GPS-BASED ADAPTIVE ESTIMATING OF TIME ERRORS FOR CLOCK SYNCHRONIZATION WITH A FIR FILTER

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Abstract

In this paper, we investigate one of the possibilities to adapt an unbiased moving average (MA) filter (finite impulse response [FIR] filter) to the slope of time error function. The linear regression coefficient is used as a statistical estimator of sample slope. We evaluate the error of the slope estimate and present two options for the adapting coefficient determination. To examine them, we generate the time error noisy signal with linear trend and estimate it with a simple MA, the optimally unbiased MA, and the two adapted filters. The particular errors of the filters, namely bias, RMSD, RMSE and maximal error are then compared. Finally we mark special features of the linear-regression-based adaptation.

INTRODUCTION

Fast and extremely accurate "on-line" GPS-based estimating of time errors in timekeeping is probably the noblest example of the optimal filtering application in Time and Frequency. Estimation deals here with the time error function of local clock, which demonstrates some special features, which are:

- Its deterministic model seems to be quadratic at least, since it is formed with an initial constant error (x_0), an initial frequency offset of local oscillator for the reference one (y_0), and a linear aging component (D) of local oscillator.
- Its noise fits well white Gaussian noise (note: the more noise is whitened, the more accurate linear optimal estimate is achieved).

Such properties of the time error function allow applying the linear optimal filtering theory (linear Kalman filtering) straightforward to recent studies [1] have showed that for the sake of accuracy, it is not enough just to use the three-space-state Kalman filter totally matched with the above-mentioned clock time model. The known computation problem here is noise produced by the state space discrete time Kalman filter (the more states, the more noise). On the other hand, both the realizable Wiener and the simple moving average (MA) filters inherently produce the bias for the nonstationary processes.

In our report [2], we show that of the filters with the same time constant an optimally unbiased MA filter (finite impulse response [FIR] filter) produces noise lower then the three-state Kalman with rather the same bias. However, the noise is bigger than that of the simple MA. Knowing that a simple MA produces the lowest possible noise among all the filters [3] and zero-bias for the stationary process ($y_0 = 0$ and $D = 0$), we then wonder if it is possible to adapt the filter [2] to the slope of the time error function? In other words, *can we design the unbiased MA filter with the same smallest noise as that of a simple MA?* The answer, unfortunately, is not exhaustively positive.

In this report we examine one of the possibilities to reduce the estimate error using a linear regression coefficient as an estimator of a time error slope. We work out the adapted filter with two possible adapting coefficients and study it for the generated noisy time error process with a linear deterministic trend.

ADAPTATION OF THE OPTIMALLY UNBIASED MA FILTER

Given the optimally unbiased MA Filter [2]

$$\boxed{\hat{x}_n = \sum_{i=0}^{N-1} \frac{2(2N-1)-6i}{N(N+1)} \xi_{n-i}}, \tag{1}$$

where N is a number of the points in the average, ξ_n is observation, and $\hat{x}_n$ is estimate of the time error x_n. Present the filter (1) in a form of

$$\boxed{\hat{x}_n = \sum_{i=0}^{N-1} \frac{1}{N}\left[1 + k_n \Phi_i(N)\right]\xi_{n-i}}, \tag{2}$$

where k_n in this case is unity, and the weighted function is

$$\Phi_i(N) = \begin{cases} \dfrac{3(N-1)-6i}{(N+1)}, & 0 \le i \le N-1 \\ 0, & otherwise \end{cases}, \tag{3}$$

Now analyze this form (2). Since an observation on the averaging interval of N points exhibits the stationary nature and a simple MA is best here in a sense of LMS, then, to get the same minimal mean-square error (MSE) for the filter (2), the coefficient should be taken as $k_n = 0$ to get from (2) a simple MA filter, this is

$$\boxed{\hat{x}_n = \frac{1}{N} \sum_{i=0}^{N-1} \xi_{n-i}}. \tag{4}$$

On the contrary,, once the observation ξ_n is a brightly pronounced nonstationary process, then the optimally unbiased MA solution (1) is preferable and the coefficient then is $k_n = 1$ to get (1) from (2).

Taking unto consideration the matter of fact that a sample slope b_n of the actual time error x_n function characterizes its nonstationarity, we show that an *adaptation task reduces, in fact, to determination of the dependence* $k_n(b_n)$.

AN ANALYSIS OF FILTERING ERRORS

In our previous report [1], we estimated the time error modeled as [4]

$$x_n = x_0 + y_0 \Delta n + \frac{D}{2} \Delta^2 n^2 + w_{xn}, \tag{5}$$

where $n = 0, 1, ..., \Delta = t_n - t_{n-1}$ is sample time, t_n is current discrete time, and $w_{xn}(t)$ is random time error deviation component. We then showed that, for the same time constant, the estimate error of the three filters, namely a simple MA, the 2[nd] order Wiener, and the 3-state Kalman differ, depending on the signal nonstationarity. We have also shown that, depending on y_0, the application range for all three filters may be separated in the following way:

- ***Simple MA** is most accurate for* $0 \le y_0 < r_1$,
- ***2[nd] Order Wiener filter** is most accurate for* $r_1 \le y_0 < r_2$,
- ***3-State Kalman filter** is most accurate for* $r_2 \le y_0$,

where reference values r_1 and r_2 are defined in by

$$r_{1,2} \cong \alpha_{1,2}\sigma_n\sqrt{\frac{\Delta}{\theta^3}}, \tag{6}$$

where σ_n^2 is variance of the observation noise, θ is the same time constant of each filter, α_1 is coefficient dependent on impulse response of the Wiener filter, and α_2 is coefficient dependent on impulse response of the Wiener and Kalman filters. Yet α_1 and α_2 either depends on the estimate error criterion. Let us mark here that theoretical determination of α_1 and α_2 is in fact a not trivial task, since one first must evaluate the discrete time error of each filter for the given time constant. Experiment also provides the result in the routine way **[1]**.

We now would like to use the same approach to determine the separating coordinate (6) for the simple MA (4) and unbiased MA (1) filters. In a linear case of (5), this is $D = 0$; a simple MA produces the bias $bias_{sMA} = 0.5y_0\theta$ and the variance $\sigma_{sMA}^2 = \sigma_n^2/N$. Under the same condition, the unbiased MA filter **[2]** for $1 \ll N$ produces zero bias and the variance $\sigma_{uMA}^2 = 4\sigma_n^2/N$. Examining an equality of the MSEs of both filters, this is

$$\frac{1}{4}y_0^2\theta^2 + \frac{\sigma_n^2}{N} = \frac{4\sigma_n^2}{N}, \tag{7}$$

we determine the coordinate

$$r \doteq 2\sqrt{3}\sigma_n\sqrt{\frac{\Delta}{\theta^3}} = \alpha\sigma_n[ns]\sqrt{\frac{\Delta[s]}{\theta_h^3[hour^3]}}\left[\frac{ns}{hour}\right] = y_r\sigma_n[ns]\sqrt{\frac{\Delta[s]}{\theta_h^3[hour^3]}}\left[10^{-12}\right], \tag{8}$$

where $\alpha \cong 0.0577$ is used if one works in ns/hours and $y_r \cong 0.01604$ if in parts of 10^{-12}. It is clear now that if $|y_0| < r$, then a simple MA is best, and once $r \le |y_0|$, then the unbiased MA must be applied.

Of course, if we know y_0 implicitly, then the filter (2) adapted under the aforementioned criterion (8) is best in a sense of minimal MSE independently on the time error function. However, we do not know it and should estimate y_0 in some approximate way. The sample n-th frequency offset y_{0n} of time error function is in fact its slope, which is readily estimated stochastically by the regression coefficient

$$\hat{y}_{0n} = b_n = \frac{Cov(\xi_n, t_n)}{\sigma_{tn}^2}, \tag{9}$$

where both a sample covariance of the observation and time and a sample time variance are determined in [2] for the model (5) in the forms of, respectively,

$$Cov(\xi_n, t_n) = \frac{\Delta}{N}\sum_{i=0}^{N-1}\left(\frac{N-1}{2} - i\right)\xi_{n-i}, \tag{10}$$

$$\sigma_{tn}^2 = \Delta^2 \frac{N^2 - 1}{12}. \tag{11}$$

Then transformation of (9) with account of (10) and (11) yields

$$\hat{y}_{0n} = \frac{12}{\Delta N(N^2-1)}\sum_{i=0}^{N-1}\left(\frac{N-1}{2} - i\right)\xi_{n-i}. \tag{12}$$

Now define the error of the time error slope estimate for the linear case (5) of $x_0 = 0$, $\xi_{n-i} = y_0\Delta(n-i) + w_{n-i}$ and $\hat{y}_{0n} = \hat{y}_0$, and get

$$\varepsilon_{yn} = y_0 - \hat{y}_{0n} = y_0 - \frac{12}{\Delta N(N^2-1)}\sum_{i=0}^{N-1}\left(\frac{N-1}{2} - i\right)\left[y_0\Delta(n-i) + w_{n-i}\right] \tag{13}$$

that easily transforms to

$$\varepsilon_{yn} = \frac{12}{\Delta N(N^2-1)}\sum_{i=0}^{N-1}\left(i - \frac{N-1}{2}\right)w_{n-i}. \tag{14}$$

Since the noise w_n is white by definition, then it follows from (14) that the average error $E\{\varepsilon_{yn}\} = 0$, and the error variance is given by

$$\sigma_{yn}^2 = E\{\varepsilon_{yn}^2\} = \frac{144}{\Delta^2 N^2 (N^2-1)^2} E\left\{\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} ij w_{n-i} w_{n-j} - (N-1)\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} i w_{n-i} w_{n-j} + \frac{(N-1)^2}{4}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} w_{n-i} w_{n-j}\right\}. \tag{15}$$

Again, since noise is white then, by definition, $E\{w_i w_j\} = \begin{cases}\sigma_n^2, & i = j \\ 0, & i \neq j\end{cases}$. It means that $\sigma_{yn}^2 = \sigma_y^2$ and (15) transfers to

$$\sigma_y^2 = \sigma_n^2 \frac{12}{\Delta^2 N(N^2-1)}. \tag{16}$$

If we take into account now that $\theta = \Delta(N-1)$ and $\Delta << \tau$ for $1 << N$, then RMSE of (16) transforms to the same formula like the case of the coordinate r (4), namely

$$\sigma_y = 2\sqrt{3}\sigma_n\sqrt{\frac{\Delta}{\theta^3}}. \tag{17}$$

It means that regression produces appreciable RMS error of the slope estimate, which equals to the estimate of a coordinate r.

ADAPTATION STRATEGY

We now can consider how to describe the coefficient $k_n(\hat{y}_{0n})$ for (2). The first case is we just think that

$$k'_n = \begin{cases} 0 & if \quad \hat{y}_{0n} \le r \\ 1 & if \quad r < \hat{y}_{0n} \end{cases}, \tag{18}$$

Actually, this is an ideal case of the slope estimated with negligible error. Since, according to (12), the error is rather big, then we may introduce an additional weight for (18) to smooth the estimate. To form this weight, let us assume the coefficient $k_n(\hat{y}_{0n})$ to be changing linearly in some range $r_1 < r < r_2$ around r. The second case then appears in a form

$$k''_n = \begin{cases} 0, & if \quad \hat{y}_{0n} < r_1 \\ -\frac{r_1}{r_2 - r_1} + \hat{y}_{0n}\frac{1}{r_2 - r_1} & if \quad r_1 \le \hat{y}_{0n} \le r_2 \\ 1 & if \quad r_2 < \hat{y}_{0n} \end{cases} \tag{19}$$

and, if to take $r_1 = r - \sigma_y$ and $r_2 = r + \sigma_y$, where σ_y is provided by (17), we get the alternative coefficient for (2).

SIMULATION

In this section we examine the adapted filter (2) with two above-defined coefficients (18) and (19). Pursuing the aim, we simulate the discrete time error process (5) with $x_0 = 0$, y_0 = var, $D = 0$, $\Delta = 100$ sec, $\sigma_n = 40$ ns, and $\theta = 6$ hours ($N = 216$). We then calculate the reference point (8); this is $r = 0.436\times10^{-12}$. Because frequency offset y_0 strongly influences the estimate error, we change it around the reference point from 0 to 2.0×10^{-12}, watching in this way the filter's adaptive facilities.

To compare the results, we define the estimate error as follows

$$\varepsilon_n = x_n^o - \hat{x}_n, \tag{20}$$

where $x_n^o = y_0\Delta n$ is simulated deterministic trend of a time error, $\hat{x}_n$ is estimate of a time error. We then evaluate (20) for a number M of estimates by the particular sample errors such as bias $\Delta\hat{x} = E\{\varepsilon_n\}$, variance $\sigma_\varepsilon^2 = E\{(\varepsilon_n - \Delta\hat{x})^2\}$, root-mean-square deviation (RMSD) $\sigma_\varepsilon = \sqrt{\sigma_\varepsilon^2}$, RMS error (RMSE) $\varepsilon_{\mathrm{RMS}} = \sqrt{E\{\varepsilon^2\}} = \sqrt{\Delta\hat{x}^2 + \sigma_\varepsilon^2}$, and maximal error $\varepsilon_{\max} = \max|\varepsilon_n|$.

Figure 1 shows the bias produced by each filter along with the ideal adaptation curve. As it had been expected, a simple MA (4) produces the biggest bias and the optimally unbiased filter (1) is best with its almost zero bias. The dotted line shows here the ideal case of adaptation, because for $y_0 \le r$ we would like the filter to be a simple MA and for $r < y_0$ the optimally unbiased MA. Because the error (17), we expect as well that the ideal case cannot be reached and, in fact, both adapted filters demonstrate strongly smoothed lines. Yet, the coefficient k'produces a bit smaller bias than that achieved with k''. We then conclude that in terms of bias, the optimally unbiased filter (1) remains best for an arbitrary offset y_0.

Figure 2 sketches the RMS deviations of the estimate noise. In contrast to bias, here a simple MA is best with its lowest constant noise; the unbiased filter generates almost two times bigger noise, so the ideal adaptation case looks like a step function at the point of r (dotted line). As well as in a bias case, here both adapted curves are also smoothed, so that only in the range of $y_0 \leq r$ the adapted filters generate noise lower than the unbiased filter does. In the range $r < y_0$ the adaptation is no superior. The conclusion is the linear-regression-based adaptation is not so efficient with respect to the estimate noise.

Figure 3 shows RMS error of each filter. Again, as well as in the case of RMSD (Figure 3), we see here that the errors of the adapted filters in the range $y_0 \leq r$ are smaller than that produced by the optimally unbiased filter. In the range $r < y_0$, however, both adapted filters are worst.

Figure 4 exhibits curves for the maximal estimate errors. Despite the error of the second adaptation, case (19) is smaller than that of the unbiased filter in the range left of the point *r;* we also come to the conclusion that effectiveness of the adaptation is poor. It is because in the range right of the point *r,* the error of each adapted filter is even bigger than that of the unbiased filter.

CONCLUDING REMARKS

We have examined in this report one of the possibilities to improve the optimally unbiased filter by adaptation to the time error function slope. The idea was to introduce for the filter an additional adapting coefficient k dependent on the function slope, so that for the stationary process ($y_0 = 0$) we assume $k = 0$, and for the brightly nonstationary process ($r << y_0$) we suppose $k = 1$. Then we estimated the slope by the regression coefficient, evaluated the slope estimate error, worked out the adaptation strategy (18) and (19), and examined the filters for the noisy time error function with linear trend. The conclusions are following:

- In terms of *bias*, the adapted filters are not superior to the optimally unbiased MA filter.
- In terms of *RMSE* and *Maximal error*, adaptation produces noise smaller in a range $y_0 \leq r$ and bigger in a range $r < y_0$.

The general conclusion is the linear regression based approach to the filter adaptation exhibits poor effectiveness, and other ways to estimate the slope with more high accuracy are needed.

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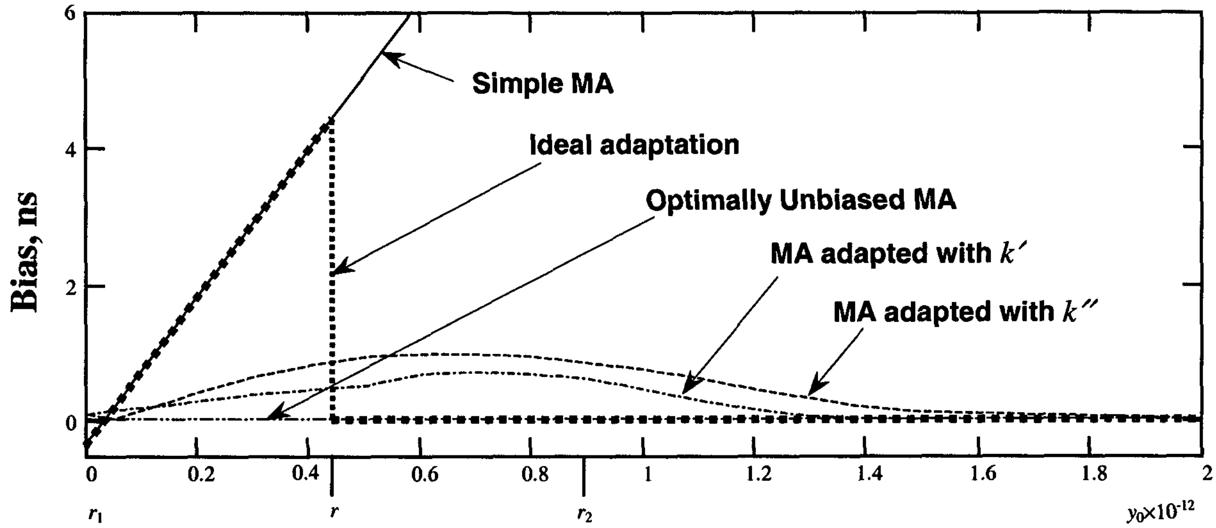

Figure 1. Bias produced by the simple MA, optimally unbiased MA, and adapted MA filter for the discrete time error linear process (1) with $x_0 = 0$, $y_0 = \text{var}$, $D = 0$, $\Delta = 100$ sec, $\sigma_n = 40$ ns, and $\theta = 6$ hours ($N = 216$).

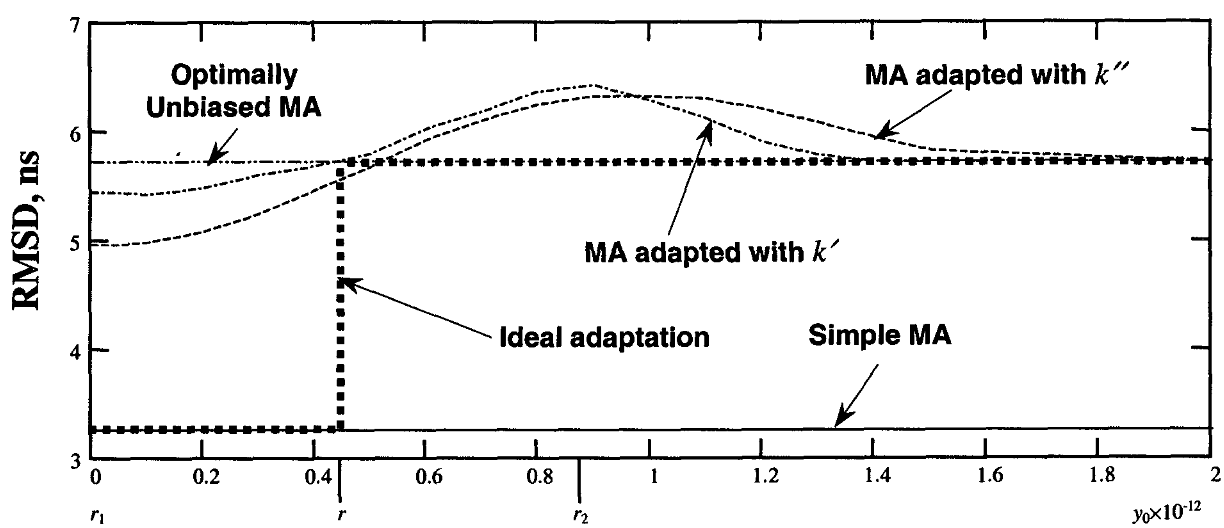

Figure 2. RMSD of the simple MA, optimally unbiased MA, and adapted MA filter for the discrete time error linear process (1) with $x_0 = 0$, $y_0 = \text{var}$, $D = 0$, $\Delta = 100$ sec, $\sigma_n = 40$ ns, and $\theta = 6$ hours ($N = 216$).

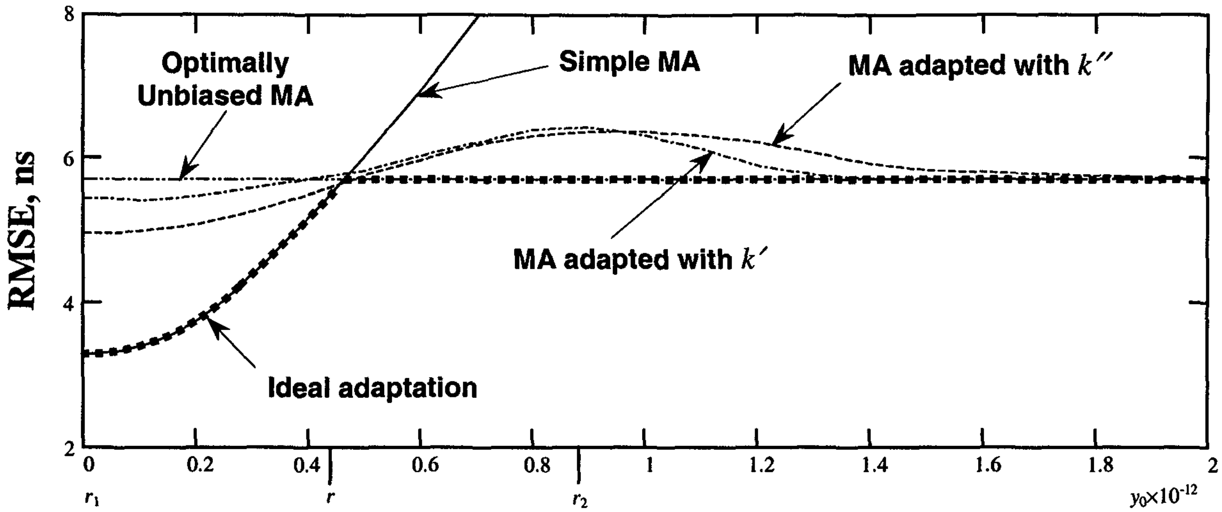

Figure 3. RMSE of the simple MA, optimally unbiased MA, and adapted MA filter for the discrete time error linear process (1) with $x_0 = 0$, $y_0 = \text{var}$, $D = 0$, $\Delta = 100$ sec, $\sigma_n = 40$ ns, and $\theta = 6$ hours ($N = 216$).

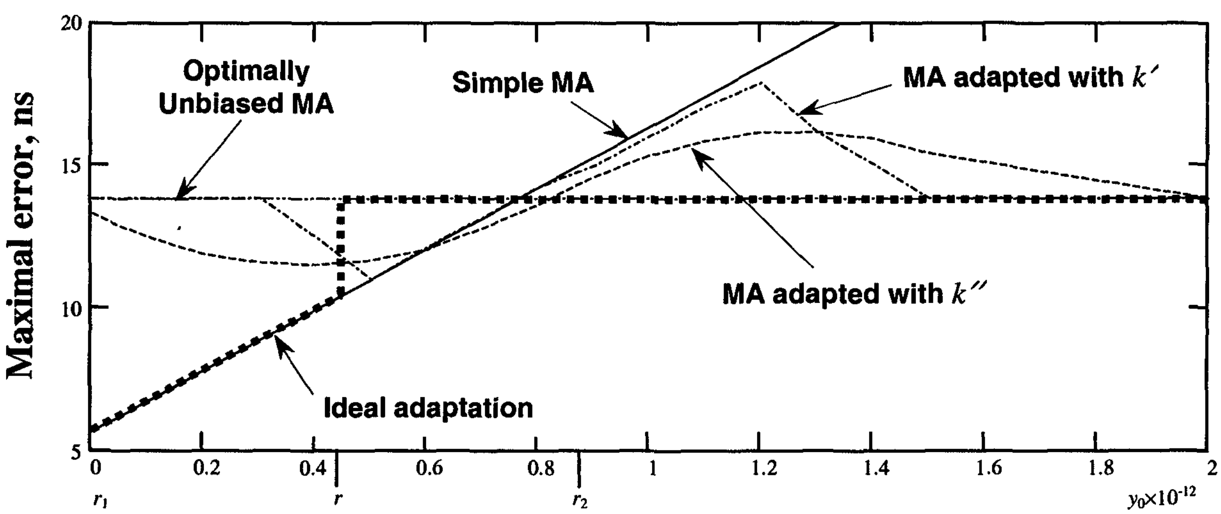

Figure 4. Maximal error of the simple MA, optimally unbiased MA, and adapted MA filter for the discrete time error linear process (1) with $x_0 = 0$, $y_0 = \text{var}$, $D = 0$, $\Delta = 100$ sec, $\sigma_n = 40$ ns, and $\theta = 6$ hours ($N = 216$).