RELATIVISTIC EFFECTS ON SV CLOCKS DUE TO ORBIT CHANGES, AND DUE TO EARTH'S OBLATENESS

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Abstract

Improvements in GPS motivate attention to some small relativistic effects which have long been known, but have previously been too small to be explicitly considered. For SV clocks, these include frequency changes due to orbit adjustments, and effects due to the earth's oblateness. For example between 25 July and 10 October 2000, SV43 occupied a transfer orbit while it was moved from slot F5 to slot F3. The fractional frequency change associated with a change da *in the semi-major axis* a *(in meters) can be estimated as* $9.429 \times 10^{-18} da$. *This yields a prediction of* -1.77×10^{-13} *for the fractional frequency change of the SV43 clock which occurred 25 July 2000. This relativistic effect has been pointed out and measured by Epstein, Fine, and Stoll [4]. On October 10, 2000 the fractional frequency change should have been* $+1.75 \times 10^{-13}$. *Also, the earth's oblateness causes a periodic fractional frequency shift with period of almost 6 hours and amplitude* 0.695×10^{-14}. *These effects will be discussed with the help of Lagrange's planetary perturbation equations.*

INTRODUCTION

The importance of relativistic contributions to atomic clock frequency shifts in the Global Positioning System (GPS) has been recognized from the early design stages of the GPS. Five distinct relativistic effects are incorporated into the System Specification Document, ICD-GPS-200 [1]. These are: the effect of the earth's mass on gravitational frequency shifts of atomic reference clocks fixed on the earth's surface relative to clocks at infinity; the effect of earth's oblate mass distribution on gravitational frequency shifts of atomic clocks fixed on the earth's surface; second-order Doppler shifts of clocks fixed on the earth's surface due to the earth rotation; gravitational frequency shifts of clocks in GPS satellites due to the earth's mass; and second-order Doppler shifts of clocks in GPS satellites due to their motion through an Earth-Centered Inertial (ECI) Frame. The combination of second-order Doppler and gravitational frequency shifts for a clock in a GPS satellite leads to the following expression for the fractional frequency shift of a satellite clock relative to a reference clock fixed on the earth's geoid [**2**]:

$$\frac{\delta f}{f} = -\frac{1}{2}\frac{v^2}{c^2} - \frac{GM}{rc^2} - \frac{\Phi_0}{c^2} \tag{1}$$

where v is the satellite speed in a local ECI reference frame, GM is the product of the Newtonian gravitational constant G and the earth's mass M, c is the defined speed of light, and Φ_0 is the effective gravitational potential on the earth's rotating geoid. The term Φ_0 includes contributions from both monopole and quadrupole moments of the earth's mass distribution, and the effective centripetal potential in an earth-fixed reference frame such as WGS-84, due to the earth's rotation. For reference we quote here the value for Φ_0[**2**]:

$$\frac{\Phi_0}{c^2} = -\frac{GM}{a_1 c^2}\left[1 + J_2/2\right] - \frac{1}{2c^2}\Omega_e^2 a_1^2 = -6.9692842 \times 10^{-10}\,, \tag{2}$$

where a_1 is the earth's equatorial radius, J_2 is the quadrupole moment coefficient of the earth, and Ω_e is the angular rotational speed of the earth.

If the GPS satellite orbit can be approximated by a Keplerian orbit of semi-major axis a, then at an instant when the distance of the clock from the earth's center of mass is r, this leads to the following expression for the fraction frequency shift of Eq. (1):

$$\frac{\Delta f}{f} = -\frac{3GM}{2ac^2} - \frac{\Phi_0}{c^2} + \frac{2GM}{c^2}\left[\frac{1}{r} - \frac{1}{a}\right]. \tag{3}$$

Eq. (3) is derived by making use of the conservation of total energy (per unit mass) of the satellite, which leads to an expression for v^2 in terms of GM/r and GM/a which can be substituted into Eq. (1):

$$\frac{1}{2}v^2 - \frac{GM}{r} = -\frac{GM}{2a}. \tag{4}$$

The first two terms in Eq. (3) give rise to the "factory frequency offset," which is applied to GPS clocks before launch in order to make them beat at a rate equal to that of reference clocks on the earth's surface. The last term in Eq. (3) is very small when the orbit eccentricity e is small; when integrated over time, these terms give rise to the so-called 'e sin E" effect or "eccentricity effect." In most of the following discussion, we shall assume that eccentricity is very small.

Clearly, from Eq. (3), if the semi-major axis should change by an amount δa due to an orbit adjustment, the satellite clock will experience a fractional frequency change

$$\frac{\delta f}{f} = +\frac{3GM\delta a}{2c^2a^2}. \tag{5}$$

The factor 3/2 in this expression arises from the combined effect of second-order Doppler and gravitational frequency shifts. If the semi-major axis increases, the satellite will be higher in the earth's gravitational potential and will be gravitationally blue-shifted more, while at the same time the satellite velocity will be reduced, reducing the size of the second-order Doppler shift (which is generally a red shift). The net effect would make a positive contribution to the fractional frequency shift.

It has long been known that orbit adjustments are associated with satellite clock frequency shifts [3], but nothing has been documented and no reliable measurements of such shifts have previously been made. Recently Marvin Epstein, Joseph Fine, and Eric Stoll [4] of ITT have evaluated the frequency shift of SV43 arising from a trajectory change applied to this satellite on 25 July 2000. The purpose of this orbit adjustment was to move the satellite from slot F5 to slot F3. A drift orbit extending from 25 July 2000 to 10 October 2000 was used to accomplish this move [6]. Epstein, Fine, and Stoll have reported that associated with the "delta-v burn" on 25 July 2000 there was a frequency shift of the rubidium clock on board SV43 of amount

$$\frac{\delta f}{f} = -1.85 \times 10^{-13} \ \text{(measured)}. \tag{6}$$

Epstein *et al.*[4] suggested that the above frequency shift is relativistic in origin, and used NIMA precise ephemerides to estimate the frequency shift arising from second-order Doppler and gravitational potential differences. They calculated separately the second-order Doppler and gravitational frequency shifts due to the orbit change. The NIMA precise ephemerides are expressed in the WGS-84 coordinate frame, which is earth-fixed. If used without removing the underlying earth rotation, the velocity would be erroneous. They, therefore, transformed the NIMA precise ephemerides to an earth-centered inertial frame by accounting for a (uniform) earth rotation rate.

The semi-major axes before and after the orbit change were calculated by taking the average of the maximum and minimum radial distances. Speeds were calculated using a Keplerian orbit model. They [4] arrived at the following numerical values of position and velocity:

$$07/22/00: \quad a = 2.656139556 \times 10^{7} \text{ m.}; \qquad v = 3.873947951 \times 10^{3} \text{ m/s.} \tag{7}$$

$$07/30/00: \quad a = 2.654267359 \times 10^{7} \text{ m.}; \qquad v = 3.875239113 \times 10^{3} \text{ m/s.} \tag{8}$$

Since the semi-major axis decreased, the frequency shift should be negative. The prediction [4] made for the frequency shift, which is based on Eq. (1), was then:

$$\frac{\delta f}{f} = -1.734 \times 10^{-13} \tag{9}$$

which is to be compared with the measured value, Eq. (6). There is, thus, fairly compelling evidence that the observed frequency shift is indeed a relativistic effect.

LAGRANGE PERTURBATION THEORY

Perturbations of GPS orbits due to the earth's quadrupole mass distribution are a significant fraction of the change in semi-major axis associated with the orbit change discussed above. This raises a question whether it is sufficiently accurate to use a

Keplerian orbit to describe GPS satellite orbits, and estimate the semi-major axis change as though the orbit were Keplerian. The purpose of the present calculation is to investigate this question. Also, it is of interest to investigate directly the effect of the earth's quadrupole moment on the frequency of GPS satellite clocks. Previously, such effects have been neglected. However, the effect may be worth considering as GPS clock performance continues to improve.

To see how large such quadrupole effects may be, the Appendix [5] quotes the perturbations on the semi-major axis a of a Keplerian satellite arising from earth's quadrupole moment. For the semi-major axis, if the eccentricity is very small the dominant contribution has a period twice the orbital period and has amplitude $3J_2a_1^2\sin^2 i_0/(2a_0) \approx 1658$ m. The following values for the constants are used here:

$J_2 = 1.0826267 \times 10^{-3}$; $GM = 3.986004415 \times 10^{14}\mathrm{m}^3/\ \mathrm{sec}^2$; $a_1 = 6.3781363 \times 10^6$ m; $\Omega_E = 7.291151467 \times 10^{-5}\ \mathrm{sec}^{-1}$; $a_0 = 2.65641046 \times 10^7$ m,

where a_0 and a_1 are the orbit semi-major axis and the earth's equatorial radius, respectively, and Ω_E is the earth's rotational angular velocity.

The oscillation in the semi-major axis would significantly affect calculations of the semi-major axis at any particular time. This suggests that Eq. (4) needs to be examined carefully in light of the periodic perturbations on the semi-major axis. Therefore in this paper we develop an approximate description of a satellite orbit, of small eccentricity, taking into account the earth's quadrupole moment to first order. Terms of order $J_2 \times e$ will be neglected. This problem is non-trivial because the perturbations themselves (see for example, the equations for mean anomaly and altitude of perigee) have factors $1/e$ which blow up as the eccentricity approaches zero. This problem is a mathematical one, not a physical one. It simply means that the observable quantities–such as coordinates and velocities–need to be calculated in such a way that finite values are obtained. Orbital elements which blow up are unobservable.

CONSERVATION OF ENERGY

The gravitational potential of a satellite at position (x, y, z) in equatorial ECI coordinates in the model under consideration here is

$$V(x,y,z) = -\frac{GM}{r}\left(1 - \frac{J_2 a_1^2}{r^2}\left[\frac{3z^2}{2r^2} - \frac{1}{2}\right]\right) . \qquad (10)$$

Since the force is conservative in this model (solar radiation pressure, thrust, etc. are not considered), the kinetic plus potential energy is conserved. Let ϵ be the energy per unit mass of an orbiting mass point. Then

$$\epsilon = \mathrm{constant} = \frac{v^2}{2} + V(x,y,z) = \frac{v^2}{2} - \frac{GM}{r} + V'(x,y,z) \qquad (11)$$

where $V'(x, y, z)$ is the perturbing potential due to the earth's quadrupole potential.

It is shown in textbooks [6] that, with the help of Lagrange's planetary perturbation theory, the conservation of energy condition can be put in the form

$$\epsilon = -\frac{GM}{2a} + V'(x,y,z) \tag{12}$$

where a is the perturbed (osculating) semi-major axis, given by Eq. (A.6) in the Appendix. In other words, for the perturbed orbit,

$$\frac{v^2}{2} - \frac{GM}{r} = -\frac{GM}{2a} \tag{13}$$

On the other hand, the net fractional frequency shift relative to a clock at rest at infinity is determined by the second-order Doppler shift (a redshift) and a gravitational redshift. The total relativistic fractional frequency shift is

$$\frac{\Delta f}{f} = -\frac{v^2}{2} - \frac{GM}{r} + V'(x,y,z) \tag{14}$$

The conservation of energy condition can be used to express the second-order Doppler shift in terms of the potential. Since in this paper we are interested in fractional frequency changes caused by changing the orbit, it will make no difference if the calculations use a clock at rest at infinity as a reference rather than a clock at rest on the earth's surface. The reference potential cancels out to the required order of accuracy.

Therefore, from perturbation theory we need expressions for the square of the velocity, for the radius r, and for the perturbing potential. We now proceed to derive these expressions.

PERTURBATION EQUATIONS

First we list some facts about an unperturbed Keplerian orbit. The eccentric anomaly E is to be calculated by solving the equation

$$E - e\sin E = M = n_0(t - t_0) \tag{15}$$

where M is the "mean anomaly" and t_0 is the time of passage past perigee, and

$$n_0 = \sqrt{GM/a^3}\,. \tag{16}$$

Then the perturbed radial distance r and true anomaly f of the satellite are obtained from

$$r = a(1 - e\cos E) \tag{17}$$

$$\cos f = \frac{\cos E - e}{1 - e\cos E}, \quad \sin f = \sqrt{1-e^2}\frac{\sin E}{1 - e\cos E}\,. \tag{18}$$

The observable x, y, z-coordinates of the satellite are then calculated from the following equations:

$$x = r(\cos\Omega\cos(f+\omega) - \cos i \sin\Omega \sin(f+\omega)) \tag{19}$$

$$y = r(\sin\Omega\cos(f+\omega) + \cos i \cos\Omega \sin(f+\omega)). \tag{20}$$

$$z = r(\sin i \sin(f+\omega)) \tag{21}$$

where ω is the angle of the ascending line of nodes, i is the inclination, and ω is the altitude of perigee.

By differentiation with respect to time, or by using the conservation of energy equation, one obtains the following expression for the square of the velocity:

$$\frac{v^2}{2} = \frac{GM}{2a}\frac{1+e\cos E}{1-e\cos E} \tag{22}$$

In these expressions v^2 and r^{-1} are observable quantities. The combination $e\cos E$, where E is the eccentric anomaly, occurs in both of these expressions. To derive expressions for v^2 and r^{-1} in the perturbed orbits, expressions for the perturbed elements a, e, E are to be substituted into the righthand sides. Therefore, we need the combination $e\cos E$ in the limit of small eccentricity.

Calculation of Perturbed Eccentricity

To leading order, from the Appendix we have for the perturbed eccentricity the following expression:

$$e = K_e + \frac{3J_2a_1^2}{2a_0^2}\left[\left(1-\frac{3}{2}\sin^2 i_0\right)\cos f + \frac{1}{4}\sin^2 i_0\cos(2\omega_0+f) + \frac{7}{12}\sin^2 i_0\cos(2\omega_0+3f)\right]. \tag{23}$$

Calculation of Perturbed Eccentric Anomaly

The eccentric anomaly is calculated from the equation

$$E = M + e\sin E \tag{24}$$

with perturbed values for M and e. Expanding to first order in e gives the following expression for $\cos E$:

$$\cos E = \cos M - e\sin M\sin E \tag{25}$$

and multiplying by e,

$$e\cos E = e\cos M - e^2\sin M\sin E \approx e\cos M \tag{26}$$

We shall neglect higher order terms in e. Referring now to the perturbed expression for mean anomaly M in the Appendix, we write

$$M = M_0 + \Delta M/e_0. \tag{27}$$

where we indicate explicitly the terms in e_0^{-1}; that is, the quantity M_0 contains all terms which do not blow up as $e \to 0$, and $\Delta M/e_0$ contains all the other terms. From the Appendix, we have to leading order

$$\Delta M/e_0 = -\frac{3J_2a_1^2}{2e_0a_0^2}\left[\left(1-\frac{3}{2}\sin^2 i_0\right)\sin f - \frac{1}{4}\sin^2 i_0 \sin(2\omega_0+f) + \frac{7}{12}\sin^2 i_0 \sin(2\omega_0+3f)\right] \tag{28}$$

and so for very small eccentricity,

$$e\cos E = e\cos M_0 - \Delta M \sin M_0\,. \tag{29}$$

Then after accounting for contributions from the perturbed eccentricity and the perturbed mean anomaly, after a few lines of algebra we obtain for $e\cos E$

$$e\cos E = e_0\cos E_0 + \frac{3J_2a_1^2}{2a_0^2}\left(1-\frac{3}{2}\sin^2 i_0\right) + \frac{5J_2a_1^2}{4a_0^2}\sin^2 i_0\cos 2(\omega_0+f). \tag{30}$$

where the first term is the unperturbed part.

The perturbation is a constant, plus a term with twice the orbital period.

Calculation of the Perturbation in Semi-major Axis

From the Appendix, the leading terms in the perturbation of the semi-major axis are

$$a = K_a + \frac{3J_2a_1^2}{2a_0}\sin^2 i_0\cos 2(\omega_0+f). \tag{31}$$

Calculation of the Perturbation in Radius

We are now in position to compute the perturbation in the radius. From the expression for r, we have after combining terms

$$r = a_0(1-e_0\cos E_0) + \Delta a - \Delta(e\cos E)$$

$$= a_0(1-e_0\cos E_0) - \frac{3J_2a_1^2}{2a_0}\left(1-\frac{3}{2}\sin^2 i_0\right) + \frac{J_2a_1^2}{4a_0}\sin^2 i_0\cos 2(\omega_0+f)\,. \tag{32}$$

Calculation of the Perturbation in the Velocity Squared

The above results, after substituting into Eq. (18), yield the expression

$$\frac{v^2}{2} = \frac{GM}{2a_0}(1 + 2e_0 \cos E_0) + \frac{3GMJ_2a_1^2}{a_0^3}\left(1 - \frac{3}{2}\sin^2 i_0\right) + \frac{GMJ_2a_1^2}{2a_0^3}\sin^2 i_0 \cos 2(\omega_0 + f)\,. \tag{33}$$

Calculation of the Perturbation in GM/r

The above expression for the perturbed r yields the following for the monopole contribution to the gravitational potential:

$$-\frac{GM}{r} = -\frac{GM}{a_0}(1 + e_0 \cos E_0) - \frac{3GMJ_2a_1^2}{2a_0^3}\left(1 - \frac{3}{2}\sin^2 i_0\right) + \frac{GMJ_2a_1^2\sin^2 i_0}{4a_0^3}\cos 2(\omega_0 + f)\,. \tag{34}$$

Calculation of the Approximate Value of the Perturbing Potential

Since the perturbing potential contains the small factor J_2, to leading order we may substitute unperturbed values for r and z into $V'(x,y,z)$ which yields the expression

$$V'(x,y,z) = -\frac{GMJ_2a_1^2}{2a_0^3}\left(1 - \frac{3}{2}\sin^2 i_0\right) - \frac{3GMJ_2a_1^2\sin^2 i_0}{4a_0^3}\cos 2(\omega_0 + f)\,. \tag{35}$$

Conservation of Energy

It is now very easy to check conservation of energy. Adding kinetic energy per unit mass, to two contributions to the potential energy, gives

$$\epsilon = \frac{v^2}{2} - \frac{GM}{r} + V' = -\frac{GM}{2a_0} - \frac{GMJ_2a_1^2}{2a_0^3}\left(1 - \frac{3}{2}\sin^2 i_0\right)\,. \tag{36}$$

This verifies that the perturbation theory gives a constant energy. The extra term in the above equation, with J_2 in it, can be neglected. This is because the nominal inclination of GPS orbits is such that the factor $(1 - 3\sin^2 i_0/2)$ is essentially zero.

Thus numerical calculations of the total energy per unit mass should give us the value of a_0.

Calculation of Fractional Frequency Shift

The fractional frequency shift calculation is very similar to the calculation of the energy, except the the second-order Doppler term contributes with a negative sign. The result is

$$\frac{\Delta f}{f} = -\frac{v^2}{2c^2} - \frac{GM}{c^2 r} + \frac{V'}{c^2}$$

$$= -\frac{3GM}{2a_0c^2} - \frac{2GM}{c^2a_0}e_0\cos E_0 - \frac{7GMJ_2a_1^2}{2a_0^3c^2}\left(1-\frac{3}{2}\sin^2 i_0\right) - \frac{GMJ_2a_1^2\sin^2 i_0}{a_0^3c^2}\cos 2(\omega_0+f)\,. \tag{37}$$

The first term, when combined with the reference potential at the earth's geoid gives rise to the "factory frequency offset." The third term can be neglected, as pointed out above. The last term has an amplitude

$$\frac{GMJ_2a_1^2\sin^2 i_0}{a_0^3c^2} = 6.95\times 10^{-15} \tag{38}$$

which may be large enough to consider when calculating frequency shifts produced by orbit changes. Therefore, this contribution may have to be considered in the determination of the semi-major axis.

The result suggests the following method of computing the fractional frequency shift: Averaging the shift over one orbit, the periodic term will average down to a negligible value. The second term is negligible. So if one has a good estimate for the nominal semi-major axis parameter, the term $-3GM/2a_0c^2$ gives the average fractional frequency shift. On the other hand, the average energy per unit mass is given by $\epsilon = -GM/2a_0$. Therefore, the precise ephemerides, specified in an ECI frame, can be used to compute the average value for ϵ, then the average fractional frequency shift will be

$$\frac{\Delta f}{f} = 3\epsilon/c^2\,. \tag{39}$$

The periodic term in Eq. (33) is of a form similar to that which gives rise to the eccentricity correction, which is applied by GPS receivers. Considering only the periodic term, the additional time elapsed on the orbiting clock will be given by

$$\delta t_{J_2} = \int_{path} dt\left[-\frac{GMJ_2a_1^2\sin^2 i_0}{a_0^3c^2}\cos(2\omega_0+2nt)\right] \tag{40}$$

where to a sufficient approximation we have replaced the quantity $2f$ in the integrand by $2n = 2\sqrt{GM/a_0^3}$; n is the approximate mean motion of GPS satellites. Upon integrating and dropping the constant of integration (assuming as usual that such constant time offsets are lumped with other contributions) gives the periodic relativistic effect on the elapsed time of the SV clock due to earth's quadrupole moment:

$$\delta t_{J_2} = -\sqrt{\frac{GM}{a_0^3}}\frac{J_2a_1^2\sin^2 i_0}{2c^2}\sin(2\omega_0+2nt)\,. \tag{37}$$

The correction which should be applied by the receiver is the *negative* of this expression

$$\delta t_{J_2}(\text{correction}) = \sqrt{\frac{GM}{a_0^3}}\frac{J_2a_1^2\sin^2 i_0}{2c^2}\sin(2\omega_0+2nt)\,. \tag{38}$$

The phase of this correction is zero when the satellite passes through the earth's equatorial plane going northwards.

If not accounted for, this effect on the SV clock time would give rise to a peak-to-peak periodic navigational error in position of approximately $2c \times \delta t_{J_2} = 1.07$ cm.

These effects were considered by Ashby and Spilker ([2], pp. 685-686]), but in that work the effect of the earth's quadrupole moment on the term GM/r was not considered; the present paper supersedes that work.

NUMERICAL CALCULATIONS

Precise ephemerides were obtained for SV43 from the Jet Propulsion Laboratories Web site *ftp://sideshow.jpl.nasa.gov/pub/2000/orbits*. These are expressed in the J2000 ECI frame. Computer code was written to compute the average value of ϵ for one day and thence the fractional frequency shift relative to infinity before and after each orbit change. The following results were obtained:

$$07/22/00 : a = 2.65611575 \times 10^7 \pm 69 \text{ m.}$$

$$07/30/00 : a = 2.65423597 \times 10^3 \pm 188 \text{ m.}$$

$$10/07/00 : a = 2.65418742 \times 10^7 \pm 95 \text{ m.}$$

$$10/12/00 : a = 2.65606323 \times 10^7 \pm 58 \text{ m.}$$

Therefore, the fractional frequency change produced by the orbit change of July 25 is calculated to be

$$\frac{\Delta f}{f} = -1.77 \times 10^{-13}, \tag{36}$$

which agrees with the measured value to within about 3.3%. We predict that the fractional frequency shift on October 10, should have been

$$\frac{\Delta f}{f} = +1.75 \times 10^{-13}. \tag{37}$$

This shift has not yet been measured.

On 9 March 2001, SV54's orbit was changed by a delta-v burn. Using the above procedures, we can calculate the fractional frequency change produced in the onboard clocks. We find

$$03/07/01 : a = 2.65597188 \times 10^7 \pm 140 \text{ m.}$$

$$03/11/01 : a = 2.65359261 \times 10^7 \pm 108 \text{ m.}$$

Using Eq. (5) yields the following prediction for the fractional frequency change of SV54 on 9 March 2001:

$$\frac{\Delta f}{f} = +2.24 \times 10^{-13} \pm 0.02 \times 10^{-13}.$$

The quoted uncertainty is due to the combined uncertainties from the determination of the energy per unit mass before and after the orbit change.

CONCLUSIONS

We note that the values of semi-major axis reported by Epstein *et al.*. differ from the values obtained by averaging as outlined above, by 200-300 m. This difference arises because of the different methods of calculation. In the present calculation, an attempt was made to account for the effect of the earth's quadrupole moment on the Keplerian orbit. It was not necessary to compute the orbit eccentricity. Agreement with measurement of the fractional frequency shift was only a few percent better than that obtained by differencing the maximum and minimum radii.

This approximate treatment of the orbit makes no attempt to consider perturbations that are non-gravitational in nature–e.g., solar radiation pressure. The work was an investigation of the approximate effect of the earth's quadrupole moment on the GPS satellite orbits, for the purpose of (possibly) accurate calculations of the fractional frequency shifts which result from orbit changes.

As a general conclusion, the fractional frequency shift can be estimated to very good accuracy from the expression for the "factory frequency offset."

$$\frac{\delta f}{f} = +\frac{3GM\delta a}{2c^2a^2}. \tag{41}$$

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APPENDIX

This Appendix quotes for convenience the results of first-order Lagrangian perturbation theory for a Keplerian orbit perturbed by a mass quadrupole centered at the origin [6]. The osculating elements are: semi-major axis a, eccentricity e, inclination i, longitude of the ascending node Ω, altitude of perigee ω, and mean anomaly M. Lagrange's planetary perturbation equations describe how these six elements change in time under the influence of a perturbation. In the present case, the perturbation is due to the earth's quadrupole mass distribution,

$$V'(x,y,z) = \frac{GMJ_2a_1^2}{r^3}\left[\frac{3z^2}{2r^2} - \frac{1}{2}\right], \qquad (A.1)$$

where GM is the product of the Newtonian gravitational constant and the earth's mass, J_2 is the earth's quadrupole moment coefficient, r is the satellite's radial distance from the earth's center of mass, and (x,y,z) are the satellite coordinates in an earth-fixed inertial reference frame with z-axis parallel to the earth's symmetry axis.

When these perturbed orbital elements are given, the eccentric anomaly E is to be calculated by solving the equation

$$E - e\sin E = M \qquad (A.2)$$

and then the perturbed radial distance r and true anomaly f of the satellite are obtained from

$$r = a(1 - e\cos E) \qquad (A.3)$$

$$\cos f = \frac{\cos E - e}{1 - e\cos E}, \quad \sin f = \sqrt{1-e^2}\frac{\sin E}{1 - e\cos E}. \qquad (A.4)$$

The observable x, y, z-coordinates of the satellite are then calculated from the following equations:

$$x = r(\cos\Omega\cos(f+\omega) - \cos\imath\sin\Omega\sin(f+\omega))$$

$$y = r(\sin\Omega\cos(f+\omega) + \cos i\cos\Omega\sin(f+\omega)). \qquad (A.5)$$

$$z = r(\sin i\sin(f+\omega))$$

The following expressions give the perturbed orbital elements correct to first order in the perturbation.

$$a = K_a + \frac{3J_2a_1^2}{a_0(1-e_0^2)^3}\left[e_0(1+e_0^2/4)\left(1 - \frac{3}{2}\sin^2 i_0\right)\cos f + \frac{e_0^3}{2}\left(1 - \frac{3}{2}\sin^2 i_0\right)\cos 2f\right.$$

$$+\frac{e_0^3}{12}\left(1-\frac{3}{2}\sin^2 i_0\right)\cos 3f+\frac{e_0^3}{16}\sin^2 i_0\cos(2\omega_0-f) \tag{A.6}$$

$$+\frac{3}{4}e_0(1+e_0^2/4)\sin^2 i_0\cos(2\omega_0+f)+\frac{1}{2}(1+3e_0^2/2)\sin^2 i_0\cos(2\omega_0+2f)$$

$$+\frac{9}{12}e_0(1+e_0^2/4)\sin^2 i_0\cos(2\omega_0+3f)+\frac{3}{8}e_0^2\sin^2 i_0\cos(2\omega_0+4f)+\frac{e_0^3}{16}\sin^2 i_0\cos(2\omega_0+5f)\Bigg].$$

$$e=K_e+\frac{3J_2a_1^2}{2a_0^2(1-e_0^2)^2}\left[(1+e_0^2/4)\left(1-\frac{3}{2}\sin^2 i_0\right)\cos f+\frac{e_0}{2}\left(1-\frac{3}{2}\sin^2 i_0\right)\cos 2f\right.$$

$$+\frac{e_0^2}{12}\left(1-\frac{3}{2}\sin^2 i_0\right)\cos 3f+\frac{e_0^2}{16}\sin^2 i_0\cos(2\omega_0-f)+\frac{1}{4}(1+11e_0^2/4)\sin^2 i_0\cos(2\omega_0+f)$$

$$+\frac{5}{4}e_0\sin^2 i_0\cos(2\omega_0+2f)+\frac{1}{12}(7+17e_0^2/4)\sin^2 i_0\cos(2\omega_0+3f) \tag{A.7}$$

$$\left.+\frac{3}{8}e_0\sin^2 i_0\cos(2\omega_0+4f)+\frac{e_0^2}{16}\sin^2 i_0\cos(2\omega_0+5f)\right]$$

$$i=K_i+\frac{3J_2a_1^2}{8a_0^2(1-e_0^2)^2}\sin 2i_0\left[\cos(2\omega_0+2f)+\frac{e_0}{3}\cos(2\omega_0+3f)+e_0\cos(2\omega_0+f)\right] \tag{A.8}$$

$$\Omega=K_\Omega-\frac{3J_2a_1^2}{4a_0^2(1-e_0^2)^2}\cos i_0\Big[2f+2e_0\sin f-e_0\sin(2\omega_0+f)$$

$$-\sin(2\omega_0+2f)-\frac{e_0}{3}\sin(2\omega_0+3f)\Big] \tag{A.9}$$

$$\omega=K_\omega+\frac{3J_2a_1^2}{2a_0^2(1-e_0^2)^2}\left[\left(2-\frac{5}{2}\sin^2 i_0\right)f+\frac{1}{e_0}\left[\left(1-\frac{3}{2}\sin^2 i_0\right)+\frac{e_0^2}{4}(7-\frac{17}{2}\sin^2 i_0)\right.\right.$$

$$\left.+\frac{e_0^2}{8}\sin^2 i_0\cos 2\omega_0\right]\sin f+\frac{1}{2}\left(1-\frac{3}{2}\sin^2 i_0\right)\sin 2f$$

$$+\frac{e_0}{12}\left(1-\frac{3}{2}\sin^2 i_0\right)\sin 3f-\frac{1}{e_0}\left[\frac{e_0^2}{2}+\frac{1}{4}\left(1-\frac{7}{2}e_0^2\right)\sin^2 i_0\right]\sin(2\omega_0+f) \tag{A.10}$$

$$-\frac{1}{2}\left(1-\frac{5}{2}\sin^2 i_0\right)\sin(2\omega_0+2f)-\frac{1}{3e_0}\left[\frac{e_0^2}{2}-\left(\frac{7}{4}+\frac{19}{16}e_0^2\right)\sin^2 i_0\right]\sin(2\omega_0+3f)$$

$$\left.+\frac{3}{8}\sin^2 i_0\sin(2\omega_0+4f)+\frac{e_0}{16}\sin^2 i_0\sin(2\omega_0+5f)\right]$$

The perturbed mean anomaly is

$$M = K_M + n_r t - \frac{3J_2 a_1^2}{2e_0 a_0^2(1-e_0^2)^{3/2}} \Bigg[\left(\left(1 - \frac{3}{2}\sin^2 i_0\right)(1 - \frac{e_0^2}{4}) + e_0^2 \sin^2 i_0 \cos 2\omega_0 \right) \sin f$$

$$+ \frac{e_0}{2}\left(1 - \frac{3}{2}\sin^2 i_0\right)\sin 2f + \frac{e_0^2}{12}\left(1 - \frac{3}{2}\sin^2 i_0\right)\sin 3f \tag{A.11}$$

$$-\frac{1}{4}(1 + 3e_0^2/2)\sin^2 i_0 \sin(2\omega_0 + f) + \frac{1}{12}(7 - e_0^2/4)\sin^2 i_0 \sin(2\omega_0 + 3f)$$

$$+ \frac{3}{8} e_0 \sin^2 i_0 \sin(2\omega_0 + 4f) + \frac{1}{16} e_0^2 \sin^2 i_0 \sin(2\omega_0 + 5f) \Bigg]$$

where

$$n_r = [-2\epsilon]^{3/2}/GM \tag{A.12}$$

and ϵ is the conserved energy per unit mass, given by

$$\epsilon = -\frac{GM}{r} + \frac{v^2}{2} + V'(x, y, z) \tag{A.13}$$

Constants such as K_a, K_e, etc., are constants of integration determined by the initial conditions.

QUESTIONS AND ANSWERS

VICTOR REINHARDT (Boeing Satellite Systems): Can you put up that next to the last slide that had the number of picoseconds change due to the varying term? I just missed the slides of that. So that was about 24 picoseconds.

NEIL ASHBY: Twenty-four picoseconds.

REINHARDT: Okay, thank you.

EDOARDO DETOMA (Alenia Aerospazio): If I remember well, at the beginning, the GPS inclination was 64.3 degrees or something like that, not 55. And 64.3 inclination is the 0,1 term of the series expansion of the simulation – I believe it was not incidental; I believe it was done on purpose.

ASHBY: I think that zeroes out the procession of the modal line.

DETOMA: Yes, but probably more than that. There is a paper which was published, I believe, in '57 or '58 in which the term for 64.3 is clearly pinpointed.

ASHBY: Yes. But the banishing of this particular term does not zero out any secular terms. I don't know what the reason was. The 55.

DETOMA: At a certain point, the constellation was reduced to 18 satellites. If you look at the literature around '86, '87, you find they change the number of satellites from 24 to 18. And this was probably due to budgetary reasons. And if you reduce the number of satellites, you must reduce the inclination to maintain the filling factor of the constellation.

DEMETRIOS MATSAKIS (U.S. Naval Observatory): I would like to just confirm that a little bit. It is a rumor, but I was told by Colonel Armour years ago that the reason for the inclination was budget. It cost more to have a higher inclination.

TOM McCASKILL (U.S. Naval Research Laboratory): Going way back to around 1968-1969, whenever Roger Easton, Jim Buisson, Don Lynch, and myself did the constellation study, we looked at about 107 different constellation configurations. And we calculated a lot of quantities, and it turned out our calculation and that you could minimize the dilution precision with an inclination of 53 degrees. And later on, with the Block I's, it went up to 63. And there was a study done by Aerospace and the Air Force, and they finally settled on 55. And there could have been some launch considerations; however, if you look at zero-degree inclination for the constellation, there is a mathematical singularity, and if you go up to 90 degrees, like Transit, you have too many satellites at the pole. So it is really a balancing operation that gives you the best dilution of precision on a worldwide basis.