Distributed Cavity Phase and the Associated Power Dependence

Ruoxin Li and Kurt Gibble
Physics Department
The Pennsylvania State University
State College, PA, the United States
kgibble@phys.psu.edu

***Abstract*—We discuss the power dependence of distributed cavity phase errors for cylindrical TE_{011} cavities in laser-cooled atomic fountain clocks. The azimuthally symmetric phase variations produce a surprisingly large distributed cavity phase error for two 2π, 4π, and 6π pulses. This is due to the correlation between the transverse variation of the Rabi frequency over the cavity aperture and a quadratic density variation of the atomic sample, along with the symmetry of the longitudinal phase variation in the cavity. We show that the large azimuthally symmetric fields and phase shifts near the walls of the endcap holes produce very small errors at optimal power for a uniform wall resistance. We also show the power variation for higher order azimuthal variations m=1, 2, and 4. These may be caused by fountain tilts, non-uniform detection of atoms, and asymmetries in the laser trapping and cooling of the atoms. We demonstrate that distributed cavity phase errors in physical cavities may have no variation with the microwave power. A combination of rigorous calculations of cavity losses, measurements of power dependence, the atomic distributions, and fountain tilts, and electrical measurements that show the lower limit of the cavity Q and the cavity symmetry, should provide stringent limits on distributed cavity phase errors for current atomic clocks.**

I. INTRODUCTION

Laser-cooled atomic fountain clocks have demonstrated accuracies better than 10^{-15}. One of the potential systematic errors is the distributed cavity phase (DCP) error [1]. We developed a theoretical model for the spatial phase variations in TE_{011} cavities and analyzed the effects on the transition probability due to the phase variation [2][3]. Here we extend our study of the power dependent effects.

In section II, we review our theoretical model. A key question is whether all DCP errors exhibit significant power dependence. In section III, we discuss this in detail and give an example of a physical cavity that produces very small power variations. In section IV we present and explain the important physics of the power dependence for the most important phase variations of physical cavities with uniform wall losses. A particularly interesting effect is a large sensitivity to azimuthally symmetric phase variations for two 2π, 4π, and 6π pulses.

II. REVIEW OF THE THEORETICAL FRAMEWORK

We use the sensitivity function s(t) to analyze the effect of the spatial phase variations of the magnetic field. The change in transition probability is [2]-[5]:

$$\delta P = -\frac{1}{2}\int_{-\infty}^{\infty} \frac{ds(t)}{dt}\phi(t)\,dt \tag{1}$$

The phase of the field is given by $\phi = -\tan^{-1}[g_z/(H_{0,z}+g_z)] \approx -g_z/H_{0,z}$ where $H_{0,z}$ is the primary standing wave field and g_z is the usually small secondary field that accounts for the wall losses of $H_{0,z}$ [2].

Our model leads to a simple expression for the change in transition probability. Throughout the paper, we discuss the transition probability instead of the frequency shift because there is an important diagnostic sensitivity for integer π pulses for which the frequency shifts are singular. Our model reduces to [3]:

$$\delta P(\vec{r}_1,\vec{r}_2) = \tfrac{1}{2}\Big(\sin\big[\theta(r_2)\big]\delta\Phi_{eff}(\vec{r}_1) - \sin\big[\theta(r_1)\big]\delta\Phi_{eff}(\vec{r}_2)\Big) \tag{2}$$

$$\delta\Phi_{eff}(\vec{r}) = b\frac{\pi}{2}\int_{-\infty}^{+\infty}\cos\big[\theta(r,z)\big]\frac{g_z(r,\phi,z)}{v(z)}dz \tag{3}$$

Here r_1 and r_2 are the radial positions of an atoms during the first and second cavity traversals, θ is the tipping angle after passing through the cavity, $\delta\Phi_{eff}$ is an effective phase, b is a multiplier for the microwave field amplitude that gives the number of π/2 pulses during a cavity traversal, and *v*(z) is the atomic velocity which we treat as constant [2].

Supported by NASA, ONR and Penn State University.

We calculate the effect on a cloud of atoms by averaging over the density distributions on both passages through the microwave cavity. Here we consider density variations of cos(mϕ) for m=0, 1, 2, and 4. For the upward cavity traversal, we take a quadratic transverse density variation for m=0 and, for m=1, a constant gradient, corresponding to a displacement of the atomic cloud or a tilt of the fountain [2][3][6]. For the downward passage, the ball expands significantly and so a uniform density distribution is a good approximation. For m=2, we treat a density variation of $n(\mathbf{r})=(N/\pi r_a^2)[1+\alpha_2 r^2/r_a^2\cos(2\phi)]$ that could arise in two ways: 1) For the upward passage, it could be caused by asymmetries in the cold-atom cooling and trapping, and then followed by a uniform density for the downward passage. 2) Non-uniform detection of the atoms following the second passage which was preceded by a uniform density on the first passage. The detection non-uniformity can be rooted in the imaging system or the transverse (Gaussian) intensity distribution of the detection laser beam. We also show the power dependence of the DCP error for m=4 with $n(\mathbf{r})=(N/\pi r_a^2)[1+\alpha_4 r^2/r_a^2\cos(4\phi)]$.

III. Can Distributed Cavity Phase Errors Be Evaluated with Power Dependent Measurements?

Other recent work has used phenomenological models to calculate DCP errors [7]-[9] and have evaluated a DCP error solely by measuring power dependent frequency shifts [10]. Here, we review our analytic solution for cavities without endcap holes [2]. The analytic solution shows that, for the cavity of [7]-[10], the dominant DCP errors have no first order power dependence. Therefore, without a detailed solution, a conclusion from a phenomenological model that all DCP errors have significant power dependence is unjustified. Further, below we show an example of a DCP error with minimal power dependence for a particular distribution of wall losses in the cavity of [10]. From this example, it is clear that power dependent measurements cannot by themselves be used to assign limits to a DCP error. A combination of modeling, electrical measurements, and clock measurements that may include changes in the atomic distribution, fountain tilts, and power dependence can be combined to set limits on DCP errors.

The analytic solution for a cylindrical cavity clearly shows how a DCP error can have no power dependence. A cavity with a large radius severely suppresses the radial penetration of modes with high frequency longitudinal (vertical) variations into the center of the cavity [2]. Thus, the largest contributions to DCP are the p=1 modes of the form $g_z=r^m\cos(m\phi)\cos(\pi z/d)$ for m>0 [2]. This has the same longitudinal dependence on z as the standing wave $H_{0z}=\cos(\pi z/d)$. Therefore, the phase distribution in the cavity is $\Phi=r^m\cos(m\phi)$ which is independent of z and produces a DCP error due to its cos(mϕ) variation. Every atom that enters the cavity experiences a field with a constant phase. The power dependence is therefore simple Rabi flopping! Yet atoms entering at different transverse positions experience a different constant phase. As an example, a fountain tilt and a

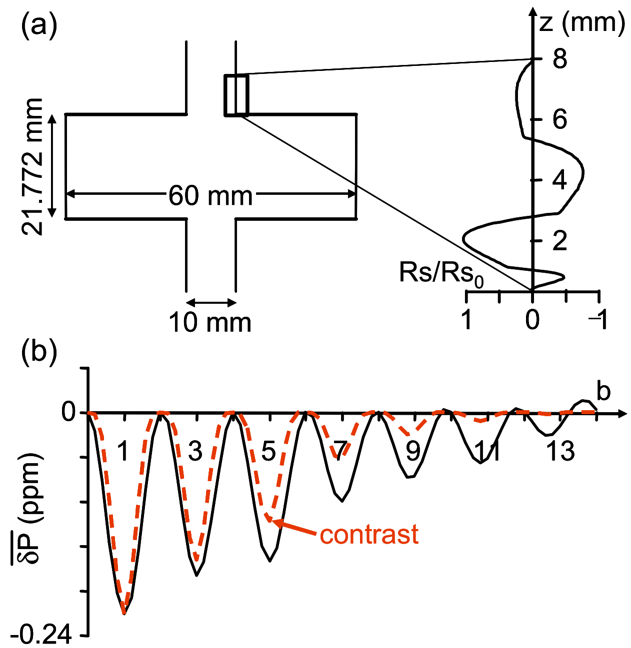

Fig. 1. (a) Cavity dimensions and a wall surface resistance Rs, with a cos(ϕ) dependence, that produces a small distributed cavity phase power dependence. Rs_0 is the resistance of copper. (b) The m=1 DCP error (solid line) due to the wall resistance in (a) versus microwave field amplitude b, where b=1 corresponds to two π/2 pulses. A normalized Ramsey fringe contrast is shown for comparison (dashed).

cavity feed imbalance will lead to a DCP error. The phenomenological fields used in [7][8] are not solutions for the actual fields in a cavity. They treated an m=0 phase variation of $\Phi=r^2z^2$, which is more than 100 times smaller than the dominant p=1 m=0 variation $\Phi=r^2-2z^2$ [2]. Therefore, after they normalize their power dependence to 1 for b=1 [7]-[9], it does not have the correct power dependence for the dominant loss modes of their cavity. Their treatment also does not consider the effects of the holes in the cavity endcaps. The holes produce locally very large fields with very large phase shifts and these have large effects on the power dependence of DCP errors as we have shown [3] and discuss in the rest of this paper. In [7]-[9], they also do not treat the 10-20% transverse variation of the tipping angle over the cavity aperture. This also has a dramatic effect on the power dependence [3] – below we explain why it leads to the largest sensitivity for some DCP errors for two 2π, 4π, and 6π pulses.

The holes in the cavity endcaps produce locally very large phase shifts and excite high p modes [2]. Therefore, does every reasonable wall loss produce the same power dependence? In Fig. 1 we show a physical cavity with endcap holes that has a m=1 DCP error with a small power dependence. These additional losses may be due to machining imperfections or inhomogeneities in the copper. To arrive at this, we began by considering m=1 phase variations because their constant gradient produces the largest effect [2]. Because the sensitivity function is linear, we calculate the DCP power dependence for wall losses in small regions throughout the cavity and then identify a reasonable linear combination of losses that shows a small power dependence. In this example, there remains some small power dependence but the frequency shift for two $7\pi/2$ pulses is less than twice the error at optimal power. One can get arbitrarily good agreement at these and higher powers by including more wall loss regions. Here we have only treated the case where m=1 terms dominate - other m terms will also contribute. For example, a small contribution from a m=0 DCP error (discussed below) will make the power dependence for two half-integer π pulses very nearly match the contrast in Fig. 1.

The example in Fig. 1 shows that DCP errors cannot be evaluated by only using measurements of power dependent frequency shifts. The model of Fig. 1 has a DCP error of $\delta\nu/\nu=1.5\times10^{-17}$ for a 2 mm offset of a laser-cooled sample that has a 1 cm diameter during the upward cavity passage. This error is small enough to give us confidence that a combination of rigorous calculations of cavity losses, measurements of power dependence, the atomic distributions, fountain tilts, and electrical measurements that show the lower limit of the cavity Q and the cavity symmetry, should provide adequately stringent limits on DCP errors for current atomic clocks.

IV. Discussion of the Distrubuted Cavity Phase Power Dependence for Cavities with Endcap Holes

The holes in the cavity endcaps, through which atoms enter and exit, produce a locally large field with a large phase shift [2][11]. Highly-peaked large phase shifts will generally produce large DCP errors at high power. In Figs. 2(a) and 3, we show the power dependent DCP population changes for m=0, 1, 2, and 4. In this section we discuss the physics of these power dependences. First we discuss the m=0 power dependence and then those for higher m.

A. m=0 Power Dependence

There are two especially notable behaviors in the m=0 power dependence in Fig. 2(a). The first is a very small DCP error at optimal power (b=1) and the second is the large DCP errors for b=4, 8, and 12, corresponding to two 2π, 4π, and 6π pulses. To understand the first, in Fig. 4(a), we show the field due to the losses $g_z(r,z)$, which is proportional to the phase, versus z for r=4.4 mm. Near the corner, $g_z(r,z)$ is nearly singular [2], as can be seen from the difference $\Delta g_z(z)$ between $g_z(r,z)$ for r=4.99 mm and 4.4 mm in Fig. 4(a) and (b). These correspond to atomic trajectories that pass 10 µm and 0.6 mm from the wall of the endcap aperture at r_a=5 mm. We show the difference because it produces the radial variation of the effective phase in this region. Ten microns from the wall, the phase shift within $z=-d/2\pm20$ µm is so large that it adds -27.4 µrad phase to the average phase for

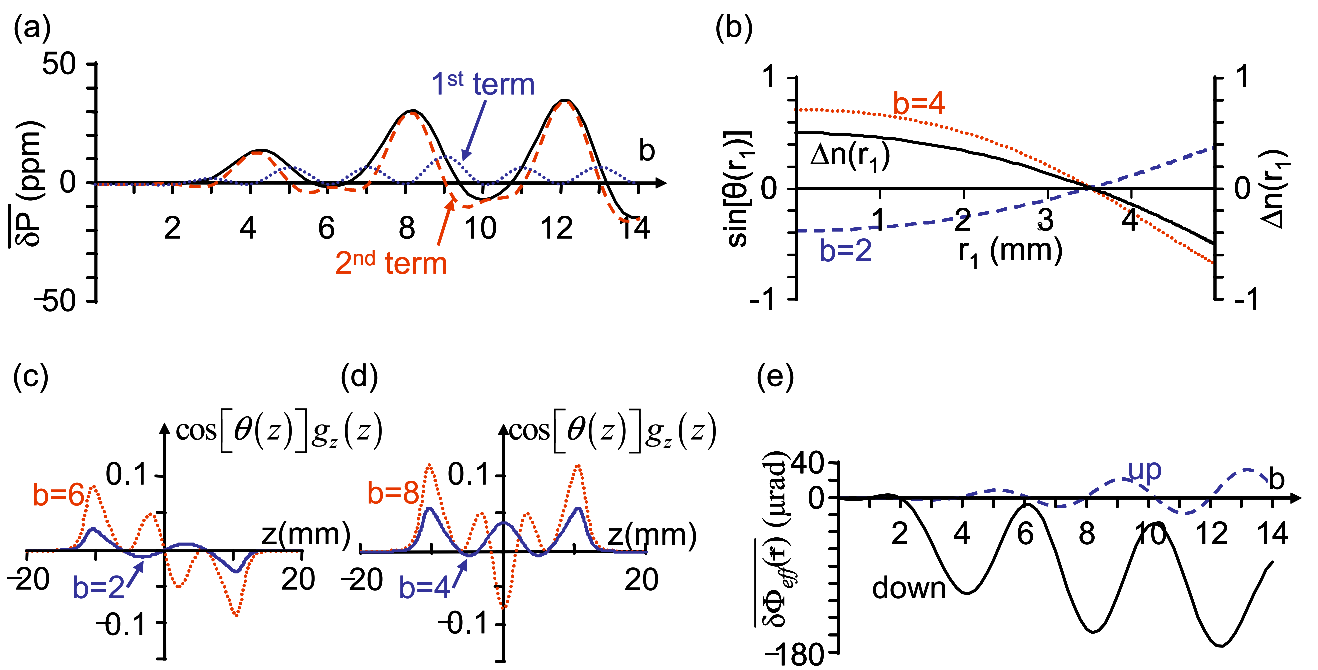

Fig. 2. (a) Power dependence of the distributed cavity phase error (solid line) for the azimuthally symmetric, m=0, phase variation of the cavity in Fig. 1 with uniform wall losses. The contribution from the first (second) term in Eq. 4 is the dotted (dashed) curve. (b) The variation of the sine of the tipping angle over the cavity aperture for a π and 2π pulse (b=2 and 4) and the non-uniform density distribution $\Delta n(r_1)$ for the first cavity passage. The average of $n(r_1)\sin[\theta(r_1)]$ is large and positive for b=2, 4, 6, … . (c). The contribution to the effective phase, $\cos[\theta(z)]g_z(z)$ at r=3.5 mm for b=2 and 6 (solid and dashed). The sum of all contributions through the cavity is nearly zero. (d) The same as (c) for b=4 and 8 (solid and dotted). The sum of all contributions is a local maximum near b=4 and 8. (e) Average of the effective phase and density for the upward (dashed) and downward (solid) cavity passages. The downward passage is large for b=4 and 8, as suggested by (d), and nearly zero for b=2 and 6 from (c).

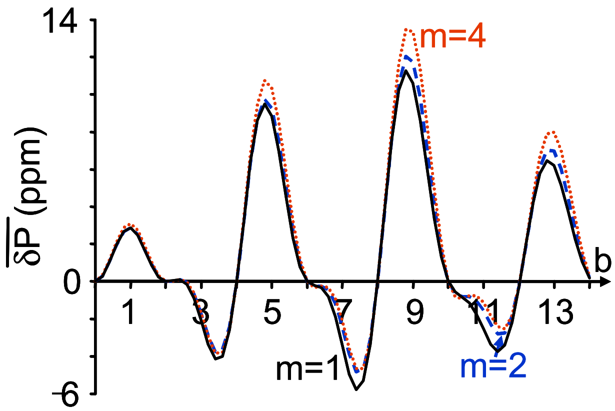

Fig 3. The power dependences of distributed cavity phase errors for the cavity in Fig. 1(a) with uniform wall losses for m=1 (solid), 2 (dashed), and 4 (dotted). For m=1, the population shifts are due to a 10% feed imbalance and a linear density variation corresponding to a 2 mm offset of a 1 cm ball (α_1=0.8). For m=2, we take a density variation corresponding to an imaging variation of ±40% (α_2=0.4). The m=4 variation is greatly magnified, corresponding to α_4=50.

the entire cavity traversal $\delta\Phi_{eff}(r_2)$, relative to that for r=4.4 mm. However, the regions in Fig. 4(a), z=-d/2 ± 2mm, nearly completely cancel the effect of the large phase shift to a very high degree. The difference in the effective phase for z=-d/2±2 mm for r=4.4 and 4.99 mm is only -1.9 μrad. Throughout the rest of the cavity, the phase is similar for r=4.99 and 4.4 mm and the details further cancel this phase shift as shown by the effective phase in Fig. 4(c). Averaging the effective phase in Fig. 4(c) over the density distributions for a 1 cm ball on the upward traversal (α=-1), a 1 Hz transition linewidth has a frequency shift at optimal power of only 1×10^{-18} due to this nearly complete cancellation. At higher powers, the population changes are equal to the population changes at optimal power for frequency shifts of order 10^{-15}.

We now explain the surprisingly large m=0 power dependence in Fig. 2(a) for b=4, 8, and 12. This large sensitivity is surprising because one expects rightfully very little sensitivity to the phase of the field for integer π pulses. The variation of the Rabi frequency over the cavity aperture is a critical component of this large sensitivity. The atoms that contribute do not in fact experience integer π pulses.

In Eq. 2, there are two terms that contribute to this power dependence and we show each of these in Fig. 2(a). The first term (dotted line) is small for integer π pulses (b=2,4,6, …) because $\sin[\theta(r_2)]$ is averaged over a uniform density distribution on the downward cavity passage. Since $\sin[\theta]=0$ defines integer π pulses, the first term in Eq. 2 is 0 for even b.

The second term in Eq. 2, the dashed line in Fig.2(a), includes an average of $\sin[\theta(r_1)]$ for the upward cavity passage. For this traversal, the density distribution of $n(r_1)=(N/\pi r_a^2)[1+\alpha(r_1^2/r_a^2-½)]$ corresponds to the radial curvature

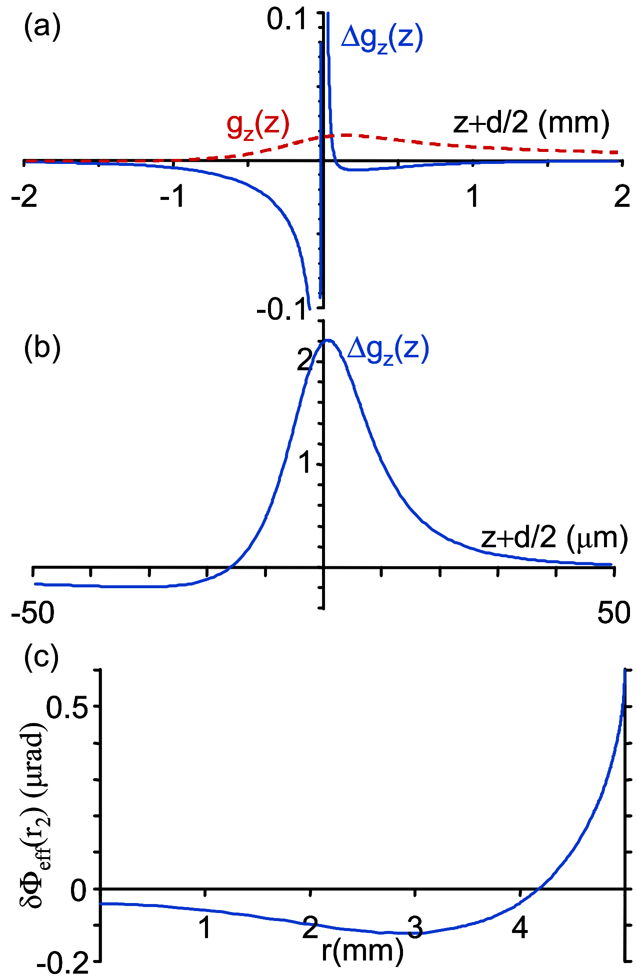

Figure 4. (a) The secondary field $g_z(z)$ for r=4.4 mm near the lower cavity endcap at z=-d/2 (dashed). The difference $\Delta g_z(z)=g_z(r=4.99\text{mm},z)-g_z(r=4.4\text{ mm}, z)$ (solid) shows that $g_z(r,z)$ is large near the corner of the endcap hole at r=5 mm and z=-d/2. The difference $\Delta g_z(z)$ gives the radial variation of the phase of the microwave field. (b) The large losses near the aperture of the endcap hole produce a large effective phase shift of −27.4 μrad within 20 μn of z=−d/2 for r=4.99 mm as compared to r=4.4 mm. The contributions from the larger region in (a) almost entirely cancel the large phase shift in (b) so that the net effect from this region is small. (c) The total effective phase shift through the cavity for the downward passage. The variation of −0.2 to +0.6 μrad is very small compared to the effect of the phase variations in (b).

that one would expect from a Gaussian ball of atoms. The only non-zero contribution to a DCP error is from the $\Delta n(r_1)=(N/\pi r_a^2)\alpha(r_1^2/r_a^2-½)$ term. This term has a positive density contribution at r=0 and a negative at $r=r_a$, for α<0 as shown in Fig. 2(b). At optimal power (b=1), $\sin[\theta(r_1)]\approx1$ so that the average of $\Delta n(r_1)\sin[\theta(r_1)]$ is essentially zero. The transverse variation of the Rabi frequency gives a larger Rabi frequency at r=0 and a smaller Rabi frequency at $r=r_a$. Therefore, near integer π pulses, $\sin[\theta(r_1)]$ reverses sign from r=0 to $r=r_a$ as does $\Delta n(r_1)$ in Fig. 2(b). Therefore essentially all contributions to the average of $\Delta n(r_1)\sin[\theta(r_1)]$ have the same sign for integer π pulses, which produces a potentially large DCP sensitivity.

The DCP error is much larger for b=4, 8, 12 than for b=2, 6, 10. This is due to the symmetry of the phase of the cavity field and the symmetry of $\cos[\theta(r_2,z)]$ in $\delta\Phi_{eff}(r_2)$ in in Eq. 3. The TE_{011} mode is symmetric about the cavity center and therefore the phase of H_z is also. For a π pulse (b=2) in Eq. 3, $\cos[\theta(r_2,z)]$ is an odd function of z. Therefore, $\delta\Phi_{eff}(r_2)$ nearly vanishes since $\cos[\theta(r_2,z)]g_z(z)$ is odd in z as shown in Fig. 2(c). For b=6, 10, …, $\cos[\theta(r_2,z)]g_z(z)$ behaves similarly and therefore $\delta\Phi_{eff}(r_2)$ is small. For b=4, 8, and 12, two 2π, 4π, and 6π pulses, $\delta\Phi_{eff}(r_2)$ in Eq. 3 is large because both $g_z(z)$ and $\cos[\theta(z)]$ are even functions of z giving Fig 2(d). Therefore, to evaluate the DCP errors, one should also measure the power dependent population changes for two 2π, 4π, 6π, … pulses. The large sensitivity is due to the transverse variation of the Rabi frequency over the cavity aperture, its correlation with a density curvature, and the symmetry of the cavity phase and the sensitivity function.

B. m=1& 2 Power Dependence

The phase deviations that have slow variations in ϕ and z most effectively propagate to the center of the cavity [2]. We therefore now consider azimuthal variations of $\cos(m\phi)$ for small m of 1, 2, and 4. The m=1 term is the largest [2] as it corresponds to a power imbalance from the feeds which causes a linear phase gradient at the center of the cavity. If atoms are launched with a small offset, or if there is a tilt of the fountain [6], the density distribution on the upward passage will have a $\cos(\phi)$ dependence and therefore any m=1 phase variation will produce a DCP error. Because the second term in Eq. 2 is negligible since $n(\mathbf{r_2})$ on the downward passage will be nearly uniform, the power dependence has the more intuitive behavior of large sensitivities for odd b, two $\pi/2$, $3\pi/2$, $5\pi/2$, … pulses as shown in Fig. 3. For this reason, the power dependences for m=2 and 4 also look very similar. The power dependence flops sign for every increase of θ by π and grows slower than linear with increasing field amplitude b. If the atomic sample is 1 cm diameter and 2 mm off-center, $n(\mathbf{r_1})=(N/\pi r_a^2)[1+\alpha_1 r_1/r_a\cos(\phi)]$ with $\alpha_1=0.8$, a 10% feed imbalance and homogenous wall losses produce a frequency shift of 2×10^{-16} at optimal power (b=1) for a 1 Hz linewidth.

Two potential sources for quadrupole DCP errors are a non-uniform density distribution $n(\mathbf{r_1})$ from the laser cooling and trapping and secondly a non-uniform illumination or imaging of the atoms in the detection region. To characterize the imaging, we imagine that a uniform density distribution after the second cavity passage is illuminated by a laser beam propagating in the x direction and the fluorescence is observed from the y direction. If the cylindrically symmetric imaging system is more efficient at the center, it will have a detection sensitivity of $1-\alpha'(x^2+z^2)/r_a^2$. The laser beam non-uniformity will be of the form $1-\beta'(y^2+z^2)/r_a^2$. The detected density then has a distribution $n(\mathbf{r_2})=(N/\pi r_a^2)[1+\alpha r_2^2/r_a^2+\alpha_2 r_2^2/r_a^2\cos(2\phi)]$ with $\alpha=-(\alpha'+\beta')/2$ and $\alpha_2=-(\alpha'-\beta')/2$.

The non-uniform initial distribution and the non-uniform detection behave identically in Eqs. 2 and 3, within a negative sign. We show the power dependence for $\alpha_2=0.4$ in Fig. 3 for a cavity with 2 feeds [12]-[14]. At optimal power, the frequency shift is 2×10^{-16} for a 1 Hz linewidth.

V. SUMMARY

We have analyzed a variety of distributed cavity phase errors as a function of microwave field amplitude. The azimuthally symmetric (m=0) phase variations produce an unexpectedly large DCP errors for two 2π, 4π, and 6π pulses. This is due to the correlation between the transverse variation of the Rabi frequency over the cavity aperture and the quadratic density variation of the atomic sample, along with the symmetry of the longitudinal phase variation in the cavity. The large fields and phase shifts near the wall of the endcap hole has an effect which is very well cancelled at optimal power. Nonetheless, it may be prudent to exclude the atoms that pass through the region of large fields with large phase shifts. Eliminating 1 to 5% of the atoms with an aperture elsewhere in the fountain may dramatically reduce the maximum phase shift experienced by any atom. We show the power variation for higher order azimuthal variations m=1, 2, and 4. These may be caused by fountain tilts, non-uniform detection of atoms, and asymmetries in the laser trapping and cooling of the atoms. We demonstrate that DCP errors in physical cavities may have no variation with the microwave power. A combination of rigorous calculations of cavity losses, measurements of power dependence, the atomic distributions, and fountain tilts, and electrical measurements that show the lower limit of the cavity Q and the cavity symmetry, should provide stringent limits on DCP errors for current atomic clocks.

REFERENCES

[1] A. Bauch, T. Heindorff, and R. Schroeder, "Measurement of the frequency shift due to distributed cavity phase difference in an atomic clock," IEEE Trans. Instrum. Meas., vol. 34, 1985, pp. 136-138.

[2] R. Li and K. Gibble, "Phase variations in microwave cavities for atomic clocks" Metrologia, vol. 41, pp. 376-386, 2004.

[3] R. Li and K. Gibble, "Microwave cavities for atomic clocks: phase variations and associated power dependences," 2005 Proc. 19th Eur. Frequency and Time Forum (Besancon), in press.

[4] G. Santarelli, P. Laurent, A. Clairon, G.J. Dick, C.A. Greenhall and C. Audoin, "Theoretical description and experimental evaluation of the effect of the interrogation oscillator frequency noise on the stability of a pulsed atomic frequency standard," Proc. Eur. Frequency and Time Forum, vol. 45, 1996, pp. 66-71.

[5] P. Lemonde, G. Santarelli, Ph. Laurent, F. Pereira, A. Clairon and Ch. Salomon, "The sensitivity function: a new tool for the evaluation of frequency shifts in atomic spectroscopy," Proceedings of Frequency Control Symp., 1998, pp. 110-115.

[6] M. Abgrall, "Evaluation des performances de la fontaine atomique PHARAO, Participation à l'étude de l'horloge spatiale PHARAO", Ph.D. thesis, 2003, p. 147.

[7] S. R. Jefferts, J. H. Shirley, E. A. Donley, T. P. Heavner an T.E. Parker, "On the power dependence of extranenous microwave fields in atomic frequency standardes," in these prodeedings, in press.

[8] S. R. Jefferts, J. H. Shirley, N. Ashby, E. A. Burt, and G. J. Dick, 2005 Proc. 19th Eur. Frequency and Time Forum (Besancon) in press.

[9] See also S. R. Jefferts, J. H. Shirley, E. A. Burt, and G. J. Dick, "Power Dependence of Distributed Cavity Phase Induced Frequency Biases in Atomic Fountain Frequency Standards," 2005 IEEE Trans. Ultrason. Ferr. in press.

[10] T. P. Heavner, S. R. Jefferts, E. A. Donley, J. H. Shirley and T. E. Parker, "NIST-F1: recent improvements and accuracy evaluations," Metrologia vol. 42, pp. 411-422, 2005.

[11] C. Fertig, R. Li, J. I. Rees, and K. Gibble, "Distributed cavity phase shifts and microwave photon recoils," Proc. 2002 IEEE Frequency Control Symp. pp 469–72 (2002).

[12] S. Bize *et al*., "Advances in atomic fountains," C. R. Physique vol. 5 829-843, (2004).

[13] S. Weyers, U. Hubner, R. Schroder, C. Tamm, A. Bauch, "Uncertainty evaluation of the atomic caesium fountain CSF1 of the PTB," Metrologia vol. 38 343-352 (2001).

[14] K. Szymaniec, W. Chalupczak, P. B. Whibberley, S. N. Lea, and D. Henderson, "Evaluation of the primary frequency standard NPL-CsF1," Metrologia vol. 42, 49-57 (2005).