# HILBERT SPECTRUM FOR TIME-DOMAIN MEASUREMENT DATA AND ITS APPLICATION

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**Abstract**

***In this paper, a new method for analyzing the time domain data is introduced. As one knows, the time domain phase measurements are nonstationary and the differencing technique is usually adopted for generating stationary data. Thus, the analysis tools for stationary data are used in such case. On the other hand, some nonstationary data processing methods could be used directly to analyze measurements, and time-varying phenomena of data will be revealed through these methods. A new algorithm based on the concept of the instantaneous frequency is described in this paper. The strategy is described as the following two parts. First, the measurements are decomposed into components with meaningful instantaneous frequencies. For this purpose, some restriction conditions for decomposition from the local viewpoint are necessary. After the process, the Hilbert transform will be applied to every component and the data will be expressed in phasor form. Under this circumstance, we can represent the amplitude and the instantaneous frequency as functions of time in a three-dimensional plot. This frequency-time distribution of the amplitude is designated as the Hilbert amplitude spectrum, and the squared values of amplitude could be substituted for energy to produce the Hilbert energy spectrum just as well. To verify our algorithm, some simulations are performed. Based on the simulation results, the time-varying spectrum characteristic can be easily identified. Furthermore, different types of noise also can be discriminated by its Hilbert spectrum. Besides, the marginal spectrum of the Hilbert spectrum offers a measure of total amplitude or energy contribution from each frequency value, which is more meaningful than the Fourier spectrum for nonstationary data. Based on the spectrum information, the optimal weighting factors between different measurements will be further investigated.***

## INTRODUCTION

For extracting the meaningful information and statistics from the experimental data, an appropriate data analysis is necessary. Generally speaking, two main purposes exist for data analysis. The first one is to determine the parameters needed to construct the necessary model, and the other is to confirm the model constructed to represent the phenomenon. Unfortunately, the experimental data are always imperfect, e.g. the total data span is too short, the data are nonstationary, or they represent nonlinear processes. Facing such data, we have limited options to use in the analysis. For example, the time-domain phase measurements are nonstationary and the differencing technique is usually adopted for generating stationary data. Thus, the analysis tools for stationary data are used in such a case **[1,2]**. On the other hand, some nonstationary data processing methods could be used directly to analyze measurements and time-varying phenomena of data will be revealed through these methods.

In this paper, we will use the data analysis method introduced in **[3]**. This scheme can be divided into two parts. In the first step, the experimental data are decomposed into a collection of intrinsic mode functions (IMF). This decomposition is viewed as an expansion of the data in terms of the IMFs. In other word, these IMFs are regarded as the basis of that expansion which can be linear or nonlinear as dictated by the data. Since the IMFs have well-behaved Hilbert transforms, the corresponding instantaneous frequencies are calculated. Thus, in the next step, we could localize any event on the time as well as the frequency axis. The local energy and the instantaneous frequency derived from the IMFs give us a full energy-frequency-time distribution of the data, and such a representation is designated as the Hilbert spectrum.

To verify the scheme, some simulations were conducted. In the first simulation, the white phase modulation and white frequency modulation noises were tested. The marginal power spectrum was generated and compared with the Fourier spectrum. In the second analysis, these two noise data were concatenated to generate the time-varying noise. Under this circumstance, the algorithm was still effective and the spectrum was able to show the time-varying characteristic.

This paper is organized as follows. In the following section, the intrinsic mode function (IMF) and the empirical mode decomposition are described. Two types of signals are used to demonstrate the results of decomposition. The Hilbert spectrum and marginal average power spectrum are introduced, too. Two simulations are conducted and the power spectrum is generated in the next section to verify the algorithm. The conclusion is given in the final section.

## SUMMARY OF THE EMPIRICAL MODE DECOMPOSITION METHOD

Before introducing the empirical mode decomposition proposed in **[3]**, the concept of instantaneous frequency must be described first. For an arbitrary time series, $x(t)$, we have its Hilbert Transform **[4]**, $y(t)$, as

$$y(t) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{x(t')}{t - t'} dt', \tag{1}$$

where $P$ indicates the Cauchy principal value. This transform exists for all functions of class $L^P$. With this definition, $x(t)$ and $y(t)$ can be used to define an analytic signal $z(t)$, as

$$z(t) = x(t) + iy(t) = a(t)e^{i\theta(t)}, \tag{2}$$

in which

$$a(t) = [x^2(t) + y^2(t)]^{1/2}, \ \theta(t) = \arctan(\frac{y(t)}{x(t)}). \tag{3}$$

In equation (2), the polar coordinate expression further clarifies the local nature of this representation: it is the best local fit of an amplitude and phase-varying trigonometric function to $x(t)$. Furthermore, the instantaneous frequency of $x(t)$ is defined as

$$\omega = \frac{d\theta(t)}{dt}. \tag{4}$$

Since the definition of the instantaneous frequency given in (4) is a single-valued function of time, some limitations on the data are necessary. Based the discussion in **[3]**, the authors proposed a class of functions designated as intrinsic mode functions for the instantaneous frequency to make sense. An intrinsic mode function is a function that satisfies two conditions. The first one is the number of extrema and the number of zero crossings must either equal or differ by one in the whole data set. The other one is that the mean value of the envelope defined by the local maxima and the envelope defined by local minima is zero at any point. Based on this definition of IMF, not only are the traditional narrow band requirements satisfactory, but the unwanted fluctuations induced by asymmetric wave forms are reduced.

For a complicated data set, the decomposition into IMF components with meaningful instantaneous frequencies is necessary. This decomposition is based on three assumptions. First, the signal has at least two extrema – one maximum and one minimum. Next, the characteristic time scale is defined by the time lapse between the extrema. Finally, if the data are totally devoid of extrema, but contain only inflection points, then it can be differentiated once or more times to reveal the extrema. Based on the discussion in **[3]**, the decomposition can be concluded with the following steps:

a. Identify the extrema of the data set $x(t)$, and form the envelopes defined by the local maxima and minima respectively by the cubic spline method.
b. Form the mean values $m_1(t)$ by averaging the upper envelope and lower envelope, and make the differences between the data and the mean values to get the first component $h_1(t) = x(t) - m_1(t)$.
c. If the first component is not an IMF, let $h_1(t)$ be the new data set. Continue the steps a and b until the first component is an IMF.
d. The first IMF component is called as $c_1(t)$. Let $r_1(t) = x(t) - c_1(t)$. Continue the steps a-c until $r_n(t)$ is smaller than a predetermined value or becomes a monotonic function that no more IMF can be extracted.

Based the above algorithm, two signals are used to demonstrate the decomposition results. The first signal is constructed by adding two cosine waves as $s_1(t) = 0.2\cos(10t) + \cos(20t)$. This signal is shown in the left-hand side of Figure 1. The corresponding IMF components $c_1$, $c_2$, and the residue $r_2$ are shown in the right-hand side of Figure 1. Figure 2 shows the corresponding instantaneous frequencies and amplitudes of these IMFs. The second signal is a time-varying signal and it is designated as $s_2(t) = \begin{cases} \cos(10t) & 0 \le t < 2 \\ \cos(20t) & 2 \le t < 4 \end{cases}$. Figures 3 and 4 show the signal, IMFs, and the corresponding instantaneous frequencies and amplitudes. From the results, it is obvious that the characteristics of the signals can be extracted through the empirical mode decomposition.

## THE HILBERT SPECTRUM

After the decomposition step, the data are reduced to several IMF components. By performing the Hilbert transform on each IMF components, the data can be expressed as follows:

$$x(t) = \sum_{j=1}^{n} a_j(t) \exp\left(i \int \omega_j(t) dt\right). \tag{5}$$

Here the residue, $r_n$, is omitted because it is either a monotonic function or it might be smaller than the predetermined threshold. Comparing to the Fourier representation, it is obvious the equation (5) is a generalized Fourier expansion. The time-varying characteristic enables us to accommodate non-stationary data. Furthermore, equation (5) enables us to represent the amplitude and the instantaneous frequency as functions of time. In other words, the amplitude can be contoured on the frequency-time plane. This frequency-time distribution of the amplitude is designated as the Hilbert amplitude spectrum, $H(\omega,t)$, or simply Hilbert spectrum.

We also can define the marginal average power spectrum $h(\omega)$ as

$$h(\omega)=\frac{1}{T}\int_0^T H^2(\omega,t)dt\ . \tag{6}$$

This marginal spectrum can be used as a measure of average power contribution from each frequency value. It is noted that the frequency in the Hilbert spectrum has a totally different meaning from Fourier spectrum analysis. In the Fourier representation, a frequency component means a sine or cosine wave persisted in the whole data set. While the data are nonstationary, the Fourier spectrum is meaningless physically. On the other hand, the frequency in the Hilbert spectrum or marginal spectrum indicates that an oscillation with such a frequency exists. The representation is still valid while the data set is nonstationary.

## THE SIMULATIONS

To verify the method, two simulations were conducted. In the first simulation, two types of noises were generated. The first one was the white phase modulation noise and the other one was white frequency modulation noise with the same noise parameter $\sigma_y = 1.0\times10^{-11}$. The sampling time is 1 second and the time span is 400 seconds. Figures 5 and 7 show the noise plots and the IMF components, respectively. In this case, the white phase modulation noise can be decomposed into nine IMF components and the white frequency phase modulation noise has seven IMFs. Using equation (6), we can generate the Hilbert marginal power spectrum and the results are shown in Figures 6 and 8, respectively. The corresponding Fourier power spectra are also shown in the same figures for comparison.

In the second simulation, we concatenate the two noises as a new time-varying noise. The IMF components are shown in Figure 9 and the Hilbert power spectrum is shown in Figure 10. From Figure 10, we can observe that the power density is increasing at low frequency with time. This feature can be used to demonstrate that the time-varying characteristics of the concatenated noise are revealed via the decomposition method.

## CONCLUSION

In this paper, we use the analysis method proposed in **[3]** to analyze the time domain measurements. The data can be decomposed into several well-behaved functions that have meaningful instantaneous frequencies. Based on the decomposition, we can observe the frequency characteristics of the experimental data. Moreover, even if the frequency components of the data are time-varying, the analysis scheme is still valid. By using this tool, we can observe the local phenomena of the measurements.

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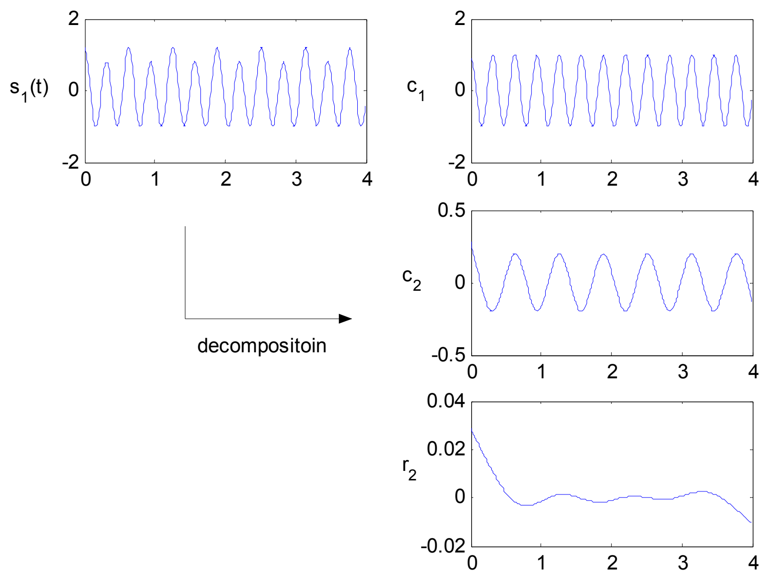


Figure 1. $s_1(t)$ and its corresponding IMF components.

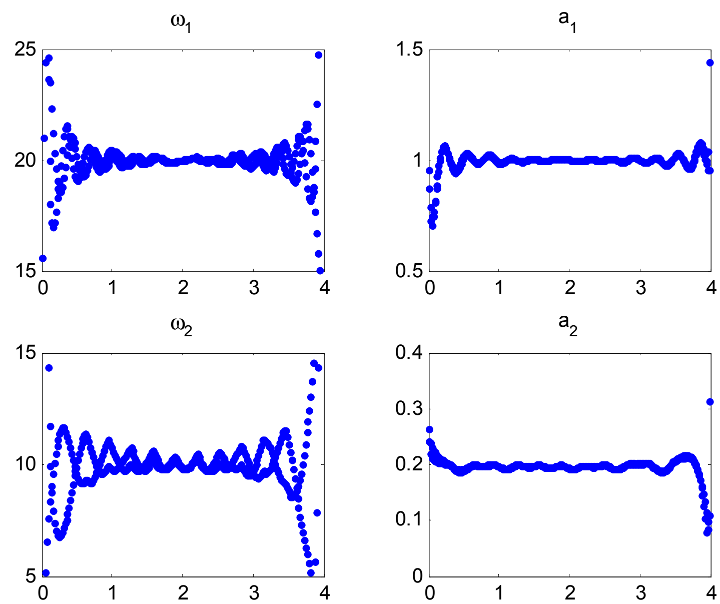


Figure 2. The instantaneous frequencies and amplitudes of IMFs in $s_1(t)$.

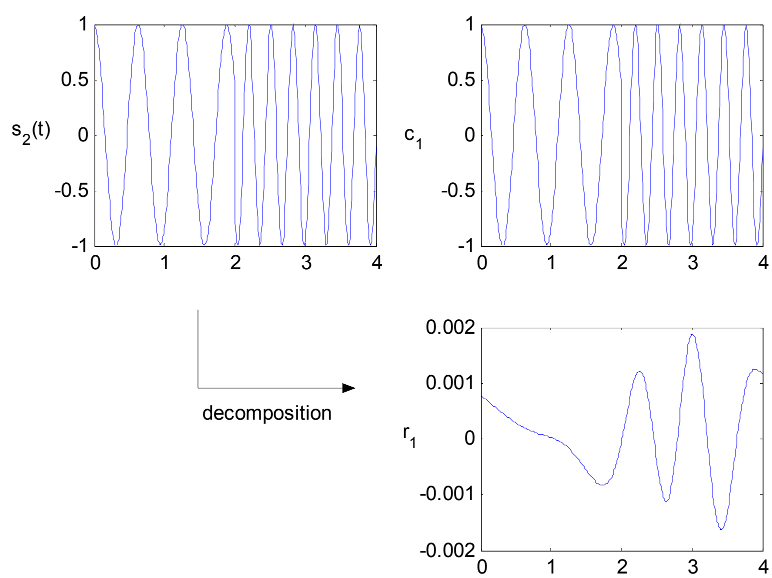


Figure 3. $s_2(t)$ and its corresponding IMF components.

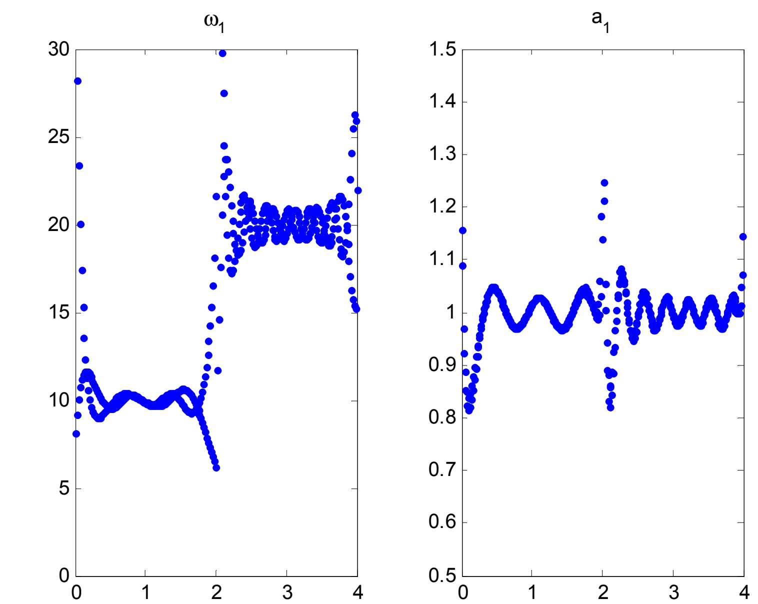


Figure 4. The instantaneous frequencies and amplitudes of IMFs in $s_2(t)$.

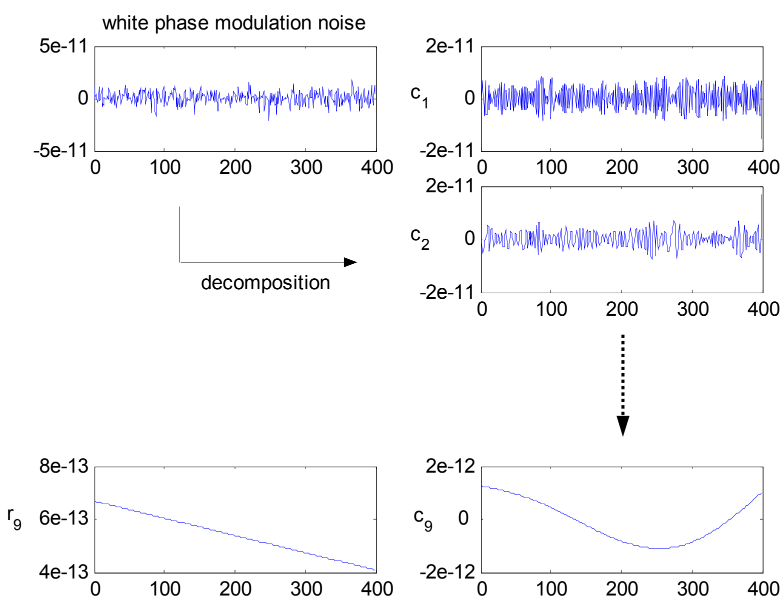


Figure 5. White phase modulation noise and its IMF components.

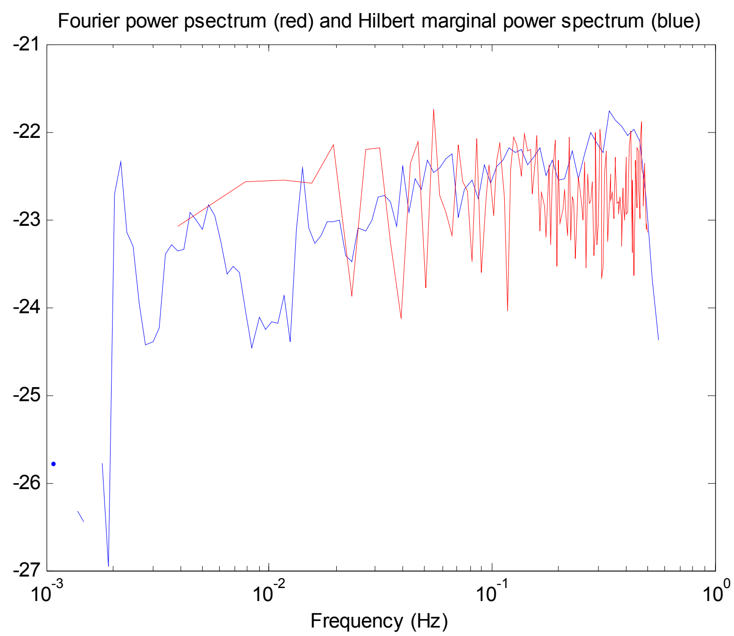


Figure 6. Marginal spectrum and power spectrum for white phase modulation noise.

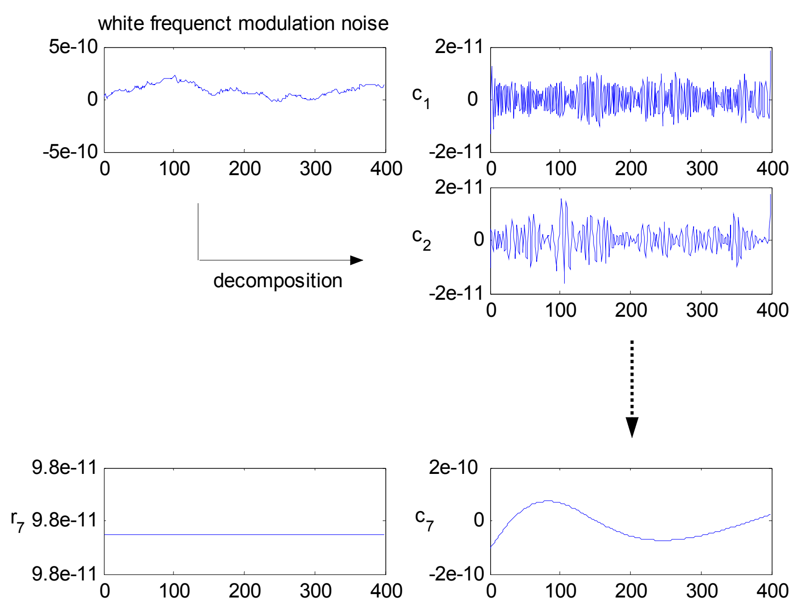


Figure 7. White frequency modulation noise and its IMF components.

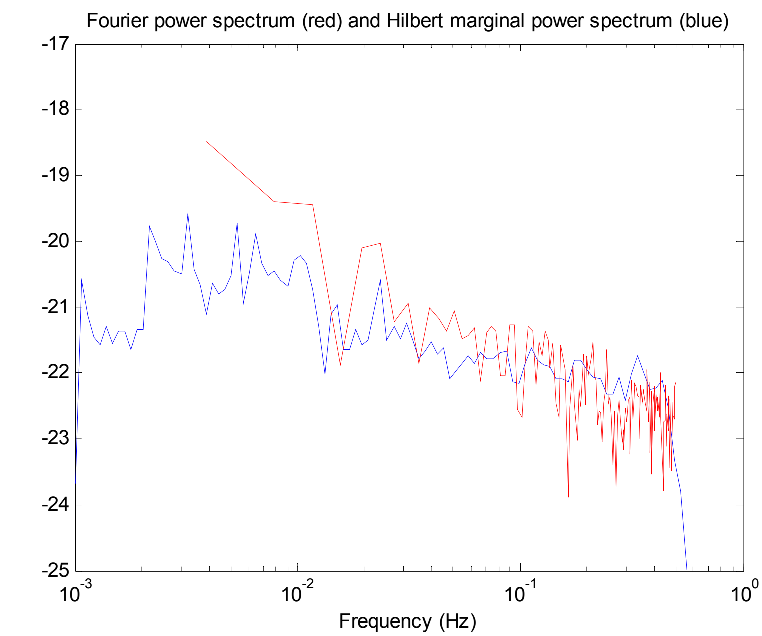


Figure 8. Marginal spectrum and power spectrum for white frequency modulation noise.

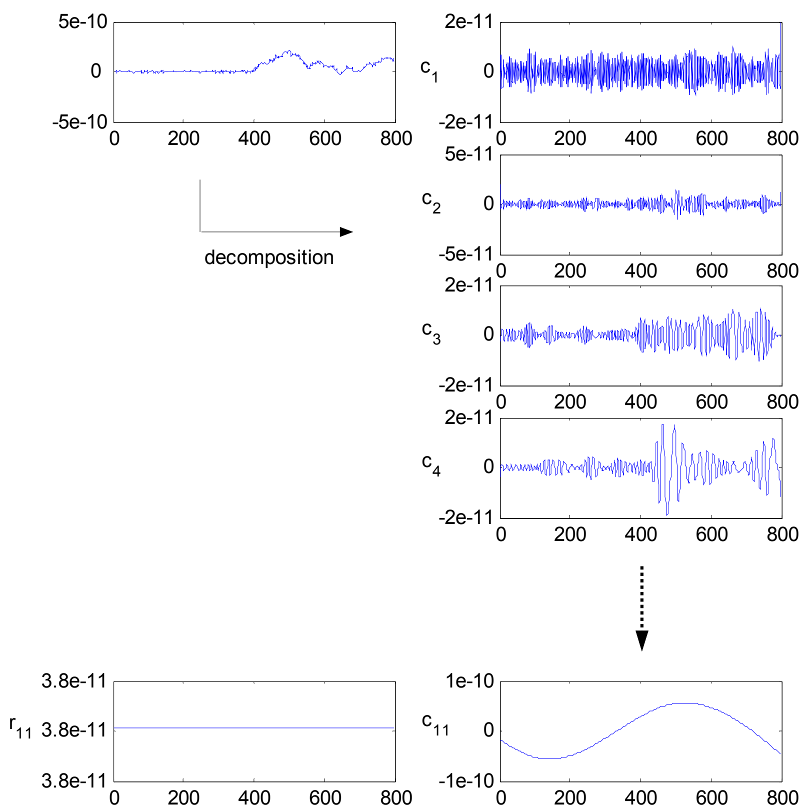


Figure 9. IMF components of the concatenated noise.

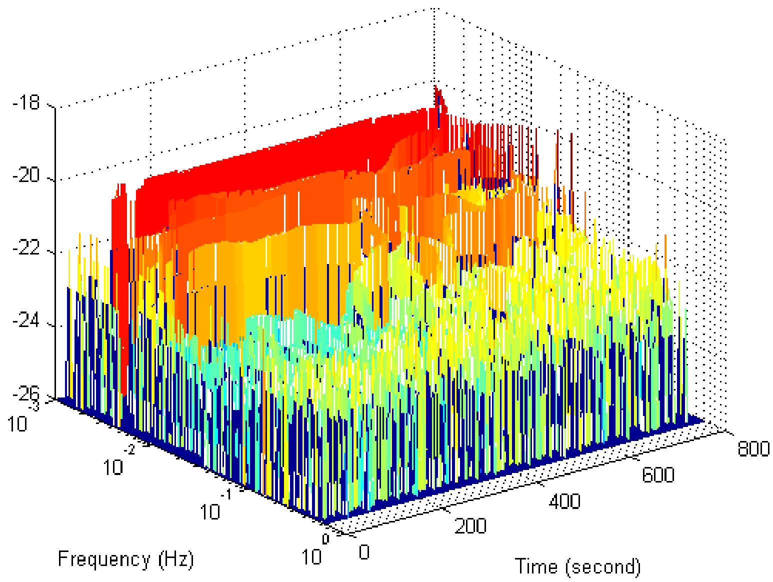


Figure 10. Hilbert power spectrum for the concatenated noise.