_{i,j}}\right)^2 u_{x_{i,j}}^2 + 2\sum_{i=1}^{N-1}\sum_{k=i+1}^{N}\frac{\partial x_j}{\partial x_{i,j}}\frac{\partial x_j}{\partial x_{k,j}}u(x_{i,j}, x_{k,j}) = \\ &= \sum_{i=1}^{N} w_i^2 u_{x_{i,j}}^2 + 2\sum_{i=1}^{N-1}\sum_{k=i+1}^{N} w_i w_k u(x_{i,j}, x_{k,j}) \end{aligned} \tag{5}$$

In this paper we obtain and compare the equations that leads to the different estimates of the uncertainty of *UTC-UTC(PTB)* by using different approaches. We can generalize the equations referring to any laboratory but for simplicity we have concentrate on the case of *UTC-UTC(PTB)*. The numerical impacts will nevertheless evaluated on all the laboratories in Sec V.

To show the matrix formulation of linked-based uncertainty we consider the real situation of the current international link network reported by Circular T 207 (March 2005). We have 56 laboratories and 55 links. The pivot laboratories are NIST, USNO, NICT, NTSC and PTB.

In order to introduce the matrix expression analogous to (4.1) and (5), having 56 laboratories and 55 links referring each laboratories to PTB, we set $W^T = (W_{AOS}\ W_{APL}\ ...\ W_{VSL})$ the row vector (1x55) of the weights of each laboratory apart from the PTB laboratory and ***LinkPTB*** (55x1) the column vector of the links refereed to PTB as:

$$Link_{PTB} = \begin{pmatrix} x_{AOS,PTB} \\ x_{APL,PTB} \\ \cdots \\ x_{VSL-PTB} \end{pmatrix}.$$

Since the prediction in (4.1) doesn't carry uncertainty, we neglect the prediction terms at the moment and continue the evaluation by considering only the link measurement. We can thus write (4.1) as:

$$EAL - UTC(PTB) = \boldsymbol{W}^{T} \cdot \boldsymbol{Link}_{PTB} \,. \tag{6}$$

In the Circular T we have the links $x_{i,j}$, not directly referred to PTB, we call ***Link*** this vector which appears as:

$$\boldsymbol{Link} = \begin{pmatrix} x_{AOS,PTB} \\ x_{APL,USNO} \\ \cdots \\ x_{VSL-PTB} \end{pmatrix}$$

The covariance matrix of the links may be written using the relation (2). In [1] we considered negligible the correlations between different links, therefore we used a covariance matrix of the links as a diagonal matrix (55x55) where the diagonal terms are simply the squared uncertainties (variances) of each link measurement as published in Circular T.

$$\boldsymbol{\Sigma}_{\boldsymbol{Link}} = \begin{pmatrix} u^2_{AOS-PTB} & 0 & 0 & 0 \\ 0 & u^2_{APL-USNO} & 0 & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & u^2_{VSL-PTB} \end{pmatrix}.$$

To estimate the uncertainty of *UTC-UTC (PTB)* from (6), we need the covariance matrix of the links referred to PTB. We need the design matrix ***A*** (55x55) such that: $\boldsymbol{Link}_{PTB} = \boldsymbol{A} \cdot \boldsymbol{Link}$. To obtain the matrix covariance of ***LinkPTB*** we can use the fundamental relationship (4):

$$\boldsymbol{\Sigma}_{\boldsymbol{Link,PTB}} = \boldsymbol{A}\boldsymbol{\Sigma}_{\boldsymbol{Link}}\boldsymbol{A}^{T} \tag{7}$$

The design matrix ***A*** is an appropriate ensemble of +1 terms which combine the link measures and refer any lab to PTB, the structure of ***A*** will be illustrated in the example of next section.

In case of link-based uncertainty [1], we observed correlation in the link measures only when a common path was partially used in two multiple links. Here we re-obtain the same result. We obtain a (55x55) covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{Link,PTB}}$ with the variance terms in the principal diagonal and the off-diagonal covariance terms only when partially overlapping links exist. To evaluate the covariance terms we calculate for example the covariance between the links *(i,PTB)* e *(j,PTB)* assuming *(j,PTB)=(j,i)+(i,PTB):*

$$\begin{aligned} &Cov((i,PTB),(j,PTB)) = \\ &= E((i,PTB)(j,PTB)) - E((i,PTB))E((j,PTB)) = Var(i,PTB) \end{aligned}$$

having used an expression for the covariance analogous to (1) and remembering that the link noise are considered as independent. The covariance term contains only the uncertainty of the common link *(i,PTB).*

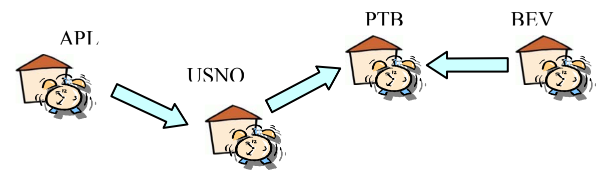
Now we have the covariance matrix of the links reported to PTB and we can obtain the uncertainty of *UTC-UTC(PTB)* applying (4) to the definition (6) as :

$$u^2_{UTC-UTC(PTB)} = \boldsymbol{W}^{T}\boldsymbol{\Sigma}_{\boldsymbol{Link,PTB}}\boldsymbol{W} \tag{8}$$

Before discussing the impact on $u^2_{UTC-UTC(PTB)}$, we want to study and understand the covariance matrix of the links called $\boldsymbol{\Sigma}_{\boldsymbol{Link,PTB}}$ to check which is the meaning of each term. In the following section we show simple examples.

## *A. Example 1*

We consider a simple case with four clocks, four labs, with two pivot laboratories (USNO and PTB), we assume the links as uncorrelated and the uncertainty are not ascribed to the laboratory equipments but to the links (the same work hypothesis of [1]).




Using the scalar expression (5) we had obtained the uncertainty of *UTC-UTC(PTB)* as:

$$\begin{aligned} u^2_{UTC-UTC(PTB)} &= W^2_{BEV}u^2_{BEV-PTB} + W^2_{APL}u^2_{APL-USNO} + \\ &+ (W_{APL} + W_{USNO})^2 u^2_{USNO-PTB} \end{aligned} \tag{9}$$

To use a matrix formulation we consider the information obtained by Circular T. We have a diagonal covariance matrix of the links

$$\boldsymbol{\Sigma}_{\boldsymbol{Link}} = \begin{pmatrix} u^2_{APL-USNO} & 0 & 0 \\ 0 & u^2_{BEV-PTB} & 0 \\ 0 & 0 & u^2_{USNO-PTB} \end{pmatrix}$$

and we can obtain the covariance matrix of the links referred to PTB. In this case we would have

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \boldsymbol{Link} = \begin{pmatrix} x_{APL,USNO} \\ x_{BEV,PTB} \\ x_{USNO,PTB} \end{pmatrix},$$

$$\boldsymbol{Link}_{PTB} = \boldsymbol{A} \cdot \boldsymbol{Link} = \begin{pmatrix} x_{APL,USNO} + x_{USNO,PTB} \\ x_{BEV,PTB} \\ x_{USNO-PTB} \end{pmatrix}$$

Using the relation $\boldsymbol{\Sigma}_{\boldsymbol{Link,PTB}} = \boldsymbol{A\Sigma}_{\boldsymbol{Link}}\boldsymbol{A}^{\boldsymbol{T}}$ we obtain the covariance matrix of the links reported to PTB:

$$\boldsymbol{\Sigma}_{\boldsymbol{Link,PTB}} = \begin{pmatrix} u^2_{APL-USNO} + u^2_{USNO-PTB} & 0 & u^2_{USNO-PTB} \\ 0 & u^2_{BEV-PTB} & 0 \\ u^2_{USNO-PTB} & 0 & u^2_{USNO-PTB} \end{pmatrix}$$

The uncertainty of *UTC-UTC(PTB)* can be obtained with (8):

$$u^2_{UTC-UTC(PTB)} = $$
$$= \begin{pmatrix} W_{APL} & W_{BEV} & W_{USNO} \end{pmatrix} \begin{pmatrix} u^2_{APL-USNO} + u^2_{USNO-PTB} & 0 & u^2_{USNO-PTB} \\ 0 & u^2_{BEV-PTB} & 0 \\ u^2_{USNO-PTB} & 0 & u^2_{USNO-PTB} \end{pmatrix} \begin{pmatrix} W_{APL} \\ W_{BEV} \\ W_{USNO} \end{pmatrix}$$

which leads to the same result of (9).

This equivalence of the formulations can be demonstrated for each laboratory. Using the complete (55x55) design matrix **A** corresponding to the current international link network, we obtain the same value for the uncertainty of *UTC-UTC(PTB)* that we obtained using (5).

## IV. Site-Based uncertainty

In this case, we assume the uncertainty is no longer due to unknowns in the "global links" as a whole. Instead it is attributed to the different laboratories assuming that the uncertainty of the link *(i,j)* is mostly given by the uncertainty added by the measuring equipment in lab *i* and by the uncertainty added by the measuring equipment in lab *j*. The assumption in this case is that the uncertainty of the link (*i,j*) may be decomposed as $u^2_{x_{i,j}} = u_i^2 + u_j^2$.

To evaluate the impact of this different point of view, we need to estimate the covariance matrices as in the previous part. The difference is in the construction of these matrices. In this case we have a vector ***Lab*** containing the contribution of each single lab to the measurement and to the uncertainty and using a design matrix ***B*** we want to obtain the matrix of original links in the Circular T. Then the covariance matrix of the links is used as in the previous case. The difference is that we want here to build the covariance matrix of the links as obtained by the summation of laboratory contributions. Clearly the contribution of each single lab is not trivial to be estimated, but at the moment we concentrate on evaluating how contributions are mixed up to obtain the final uncertainty. Later we will deal with the estimation of single lab contribution. We define a column vector called ***Lab*** which in the current TAI configuration is given by 56 laboratories therefore the vector has dimensions (56x1):

$$\boldsymbol{Lab} = \begin{pmatrix} AOS \\ APL \\ AUS \\ \dots \\ VSL \end{pmatrix}$$

using a design matrix ***B*** (55x56) we obtain the column vector of the original links ***Link*** (55x1) as reported in the Circular T by

$$\boldsymbol{Link} = \boldsymbol{B} \cdot \boldsymbol{Lab}\,.$$

The design matrix ***B*** is a matrix with 1 and –1 in the columns of the laboratory *i* and *j* respectively if the link is $x_{i,j}$. The row is the position of the link $x_{i,j}$. In our case we have considered them in alphabetical order as the Circular T. The relationship of the respective covariance matrices would be, using (4):

$$\boldsymbol{\Sigma}_{\boldsymbol{Link}} = \boldsymbol{B\Sigma}_{\boldsymbol{Lab}}\boldsymbol{B}^{\boldsymbol{T}}$$

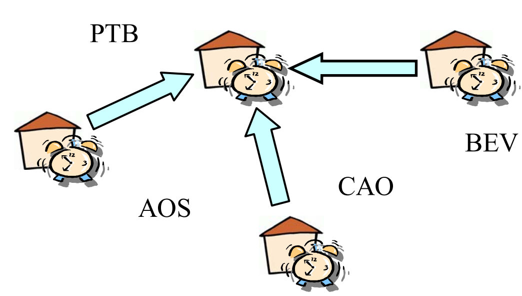
In this case, $\boldsymbol{\Sigma}_{\boldsymbol{Link}}$ would not be a diagonal matrix because the uncertainty contribution due to the equipment in the common lab is a common correlation factor. Having evaluated $\boldsymbol{\Sigma}_{\boldsymbol{Link}}$, we can use the formulae (7) and (8) as the previous paragraph to calculate the uncertainty of *UTC-UTC (PTB).*

we now illustrate some simple examples to understand the correlations and the matrix composition.

The dimension of the ***Lab*** vector can be different if we consider different equipments for the links in the pivot laboratories. For example the PTB laboratory uses the TWSTFT equipment for the link with IEN, OP, USNO etc. but it also uses the GPS P3 equipment for the link with NICT. In this case the vector ***Lab*** would have two type of equipment for the PTB lab so it would have dimension (57x1). This consideration can be generalized.

### *A. Example 2*

The first example is the case of the laboratory PTB linked to 3 laboratories with the same equipment (only one pivot, PTB, using only one piece of equipment):




The vector of the laboratories is:

$$\boldsymbol{Lab} = (AOS, BEV, CAO, PTB)^T$$

the column vector ***Link*** in the Circular T can be obtained using the design matrix ***B*** by:

$$\boldsymbol{Link} = \begin{pmatrix} AOS - PTB \\ BEV - PTB \\ CAO - PTB \end{pmatrix} = \boldsymbol{B} \cdot \boldsymbol{Lab} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} AOS \\ BEV \\ CAO \\ PTB \end{pmatrix}$$

Considering the covariance matrix of ***Lab*** (the different equipment either in the same or different labs are assumed uncorrelated)

$$\boldsymbol{\Sigma}_{\boldsymbol{Lab}} = \begin{pmatrix} u_{AOS}^2 & 0 & 0 & 0 \\ 0 & u_{BEV}^2 & 0 & 0 \\ 0 & 0 & u_{CAO}^2 & 0 \\ 0 & 0 & 0 & u_{PTB}^2 \end{pmatrix}$$

we obtain the covariance matrix of ***Link*** using the relation $\boldsymbol{\Sigma}_{\boldsymbol{Link}} = \boldsymbol{B\Sigma}_{\boldsymbol{Lab}}\boldsymbol{B}^{\boldsymbol{T}}$ and, in this particular case we have $\boldsymbol{\Sigma}_{\boldsymbol{Link,PTB}} = \boldsymbol{\Sigma}_{\boldsymbol{Link}}$, therefore:

$$\boldsymbol{\Sigma}_{\boldsymbol{Link,PTB}} = \boldsymbol{\Sigma}_{\boldsymbol{Link}} = \begin{pmatrix} u_{AOS}^2 + u_{PTB}^2 & u_{PTB}^2 & u_{PTB}^2 \\ u_{PTB}^2 & u_{BEV}^2 + u_{PTB}^2 & u_{PTB}^2 \\ u_{PTB}^2 & u_{PTB}^2 & u_{CAO}^2 + u_{PTB}^2 \end{pmatrix}.$$

**Remark:** We can observe the difference between the covariance matrix associated with link-based and site-based uncertainties. Both the matrixes have the same values on the main diagonal but with the off-diagonal terms are zero for the link-based uncertainties. Indeed for the site-based uncertainty the off-diagonal terms represent the correlations due to the PTB equipment that is common in all the links.

Finally we calculate the uncertainty of *UTC-UTC (PTB)* using (8) with the new covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{Link,PTB}}$:

$$u_{UTC-UTC(PTB)}^2 =$$

$$\begin{pmatrix} W_{AOS} & W_{BEV} & W_{USNO} \end{pmatrix} \boldsymbol{\Sigma}_{\boldsymbol{Link,PTB}} \begin{pmatrix} W_{AOS} \\ W_{BEV} \\ W_{USNO} \end{pmatrix} =$$

$$= W_{AOS}^2 u_{AOS}^2 + W_{BEV}^2 u_{BEV}^2 + W_{CAO}^2 u_{CAO}^2 +$$

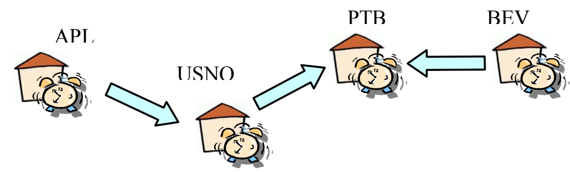
$$(W_{AOS} + W_{BEV} + W_{CAO})^2 u_{PTB}^2 =$$

$$= W_{AOS}^2 u_{AOS}^2 + W_{BEV}^2 u_{BEV}^2 + W_{CAO}^2 u_{CAO}^2 + (1 - W_{PTB})^2 u_{PTB}^2$$

where we have used the relationship $W_{AOS} + W_{BEV} + W_{CAO} + W_{PTB} = 1$ in the last passage.

## B. *Example 3*

To investigate more deeply the effect of correlations we need to consider a more complex system. Therefore we consider the case with 2 pivots (PTB and USNO), as illustrated below (some case as in Example 1)




In this case the column vector ***Lab*** is the following. In any lab only one piece of equipment is used for the links:

$$\boldsymbol{Lab} = \begin{pmatrix} APL \\ BEV \\ USNO \\ PTB \end{pmatrix}$$

the column vector ***Link*** in the Circular T can be obtained using the design matrix ***B*** as:

$$\boldsymbol{Link} = \begin{pmatrix} APL - USNO \\ BEV - PTB \\ USNO - PTB \end{pmatrix} = \boldsymbol{B} \cdot \boldsymbol{Lab} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} APL \\ BEV \\ USNO \\ PTB \end{pmatrix}$$

Considering the covariance matrix of ***Lab*** (again only one piece of equipment in each lab, all uncorrelated)

$$\boldsymbol{\Sigma}_{\boldsymbol{Lab}} = \begin{pmatrix} u_{APL}^2 & 0 & 0 & 0 \\ 0 & u_{BEV}^2 & 0 & 0 \\ 0 & 0 & u_{USNO}^2 & 0 \\ 0 & 0 & 0 & u_{PTB}^2 \end{pmatrix}$$

we obtain the covariance matrix of the ***Link*** using the relation $\boldsymbol{\Sigma}_{\boldsymbol{Link}} = \boldsymbol{B\Sigma}_{\boldsymbol{Lab}}\boldsymbol{B}^{\boldsymbol{T}}$:

$$\boldsymbol{\Sigma}_{\boldsymbol{Link}} = \begin{pmatrix} u_{APL}^2 + u_{USNO}^2 & 0 & -u_{USNO}^2 \\ 0 & u_{BEV}^2 + u_{PTB}^2 & u_{PTB}^2 \\ -u_{USNO}^2 & u_{PTB}^2 & u_{USNO}^2 + u_{PTB}^2 \end{pmatrix} \quad (10)$$

In this matrix we have different terms off-diagonal.

**Remark:** In this case we can see two different terms of correlation. The uncertainty of PTB equipment denoted by $u_{PTB}^2$ is involved in the covariance of the links USNO-PTB and BEV-PTB because both links are based on the same PTB equipment. A negative covariance given by the uncertainty of USNO equipment denoted by $-u_{USNO}^2$ is due to use the same USNO equipment for USNO-PTB and APL-USNO links but it appears with the negative sign due to the different position of USNO in the two links. If we consider these relations:

$$\begin{cases} Cov(APL-USNO, USNO-PTB) = -Var(USNO) \\ Cov(USNO-PTB, BEV-PTB) = Var(PTB) \end{cases}$$

we can understand the sign for uncertainty of USNO in the covariance matrix.

To obtain the uncertainty of *UTC-UTC (PTB)* we need the covariance matrix of the links referred to PTB. We can obtain it using again the relation $\boldsymbol{\Sigma}_{\boldsymbol{Link,PTB}} = \boldsymbol{A\Sigma}_{\boldsymbol{Link}}\boldsymbol{A}^{\boldsymbol{T}}$ with the appropriate matrix **A** yielding to:

$$\boldsymbol{\Sigma}_{\boldsymbol{Link,PTB}} = \begin{pmatrix} u_{APL}^2 + u_{PTB}^2 & u_{PTB}^2 & u_{PTB}^2 \\ u_{PTB}^2 & u_{BEV}^2 + u_{PTB}^2 & u_{PTB}^2 \\ u_{PTB}^2 & u_{PTB}^2 & u_{USNO}^2 + u_{PTB}^2 \end{pmatrix}$$

**Remark:** Off- diagonal we always have the covariance due to the uncertainty of PTB equipment and we don't have anymore the covariance due to USNO. In order to understand that we consider that, using the same USNO equipment we can write:

*(UTC(APL)-UTC(USNO))+(UTC(USNO)-UTC(PTB))=UTC(APL)-UTC(PTB).*

If the measurements are simultaneous, performed with the same USNO equipment, we may assume that the noise added by the common USNO equipment is cancelled in the link composition, so we don't have anymore the contribution of the USNO equipment. We see that the uncertainty of the USNO equip disappears and in this case it would not contribute to uncertainty of *UTC-UTC(PTB).* We call this topological case as the "sandwich case" in which the uncertainty due to the central lab equipment is cancelled.

The uncertainty of *UTC-UTC(PTB)* can be estimated using (8):

$$u_{UTC-UTC(PTB)}^2 =$$

$$= (W_{APL}\ W_{BEV}\ W_{USNO}) \begin{pmatrix} u_{APL}^2 + u_{PTB}^2 & u_{PTB}^2 & u_{PTB}^2 \\ u_{PTB}^2 & u_{BEV}^2 + u_{PTB}^2 & u_{PTB}^2 \\ u_{PTB}^2 & u_{PTB}^2 & u_{USNO}^2 + u_{PTB}^2 \end{pmatrix} \begin{pmatrix} W_{APL} \\ W_{BEV} \\ W_{USNO} \end{pmatrix} =$$

$$= W_{APL}^2 u_{APL}^2 + W_{BEV}^2 u_{BEV}^2 + W_{USNO}^2 u_{USNO}^2 + (1 - W_{PTB})^2 u_{PTB}^2$$

where we have used $W_{AOS} + W_{BEV} + W_{CAO} + W_{PTB} = 1$.

*C. Example 4*

To evaluate the real case of the correlations among the laboratories we have to consider the different lab equipments used in the different link. The situation is the same of the Example 1 and 3 with USNO e PTB pivot laboratories.

We can consider for example the link APL-USNO with USNO GPS receiver, USNO-PTB with TWSTFT receivers, and PTB-BEV with PTB GPS receiver. The column vector of the lab equipment in this case is:

$$\boldsymbol{Lab} = (APL, BEV, USNO_{GPS}, USNO_{TW}, PTB_{TW}, PTB_{GPS})^T$$

in this case the column vector ***Link*** reported in the Circular T can be written as:

$$\boldsymbol{Link} = \begin{pmatrix} APL - USNO_{GPS} \\ BEV - PTB_{GPS} \\ USNO_{TW} - PTB_{TW} \end{pmatrix} = \boldsymbol{B} \cdot \boldsymbol{Lab} =$$

$$= \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} APL \\ BEV \\ USNO_{GPS} \\ USNO_{TW} \\ PTB_{TW} \\ PTB_{GPS} \end{pmatrix}$$

using the design matrix ***B***.

Considering the covariance matrix of the ***Lab*****,** where any piece of equipment is considered, at the moment, independent from any other in the same or different lab:

$$\boldsymbol{\Sigma}_{\boldsymbol{Lab}} = \begin{pmatrix} u_{APL}^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & u_{BEV}^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & u_{USNO_{GPS}}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & u_{USNO_{TW}}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & u_{PTB_{TW}}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & u_{PTB_{GPS}}^2 \end{pmatrix}$$

we obtain the covariance matrix of the ***Link*** using the relation $\boldsymbol{\Sigma}_{\boldsymbol{Link}} = \boldsymbol{B\Sigma}_{\boldsymbol{Lab}}\boldsymbol{B}^{\boldsymbol{T}}$:

$$\boldsymbol{\Sigma}_{\boldsymbol{Link}} = \begin{pmatrix} u_{APL}^2 + u_{USNO_{GPS}}^2 & 0 & 0 \\ 0 & u_{BEV}^2 + u_{PTB_{GPS}}^2 & 0 \\ 0 & 0 & u_{USNO_{TW}}^2 + u_{PTB_{TW}}^2 \end{pmatrix}$$

In order to obtain the uncertainty of *UTC-UTC (PTB)* we need to obtain the covariance matrix of the links referred to PTB with this relation $\boldsymbol{\Sigma}_{\boldsymbol{Link,PTB}} = \boldsymbol{A\Sigma}_{\boldsymbol{Link}}\boldsymbol{A}^{\boldsymbol{T}}$

$$\boldsymbol{\Sigma}_{\boldsymbol{Link,PTB}} =$$

$$= \begin{pmatrix} \left(u_{APL}^2 + u_{USNO_{GPS}}^2\right) + \left(u_{USNO_{TW}}^2 + u_{PTB_{TW}}^2\right) & 0 & u_{USNO_{TW}}^2 + u_{PTB_{TW}}^2 \\ 0 & u_{BEV}^2 + u_{PTB_{GPS}}^2 & 0 \\ u_{USNO_{TW}}^2 + u_{PTB_{TW}}^2 & 0 & u_{USNO_{TW}}^2 + u_{PTB_{TW}}^2 \end{pmatrix}$$

The uncertainty of *UTC-UTC (PTB)* can be obtained as:

$$u^2_{UTC-UTC(PTB)} =$$

$$= \left(W_{APL}\ W_{BEV}\ W_{USNO}\right) \boldsymbol{\Sigma}_{\boldsymbol{Link,PTB}} \begin{pmatrix} W_{APL} \\ W_{BEV} \\ W_{USNO} \end{pmatrix} =$$

$$= W^2_{BEV}\left(u^2_{BEV} + u^2_{PTB_{GPS}}\right) + W^2_{APL}\left(u^2_{APL} + u^2_{USNO_{GPS}}\right) +$$

$$+ \left(W_{APL} + W_{USNO}\right)^2 \left(u^2_{USNO_{TW}} + u^2_{PTB_{TW}}\right)$$

In this case we obtain the same solution (9) presented in the Example 1 considering the assumption $u^2_{x_{i,j}} = u_i^2 + u_j^2$. In fact the pivot labs have different equipment dedicated to the different links and no correlation due to common equipment is present. Conversely, the result is different to the result of Example 3. In Example 3 USNO used the same equipment for the two links so the contribution of the USNO common equipment was cancelled, here the two USNO equipment are different and independent and they play independent role.

## V. Extending the Evaluation to Any Lab

In the previous parts we have considered only the uncertainty of *UTC-UTC(PTB)* in order to show the differences between the side-based the link-based uncertainties. Now we want to determine the uncertainty of *UTC-UTC(k)* for every *k* laboratory. This can be done using the relationship (4.2) and the corresponding uncertainty expression as reported in [1] paying attention to the role played by covariance. This is done in the section V.B. Alternatively, we can extend the matrix approach and evaluate the transformation of the covariance matrices by (4). This is illustrated in sec V.A.

### A. Using a matrix formulation

We consider the *(N x N)* system solved by ALGOS in order to determine the difference between *EAL* and *UTC(k)*:

$$\begin{cases} \sum_{i=1}^{N} w_i x_i(t) = \sum_{i=1}^{N} w_i h_i'(t) \\ x_i(t) - x_j(t) = x_{i,j}(t) \end{cases}.$$

Let's suppose that all the measures are referred to the PTB lab. We can write this system in the matrix formulation:

$$\begin{pmatrix} \sum_{i=1}^{N} W_i h_i' \\ x_{1,PTB} \\ x_{2,PTB} \\ \cdots \\ x_{N-1,PTB} \end{pmatrix} = \begin{pmatrix} W_1 & W_2 & \cdots & \cdots & W_{PTB} \\ 1 & 0 & \cdots & \cdots & -1 \\ 0 & 1 & \cdots & \cdots & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ \cdots \\ x_{PTB} \end{pmatrix} \quad \text{(10bis).}$$

where we have introduced for simplicity a single contribution from any lab, independently of the actual number of clocks in the lab, basing on the fact that [1] the ALGOS algorithm would generate the same results if each laboratory's clocks were replaced by a single "equivalent" clock whose reading was the weighted average of the individual clocks and whose weight $W_i$ in EAL was the sum of the individual clock weights.

The column vector of the links referred to PTB in the l.h.s. can be written as a block matrix, where the summation of the predictions is also imbedded:

$$\boldsymbol{Link'_{PTB}} = \begin{pmatrix} \sum_{i=1}^{N} W_i h_i' \\ \boldsymbol{Link_{PTB}} \end{pmatrix}$$

where $\boldsymbol{Link_{PTB}}$ is the usual vector of the links referred to PTB. Since we consider that the contribution of prediction $h_i'$ is negligible when evaluating uncertainties, the first term in the block matrix can be set to zero, without loosing generality, therefore $\boldsymbol{Link'_{PTB}} = \begin{pmatrix} 0 \\ \boldsymbol{Link_{PTB}} \end{pmatrix}$

We may interpret (10bis) as a linear transformation $\boldsymbol{Link'_{PTB}} = \boldsymbol{C}\,\boldsymbol{Z}$ where the vector $\boldsymbol{Z}$ is

$$\boldsymbol{Z} = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ \cdots \\ x_{PTB} \end{pmatrix} = \begin{pmatrix} EAL - UTC(1) \\ EAL - UTC(2) \\ \cdots \\ \cdots \\ EAL - UTC(PTB) \end{pmatrix}$$

and $\boldsymbol{C}$ is the design matrix appearing in (10bis). Actually the vector $\mathbf{Z}$ contains the difference between *EAL* and the appropriate laboratory average clock that maybe directly *UTC(k)* or, in any case, may be related to *UTC(k)* with a negligible uncertainty by means of a measurement internal to the *k* lab. Therefore here we consider directly *UTC(k)* in the $\mathbf{Z}$ vector when the aim is evaluating uncertainties.

The quantities in the vector $\boldsymbol{Z}$ are the unknowns and since $\boldsymbol{C}$ is invertible, the ALGOS system is solved as $\boldsymbol{Z} = \boldsymbol{C}^{-1}\boldsymbol{Link'_{PTB}}$ so, using again (4), we obtain: :

$$\boldsymbol{\Sigma_Z} = \left(\boldsymbol{C}^{-1}\right)\boldsymbol{\Sigma}'_{\boldsymbol{Link_{PTB}}}\left(\boldsymbol{C}^{-1}\right)^{\boldsymbol{T}}.$$

In the principal diagonal of $\boldsymbol{\Sigma_Z}$ we have the squared uncertainty of *UTC-UTC(k)* for any lab *k*, in the last diagonal element we have the squared uncertainty of *UTC-UTC(PTB)*, the same that we found before by means of (8) Since predictions are not considered when evaluating uncertainties, the covariance matrix $\boldsymbol{\Sigma}'_{\boldsymbol{Link_{PTB}}}$ can be written as a block matrix:

$$\boldsymbol{\Sigma}'_{Link_{PTB}} = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Sigma}_{Link_{PTB}} \end{pmatrix}$$

where the only non zero term is the covariance matrix of the link reported to PTB.

*B. Using a scalar expression of the uncertainty for site – based uncertainty*

In [1] we presented the study of the uncertainty of *UTC-UTC(k)* using a scalar expression. Here we want to describe the scalar expression for the site-based uncertainty to check it results in agreement with the matrix approach presented in the previous section. In particular in [1] we have used the following relationship:

$$u^2_{x_i} = u^2_{x_j} + u^2_{x_{i,j}} - 2\ u^2_{x_{i,j}} W_{eq_i}$$

to calculate the uncertainty of each clock $H_i$, given the uncertainty of any clock $H_j$ and the uncertainties of the chains of measures linking clock $H_i$ to clock $H_j$.

In the site-based uncertainty we use the same relation obtained from (4.2):

$$u^2_{x_i} = u^2_{x_j} + u^2_{x_{i,j}} + 2u_{x_j, x_{i,j}} \qquad \text{(10ter)}$$

but we have to calculate the new expression of the correlation term:

$$u_{x_j, x_{i,j}} = u_{\left(\sum_{\ell=1}^{N} w_\ell \left(h'_\ell - x_{\ell,j}\right), x_{i,j}\right)} = -u_{\left(\sum_{\ell=1}^{N} w_\ell x_{\ell,j}, x_{i,j}\right)}$$

the last term can be written using the differences of the clocks.

$$\begin{aligned} -u_{\left(\sum_{\ell=1}^{N} w_\ell x_{\ell,j}, x_{i,j}\right)} &= -u_{\left(\sum_{\ell=1}^{N} w_\ell \left(h_j - h_l\right), \left(h_j - h_i\right)\right)} = \\ &= -u_{\left(\sum_{\ell=1}^{N} w_\ell \left(h_j - h_l\right), h_j\right)} + u_{\left(\sum_{\ell=1}^{N} w_\ell \left(h_j - h_l\right), h_i\right)} = \\ &= -(1-W_j)Var(h_j) - W_i Var(h_i) = -(1-W_j)u_j^2 - W_i u_i^2 \end{aligned}$$

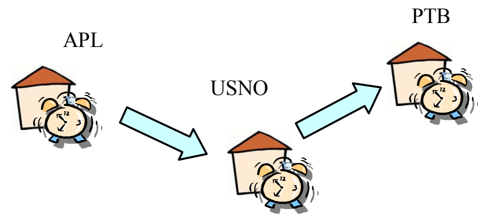
so the previous relation (10ter) becomes

$$u^2_{x_i} = u^2_{x_j} + u^2_{x_{i,j}} - 2((1-W_j)u_j^2 + W_i u_i^2)$$

in our case we assume $u^2_{x_{i,j}} = u_i^2 + u_j^2$ so

$$u^2_{x_i} = u^2_{x_j} + (u_i^2 + u_j^2) - 2((1-W_j)u_j^2 + W_i u_i^2)\ . \qquad (11)$$

This expression is very different from the corresponding expression based on the link-based assumption reported in [1]. We will present a brief example to show this relation. We consider three clocks with this topology:




By the matrix analysis we obtain the uncertainty for each laboratory:

$$\begin{cases} u^2_{UTC-UTC(PTB)} = \\ W^2_{APL}\left(u^2_{APL} + u^2_{PTB}\right) + W^2_{USNO}\left(u^2_{USNO} + u^2_{PTB}\right) + 2W_{APL}W_{USNO}u^2_{PTB} \\ u^2_{UTC-UTC(APL)} = W^2_{PTB}u^2_{PTB} + W^2_{USNO}u^2_{USNO} + \left(W_{USNO} + W_{PTB}\right)^2 u^2_{APL} \\ u^2_{UTC-UTC(USNO)} = W^2_{PTB}u^2_{PTB} + W^2_{APL}u^2_{APL} + \left(W_{APL} + W_{PTB}\right)^2 u^2_{USNO} \end{cases}$$

.

If we know the uncertainty $u^2_{UTC-UTC(PTB)}$ we can calculate the uncertainty for USNO laboratory using (11) and the scalar method:

$$\begin{aligned} &u^2_{UTC-UTC(USNO)} = \\ &= u^2_{UTC-UTC(PTB)} + \left(u^2_{PTB} + u^2_{USNO}\right) - 2\left(\left(1 - W_{PTB}\right)u^2_{PTB} + W_{USNO}u^2_{USNO}\right) = \\ &= W^2_{PTB}u^2_{PTB} + W^2_{APL}u^2_{APL} + \left(W_{APL} + W_{PTB}\right)^2 u^2_{USNO} \end{aligned}$$

and we obtain the same result of the matrix method.

## VI. The application to UTC

Starting from the examples above, the uncertainty of *UTC-UTC(k)* could be computed under the assumption that all uncertainties are link-based or under the assumption that they are entirely site-based. The data input are the values of uncertainty of the links presented in the Circular T (sec 6). To apply the site-based assumption and to obtain the uncertainty receiver by receiver we consider the relation $u^2_{x_{i,j}} = u_i^2 + u_j^2$. It has to be said that this equation is not always strictly valid because the same lab may act in different links appearing in Circular T and the listed uncertainty values are not always compatible with the complete list of corresponding equations $u^2_{x_{i,j}} = u_i^2 + u_j^2$. When that happened, we took the minimal value of any lab uncertainty compatible with the series of equations $u^2_{x_{i,j}} = u_i^2 + u_j^2$.

Using the Circular T 207 we have obtained the values for the uncertainty of *UTC-UTC(k)* with both methods as shown in Table 1. The difference for PTB laboratory is about 10%. Both methods agree to a very large extent.

TABLE I. THE UNCERTAINTIES OF ALL LABORATORIES PARTICIPATING TO UTC COMPUTED ASSUMING THE UNCERTAINTIES ARE EITHER ENTIRELY SITE-BASED OR ENTIRELY BASELINE-BASED . DATA TAKEN FROM CIRCULAR T 207.

| k | u[UTC – UTC (k)]/ns | | k | u[UTC – UTC (k)]/ns | |
|---|---|---|---|---|---|
| | Link-based | Site-based | | Link-Based | Site-based |
| AOS | 5.4 | 5.4 | NIST | 4.9 | 4.9 |
| APL | 5.5 | 5.5 | NMC | 20.4 | 20.4 |
| AUS | 7.2 | 7.0 | NMIJ | 6.9 | 6.8 |
| BEV | 5.4 | 5.4 | NMLS | 20.6 | 20.6 |
| BIRM | 20.6 | 20.5 | NPL | 2.6 | 2.6 |
| CAO | 7.4 | 7.3 | NPLI | 20.2 | 20.2 |
| CH | 5.3 | 4.6 | NRC | 15.1 | 15.1 |
| CNM | 20.9 | 20.8 | NTSC | 6.9 | 6.8 |
| CNMP | 8.2 | 8.2 | OMH | 20.2 | 20.2 |
| CSIR | 20.3 | 20.2 | ONBA | 8.8 | 8.8 |
| DLR | 5.3 | 4.7 | ONRJ | 21.2 | 20.9 |
| DTAG | 10.5 | 10.5 | OP | 1.9 | 1.9 |
| HKO | 7.2 | 7.0 | ORB | 5.3 | 4.7 |
| IEN | 2.0 | 2.0 | PL | 5.3 | 5.4 |
| IFAG | 5.3 | 4.6 | PTB | 1.7 | 1.9 |
| IGMA | 20.5 | 20.5 | ROA | 5.3 | 5.3 |
| INPL | 10.9 | 10.9 | SCL | 11.5 | 11.4 |
| JATC | 21.2 | 21.1 | SG | 20.4 | 20.2 |
| JV | 20.7 | 20.7 | SMU | 20.7 | 20.6 |
| KRIS | 7.0 | 6.8 | SP | 9.9 | 9.6 |
| LDS | 20.2 | 20.2 | SU | 6.0 | 6.0 |
| LT | 5.4 | 5.4 | TCC | 21.0 | 20.8 |
| MSL | 20.5 | 20.0 | TL | 6.6 | 6.5 |
| NAO | 20.3 | 20.2 | TP | 5.8 | 5.8 |
| NICT | 4.3 | 4.1 | UME | 25.1 | 25.0 |
| NIM | 20.2 | 20.0 | USNO | 1.7 | 1.8 |
| NIMT | 20.5 | 19.9 | VSL | 2.0 | 2.0 |

**Remark:** The difference of the results based on the two assumptions is very small. The largest difference is about 13%. The main difference comes from the different treatment of the "sandwich cases" (Example 3). In the site-based assumption the contribution of the central lab in the sandwich disappears, in the link-based assumption it remains. This difference is due to effect shown in the example 3 where two important pivots are linked with GPS receiver. In the real current network configuration we have the NICT laboratory linked with PTB using the GPSP3 equipment and NIMT, MSL etc. linked with NICT using the same equipment. In the site–based case the uncertainty due to the common NICT equipment, due to the sandwich case, completely disappears in the computation of the uncertainty of *UTC-UTC(NIMT)* indeed, in the link-based case, we have to maintain this contribution. Under the site-based assumption, the uncertainty of the laboratories connected by a sandwich case is smaller than the evaluation obtained assuming the uncertainty is link-based. The other differences between the two assumption results are due to the uncertainty of PTB which plays different role as shown in the Examples 1 and 3 but, since PTB equipment uncertainty is very small, the resulting effect is in any case very small.

## VII. COMMENTS AND CONCLUSIONS

In this work we have presented the refinement of the uncertainty of [*UTC – UTC (k)*] calculation in a matrix formulation and distinguishing between site-based and baseline-based uncertainties by their correlation properties. With some simple examples we have shown the different contributions of the correlations and we have applied this method to real situation of *UTC* calculation. Using this different approach the results are numerically very similar to previous work [1]. This demonstrate that the presented analysis is quite sound and that the numerical computation of [*UTC – UTC (k)*] uncertainty is fairly insensitive to the refinement in the interpretations of the contributions to the link uncertainties.

However, additionally we should stress that:

- Link-based and site-based noise are both present. They are not mutually exclusive approximations, and their combined effect may be rather complex.
- Link equipment noise may or may not be dominant in the link uncertainty, it depends on each case.
- Further possible correlation between different equipment located in the same laboratory may exist.
- Further possible correlation between different links involving the same lab may exist.
- The pivot uncertainty has to be very small since it plays a very important role in two types of uncertainties described here.

## REFERENCES


[1] W. Lewandowski, D. Matsakis, G. Panfilo, and P. Tavella, "First evaluation and experimental results on the determination of uncertainties in [*UTC – UTC (k)*]", 2005, Proceedings of the 36th Annual Precise Time and Time Interval (PTTI) Meeting, 247-261, December 2004, Washington, D.C.

[2] R. G. Brown and P. Y. C. Hwang. Introduction to Random Signal and Applied Kalman filtering. Chapter 1. John Wiley & Sons. New York. 1997.

[3] C. Radhakrishna Rao. Linear statistical inference and its applications. Second Edition. Chapter 2, 3, 4. John Wiley & Sons. New York. 1973.

[4] GUM – Supplement 2: Models with any Number of Output Quantities (Joint Committee for Guides in Metrology) (in progress).

[5] J. Hahn and P. Tavella, 2000, "A time scale for satellite navigation systems: why and how?" International Journal of Satellite Communication, 18, No. 4 & 5, 305-324.

[6] F. J. Galindo, P. Tavella, "Optimal weighting of correlated clocks in the definition of an ensemble time-scale", in proc. European Time and Frequency Forum, Turin, March 2000, pp. 418-422.

[7] Guide to the Expression of Uncertainty in Measurement (International Organization for Standardization, Switzerland), 1993.