LEAST-SQUARES ANALYSIS OF TWO-WAY SATELLITE TIME AND FREQUENCY TRANSFER MEASUREMENTS

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Abstract

The least-squares analysis of Two-Way Satellite Time and Frequency (TWSTFT) data, with the aim of determining optimal estimates of phase and frequency offsets, is considered. An overview of the application of Gauss-Markov estimation to the analysis of a uniformly spaced time series is presented. Two aspects of TWSTFT data analysis are examined in depth. Firstly, measures based on second difference statistics for characterizing the consistently but unevenly spaced TWSTFT measurements are introduced. Secondly, an approach to the estimation of the occasional and unknown delay steps in TWSTFT data, for example due to hardware replacement, is presented.

1 INTRODUCTION

Two-Way Satellite Time and Frequency Transfer (TWSTFT) [1] has been used operationally by primary timing laboratories for several years. TWSTFT links between laboratories have proved to be both stable and reliable, and consequently TWSTFT measurements are used for several of the main links in the computation of International Atomic Time (TAI). This paper describes new analysis techniques being developed for the processing of TWSTFT measurements. The overall aim of the study is to be able to determine, at any measurement epoch, estimates of the phase and normalized frequency offsets, together with their uncertainties, between two clocks or time scales being compared using TWSTFT.

In Section 2, we describe an approach, based on Gauss-Markov estimation, to solving this problem for the case of measurements of the phase difference between two clocks or time scales made at *uniformly* spaced times and *assuming knowledge* of the random noise processes underlying the measurements. For time series corresponding to uniformly spaced times, second difference statistics, such as the Allan and modified Allan variances, provide a means to obtain information about the noise processes.

However, the application of this approach to TWSTFT is made difficult for two reasons. Firstly, for many operational links based on TWSTFT, measurements are available only on Mondays, Wednesdays, and Fridays. The measurements constitute a time series that is *unevenly* spaced (with a 2-, 2- and 3-day spacing), but *consistent*, in that this spacing pattern is repeated. Conventional second difference statistics, such as the Allan and modified Allan variances, are not applicable to such time series. A new second-difference statistic, designed for consistently but unevenly spaced time series, is described in Section 3. Secondly, it is well known that TWSTFT data can be subject to occasional phase offsets or steps resulting from replacement of the hardware underlying the collection of the data. In Section 4, a method is described for analyzing TWSTFT measurements to provide estimates of these steps, together with their associated uncertainties. A summary is given in Section 5.

2 ANALYSIS OF A UNIFORMLY SPACED TIME SERIES

Let $x(t)$ denote the phase difference, as a function of time t, between two clocks and $\{x_i: i = 1, \ldots, m\}$ a time series of measurements of x made at uniformly-spaced times $\{t_i: i = 1, \ldots, m\}$. Suppose $x(t)$ is modeled in terms of a phase offset p_0 and a normalized frequency offset f_0, i.e.,

$$x(t) = p_0 + f_0 t, \tag{1}$$

and the measurements as

$$x_i = p_0 + f_0 t_i + e_i, \tag{2}$$

where e_i denotes a random error. Using matrix notation,

$$\mathbf{x} = X\mathbf{p} + \mathbf{e}, \tag{3}$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}, \quad X = \begin{pmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \vdots \\ 1 & t_m \end{pmatrix}, \quad \mathbf{p} = \begin{pmatrix} p_0 \\ f_0 \end{pmatrix}, \quad \mathbf{e} = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{pmatrix}.$$

We suppose that the elements of **e** have zero expectation ($E(\mathbf{e}) = \mathbf{0}$) and a known covariance matrix ($V(\mathbf{e}) = V$) **[2]**. Typically, the elements e_i are a linear combination of (independent) samples of known noise types. For simplicity of presentation, we restrict the analysis to the well-known noise types of white phase modulation (WPM), white frequency modulation (WFM), and random-walk frequency modulation (RWFM). The extension of the work to other noise types, such as flicker phase modulation (FPM) and flicker frequency modulation (FFM), given a characterization of these noise types, is straightforward.

We express V in terms of known covariance matrices that relate to each noise type and parameters describing the "magnitude" of each noise type present. Each noise type is characterized by a transformation of a WPM process. The covariance matrices V for *standardized* WPM, WFM, and RWFM, for which the transformed WPM process has *unit* standard deviation, are given by, respectively,

$$V_{\mathrm{WPM}} = I, \quad V_{\mathrm{WFM}} = TT^{\mathrm{T}}, \quad V_{\mathrm{RWFM}} = \left(T^2\right)\left(T^2\right)^{\mathrm{T}},$$

where I is the identity matrix, and T defines the "summation operator"

$$\mathbf{T} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 1 & \cdots & 1 & 1 \end{pmatrix}$$

[3, Appendix A]. The covariance matrix $V(\mathbf{e})$ for noise arising as a linear combination of (independent) WPM, WFM, and RWFM is then expressed as

$$V(\mathbf{e}) = \sigma^2_{\mathrm{WPM}} V_{\mathrm{WPM}} + \sigma^2_{\mathrm{WFM}} V_{\mathrm{WFM}} + \sigma^2_{\mathrm{RWFM}} V_{\mathrm{RWFM}}, \tag{4}$$

where σ^2_{WPM}, σ^2_{WFM}, and σ^2_{RWFM} denote the variances of the WPM underlying each noise type.

Given the model (2) for the measurements, and assuming the covariance matrix $V(\mathbf{e})$ is also given, for example as defined by (4), a *leas-squares analysis* or *Gauss-Markov regression* [2] may be applied to obtain estimates of the phase offset p_0 and the normalized frequency offset f_0, together with their associated uncertainties. The estimates are *optimal* in the sense of being the estimates of minimum variance from the class of linear unbiased estimates of p_0 and f_0 [2].

The results of this analysis for the examples of (pure) WPM, WFM, and RWFM are given in Table 1. For ease of presentation of results, we suppose $t_i = i\tau_0$ with $i = -m, \ldots, +m$ for WPM and $i = 0, \ldots, m$ for WFM and RWFM. Other cases can also straightforwardly be handled. The results show that for WFM and RWFM, the estimate p_0 of the phase offset at $t = 0$ is the measurement x_0, whereas for WPM and data distributed symmetrically about $t = 0$, it is the mean of the measured phase offset values. Furthermore, for WFM and RWFM, the estimate f_0 of the normalized frequency offset is the slope of the chord joining the first and last measured values (for WFM) and the first and second values (for RWFM).

The law of propagation of uncertainty [4] is applied to the model defined by the Gauss-Markov regression and the statistical model (4) for the measurements to provide the standard uncertainties of the estimates p_0 and f_0 and their covariance. (For the cases considered in Table 1, the estimates p_0 and f_0 are uncorrelated for WPM and WFM, but they are correlated for RWFM.) The estimates of p_0 and f_0, together with the model (1), can be used to predict the phase offset x at any given time t. A further application of the law of propagation of uncertainty permits the standard uncertainty of any quantity derived from p_0 and f_0, such as a predicted phase offset, to be evaluated.

Noise type	Estimate of p_0	Standard uncertainty of p_0	Estimate of f_0	Standard uncertainty of f_0
WPM $\mathbf{x} = (x_{-m}, \ldots, x_m)^{\mathrm{T}}$	$\dfrac{\sum_{j=-m}^{m} x_j}{2m+1}$	$\dfrac{\sigma_{WPM}}{\sqrt{2m+1}}$	$\dfrac{\sum_{j=-m}^{m} j\tau_0 x_j}{\sum_{j=-m}^{m} j^2\tau_0^2}$	$\dfrac{\sigma_{WPM}}{\sqrt{\sum_{j=-m}^{m} j^2\tau_0^2}}$
WFM $\mathbf{x} = (x_0, \ldots, x_m)^{\mathrm{T}}$	x_0	σ_{WFM}	$\dfrac{x_m - x_0}{m\tau_0}$	$\dfrac{\sigma_{WFM}}{\tau_0\sqrt{m}}$
RWFM $\mathbf{x} = (x_0, \ldots, x_m)^{\mathrm{T}}$	x_0	σ_{RWFM}	$\dfrac{x_1 - x_0}{\tau_0}$	$\dfrac{\sigma_{RWFM}\sqrt{2}}{\tau_0}$

Table 1: Results of Gauss-Markov estimation for the examples of (pure) WPM, WFM and RWFM.

3 SECOND DIFFERENCE STATISTICS

For several years TWSTFT measurements made using Intelsat satellites between European primary timing laboratories, and between European and North American laboratories, have been available on Mondays, Wednesdays, and Fridays. The measurements constitute a *consistently* but *unevenly spaced* time series.

Time transfer measurements are usually characterized through the use of the well-known second difference statistics *AVAR*, *MVAR* and *TVAR* **[5]**. However, these statistics do not lend themselves directly to the characterization of unevenly spaced measurements. The characterization of unevenly spaced TWSTFT data has been considered previously, using techniques based on interpolating missing data followed by the application of second difference statistics to the resulting "reconstructed" time series **[3, 6]**. The approach considered here is to develop a second-difference statistic that may be used directly on unevenly spaced TWSTFT data, thus avoiding any dependence on the form of interpolation used.

3.1 Allan Variance for a Uniformly Spaced Time Series

In order to motivate the construction of a second-difference statistic for a nonuniformly spaced time series, we begin by reviewing the Allan variance for a uniformly spaced time series and, in particular, its definition in terms of a second-difference operator.

Consider the time series $\mathbf{x} = (x_1, \ldots, x_m)^{\mathrm{T}}$ with x_i denoting a measurement of $x(t)$ at $t_i = i\tau_0$, $i = 1, \ldots, m$. Here, $\tau = \tau_0$ denotes the spacing (in time) between the measurements $\mathbf{x}$ of phase difference. To determine the "single-spaced" ($\tau = \tau_0$) Allan variance statistic, we first form the time series of second differences $\mathbf{y} = (y_1, \ldots, y_{m-2})^{\mathrm{T}}$ of $\mathbf{x}$, viz.,

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{m-3} \\ y_{m-2} \end{pmatrix} = \begin{pmatrix} 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & -2 & 1 & \\ & & & & 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m-1} \\ x_m \end{pmatrix}.$$

In matrix form,

$$\mathbf{y} = A_1\mathbf{x},$$

where A_1 is a band diagonal matrix with band defined by

$$\mathbf{a}_1^{\mathrm{T}} = \begin{pmatrix} 1 & -2 & 1 \end{pmatrix}.$$

The time series $\mathbf{y}$ is *stationary* for the five well known noise types listed in Section 2 as well as linear frequency drift (the latter modeled by behavior of the form $f_D t^2$), and is independent of the phase and normalized frequency offsets p_0 and f_0 (but not f_D) in a model for the original time series $\mathbf{x}$. The Allan variance $AVAR(\tau_0)$ is then calculated from $\mathbf{y}$ using

$$AVAR(\tau_0) = \frac{1}{2\tau_0^2} E\left[y_i^2\right],$$

where $E[y_i^2]$ denotes the sample expectation (or arithmetic mean) of the elements y_i^2 derived from $\mathbf{y}$.

The Allan variance for a longer averaging time $\tau = n\tau_0$ may similarly be constructed using a second-difference operator $\mathbf{a}_n^{\mathrm{T}}$ that is a linear combination of the single space $\tau = \tau_0$ operator $\mathbf{a}_1^{\mathrm{T}}$ given above. For example, for the case $n = 2$:

$$\mathbf{a}_2^{\mathrm{T}} = \begin{pmatrix} 1 & 0 & -2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{pmatrix}.$$

Let A_2 be a band diagonal matrix with band defined by $\mathbf{a}_2^{\mathrm{T}}$, and

$$\mathbf{z} = A_2\mathbf{x}.$$

Because the second-difference operator $\mathbf{a}_2^{\mathrm{T}}$ is constructed as a linear combination of second-difference operators $\mathbf{a}_1^{\mathrm{T}}$, the time series $\mathbf{z}$ is also stationary for the five well known noise types and linear frequency drift, and is independent of both the phase and normalized frequency offsets for the time series $\mathbf{x}$. The Allan variance for the averaging time $\tau = 2\tau_0$ is given by:

$$AVAR(\tau) = \frac{1}{2\tau^2} E\left[z_i^2\right]$$

3.2 Allan Variance for a Nonuniformly Spaced Time Series

Let $(x_1, \ldots, x_6)^{\mathrm{T}}$ denote six consecutive (uniformly spaced) measurements of phase offset and consider the situation that the *only values* that are available are x_1, x_4, and x_6. Here, x_2, x_3 and x_5 denote "placeholders" for missing measurements. We can derive a second-difference operator $\mathbf{a}_{32}^{\mathrm{T}}$ (the "3" and "2" represent the 3- and 2-τ_0 spacing between the measurements x_1 and x_4, and x_4 and x_6, respectively) as a linear combination of single spaced $\tau = \tau_0$ second-difference operators $\mathbf{a}_1^{\mathrm{T}}$ that has zero second, third, and fifth elements, as follows:

$$\mathbf{a}_{32}^{\mathrm{T}} = \begin{pmatrix} \frac{4}{5} & 0 & 0 & -2 & 0 & \frac{6}{5} \end{pmatrix} = \begin{pmatrix} \frac{4}{5} & \frac{8}{5} & \frac{12}{5} & \frac{6}{5} \end{pmatrix} \begin{pmatrix} 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{pmatrix}.$$

The operator $\mathbf{a}_{32}^{\mathrm{T}}$ is determined uniquely except for a multiplying scale factor that is chosen so that the "central" coefficient is -2 (as for the uniformly spaced second-difference operator $\mathbf{a}_n^{\mathrm{T}}$).

Now suppose that $\mathbf{x}$ is a time series for which the only available measurements follow the above 3- and 2-τ_0 spacing, i.e., the available measurements are x_1, x_4, x_6, x_9, x_{11}, x_{14}, etc. Let A_{32} be the matrix

$$A_{32} = \begin{pmatrix} \mathbf{a}_{32}^{\mathrm{T}} & & & \\ & \mathbf{a}_{32}^{\mathrm{T}} & & \\ & & \ddots & \\ & & & \mathbf{a}_{32}^{\mathrm{T}} \end{pmatrix},$$

and

$$\mathbf{z} = A_{32}\mathbf{x}.$$

$\mathbf{z}$ is calculated *only* in terms of the available measurements in $\mathbf{x}$. In addition, being a linear combination of single spaced $\tau = \tau_0$ second-difference operators $\mathbf{a}_1^T$, $\mathbf{z}$ is stationary for the five well known noise types and linear frequency drift, and is independent of both the phase and normalized frequency offsets for the time series $\mathbf{x}$.

For this time series we can also construct a second-difference operator $\mathbf{a}_{23}^T$ that applies to measurements exhibiting a 2- and 3-τ_0 spacing, e.g., the measurements x_4, x_6, and x_9. Applying both $\mathbf{a}_{32}^T$ and $\mathbf{a}_{23}^T$ operators to the available measurements in $\mathbf{x}$ gives the more natural counterpart to the analysis for a uniformly spaced time series where there is "overlap" between successive second-difference operators.

An expression for a general spacing of available measurements is given as follows. Let $(x_1, \ldots, x_{1+p+q})^T$ denote measurements of phase offset for which the only values that are available are x_1, x_{1+p}, and x_{1+p+q}, i.e., the spacing between available measurements is $\tau_1 = p\tau_0$ and $\tau_2 = q\tau_0$. A second-difference operator $\mathbf{a}_{pq}^T$, that is a linear combination of single spaced $\tau = \tau_0$ second-difference operators $\mathbf{a}_1^T$ and operates only on the available measurements, is defined in terms of these measurements by:

$$\mathbf{a}_{pq}^T\mathbf{x} = \frac{2q}{p+q}x_1 - 2x_{1+p}\frac{2p}{p+q}x_{1+p+q} =: z.$$

Now suppose, for a given time series $\mathbf{x}$, that $\mathbf{z}$ contains all possible values z computed by applying $\mathbf{a}_{pq}^T$ and $\mathbf{a}_{qp}^T$ to $\mathbf{x}$, i.e, from available measurements in $\mathbf{x}$ separated by $\tau_1 = p\tau_0$ and $\tau_2 = q\tau_0$. Then we generalize the definition of the Allan variance for a uniformly spaced time series to be applicable to a nonuniform spacing as follows:

$$GAVAR(\tau_1, \tau_2) = \frac{1}{2\tau^2}E\left[z_i^2\right], \quad \tau = \frac{\tau_0(p+q)}{2}.$$

In terms of its dependence on τ_1 and τ_1, this generalized Allan variance $GAVAR(\tau_1,\tau_2)$ has the following properties:

a) If $\tau_1 = \tau_2 = \tau$, then $GAVAR(\tau_1, \tau_2)$ is identical to $AVAR(\tau)$ with an averaging time τ.

b) The underlying time series $\mathbf{z}$ on which the $GAVAR(\tau_1, \tau_2)$ variance is based is stationary for the five well-known noise types and for linear frequency drift.

c) $GAVAR(\tau_1, \tau_2)$ is independent of the phase and normalized frequency offsets for the time series $\mathbf{x}$.

Figure 1 compares values of $GAVAR(\tau_1, \tau_2)$ with those of $AVAR(\tau)$ for WPM, WFM, RWFM, and linear frequency drift (LFD). For various choices of p and q the ratio of $GAVAR(p\tau_0, q\tau_0)$ to $AVAR(\tau)$, with $\tau = \tau_0(p + q)/2$, is calculated and the figure displays values of this ratio as a function of $r = p/(p + q)$. As expected, when $p = q$ ($r = ½$), the ratio is unity, showing that the two measures are identical. Furthermore, for p close to q, the ratio remains close to unity. However, as r departs from ½ the ratio increases for WPM and decreases for WFM, RWFM, and LFD. The curves for RWFM and LFD are very similar.

3.3 ALLAN VARIANCE FOR TWSTFT

A time series of TWSTFT measurements consists of measurements made on the Monday, Wednesday, and Friday of each week. For this spacing of measurements, we can apply the second-difference operators $\mathbf{a}_{22}^{\mathrm{T}}$ (= $\mathbf{a}_2^{\mathrm{T}}$) to the measurements on Monday, Wednesday, and Friday of each week, $\mathbf{a}_{23}^{\mathrm{T}}$ to the measurements on Wednesday, Friday, and Monday, and $\mathbf{a}_{32}^{\mathrm{T}}$ to the measurements on Friday, Monday, and Wednesday to obtain a vector $\mathbf{z}$. An Allan variance is then calculated in terms of the elements of $\mathbf{z}$.

3.4 MODIFIED ALLAN VARIANCE

In many situations, particularly in the presence of WPM, the use of a modified Allan variance is preferred to the Allan variance for the characterization of TWSTFT and other time transfer measurements. The reason is that it is not possible to use the Allan variance to distinguish between the noise types of WPM and WFM [7].

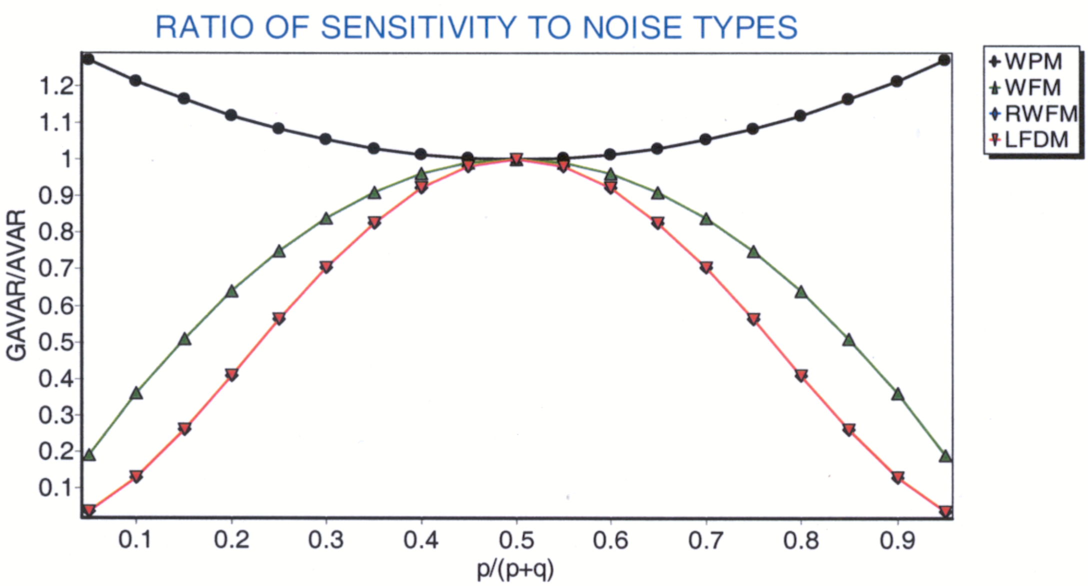

Figure 1: Relative sensitivity of GAVAR and AVAR to the five standard noise types and to linear frequency drift.

The second-difference operator used as the basis of the calculation of a modified Allan variance is constructed as an "average" of Allan variance second-difference operators. For example, for a uniformly spaced time series and averaging time $\tau = 2\tau_0$, we form

$$\mathbf{m}_2^{\mathrm{T}} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -1 & -1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 \end{pmatrix}.$$

Let M_2 be a band diagonal matrix with band defined by $\mathbf{m}_2^{\mathrm{T}}$, and

$$\mathbf{w} = M_2 \mathbf{x}.$$

Then, the modified Allan variance $MVAR(\tau)$ is defined by:

$$MVAR(\tau) = \frac{1}{2\tau^2} E\left[w_i^2\right].$$

In a similar way, a modified Allan variance may be defined for a nonuniformly spaced time series. Consider the case of TWSTFT measurements made on the Wednesday (W), Friday (F), and Monday (M) of two consecutive weeks. Two Allan variance second-difference operators for the available TWSTFT measurements are as follows:

$$\begin{pmatrix} \mathrm{W} & & \mathrm{F} & & & \mathrm{M} & & \mathrm{W} & & \mathrm{F} & & & \mathrm{M} \\ \frac{8}{9} & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & \frac{10}{9} & 0 & 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} \mathrm{W} & & \mathrm{F} & & & \mathrm{M} & & \mathrm{W} & & \mathrm{F} & & & \mathrm{M} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The first of these is for the spacing $p = 5$ and $q = 4$. The second is for the spacing $p = q = 5$ and is identical to a second-difference operator for a uniformly spaced time series with $n = 5$. The second-difference operator used as the basis of the calculation of a modified Allan variance for the TWSTFT measurement is then given by averaging these operators, viz.,

$$\begin{aligned} &\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -\frac{8}{9} & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & -\frac{10}{9} & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \\ &\quad = \begin{pmatrix} -\frac{4}{9} & 0 & -\frac{1}{2} & 0 & 0 & 1 & 0 & 1 & 0 & -\frac{5}{9} & 0 & 0 & -\frac{1}{2} \end{pmatrix} \end{aligned}$$

Properties of a modified Allan variance for a nonuniformly spaced time series calculated in terms of such second-difference operators are under investigation.

4 ESTIMATING STEPS IN TWSTFT TIME SERIES

Delay steps of unknown magnitude may occasionally be observed in TWSTFT time series. These are usually due to the replacement of a failed component within the TWSTFT instrumentation. In some cases it is possible independently to determine the magnitude of a delay step, but usually, for example when the step is the result of changing a satellite transponder, this is not possible. We examine here a method to determine an estimate of the delay step from the time series of TWSTFT measurements.

Figure 2 shows a time series of (UTC(NPL) – UTC(USNO)) TWSTFT measurements. Three delay steps of unknown magnitude are observed, at MJD 51968, MJD 51984, and MJD 52032. The first two of these steps are due to unknown transponder delay changes, while the third is due to a modem replacement.

4.1 Analysis Method

Let $\mathbf{x} = (x_1, \ldots, x_m)^{\mathrm{T}}$ be a time series of TWSTFT measurements corresponding to (not necessarily uniformly spaced) times $\mathbf{t} = (t_1, \ldots, t_m)^{\mathrm{T}}$. Suppose delay steps in the time series occur at the three known times T_1, T_2, and T_3. We model these measurements in terms of an nth order polynomial $p(t, \mathbf{c})$ with coefficients $\mathbf{c} = (c_1, \ldots, c_n)^{\mathrm{T}}$ and parameters $\mathbf{d} = (d_1, d_2, d_3)^{\mathrm{T}}$ representing the (unknown) delay steps, as follows:

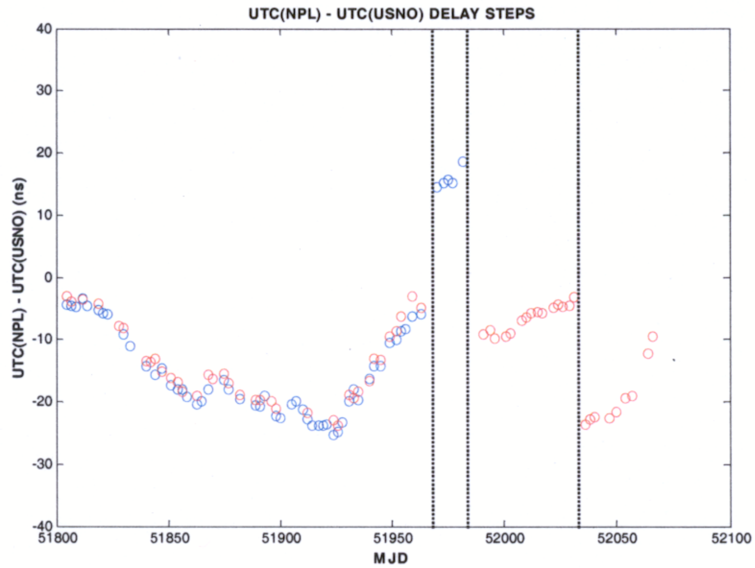

Figure 2: TWSTFT measurements showing delay steps on MJD 51968, 51984, and 52032.

$$
\begin{aligned}
x_i &= p(t_i, \mathbf{c}) + e_i, & t_i &< T_1, \\
x_i &= p(t_i, \mathbf{c}) + d_1 + e_i, & T_1 &\le t_i < T_2, \\
x_i &= p(t_i, \mathbf{c}) + d_1 + d_2 + e_i, & T_2 &\le t_i < T_3, \\
x_i &= p(t_i, \mathbf{c}) + d_1 + d_2 + d_3 + e_i, & T_3 &\le t_i.
\end{aligned} \tag{5}
$$

If there were additional delay steps, the number of parameters $\mathbf{d}$ and the model (5) would be modified in a natural way. The measurement errors e_i, $i = 1, \ldots, m$, are assumed to be samples of WPM (although other noise types and combinations of noise types may be considered: see Section 2). In this case, estimates of the parameters $\mathbf{c}$ and $\mathbf{d}$ are obtained by solving the linear least-squares problem:

$$
\min_{\mathbf{c},\mathbf{d}} \sum_{i=1}^{m} e_i^2 .
$$

Equivalently, the problem is to find the least-squares solution $\mathbf{b} = (\mathbf{c}^{\mathrm{T}}, \mathbf{d}^{\mathrm{T}})^{\mathrm{T}}$ to the linear system of equations

$$
\mathbf{x} = B\mathbf{b} + \mathbf{e} \tag{6}
$$

(cf. equation (3)), which is overdetermined provided m (the number of measurements) exceeds $n + 3$ (the number of polynomial coefficients plus the number of delay step parameters). Here, the ith row of the matrix B contains in its first n columns the values at t_i of basis functions $\phi_j(t)$ used in the representation of the polynomial $p(t, \mathbf{c})$, viz.,

$$p(t,\mathbf{c}) = \sum_{j=1}^{n} c_j \phi_j(t),$$

and in each of its last three columns the value zero or one according to whether the offset d_j, $j = 1, 2, 3$, appears in the (corresponding) ith model equation (5). One choice for the basis functions $\phi_j(t)$ are the monomials x^{j-1}, although it is preferable to use alternative functions that ensure the reliability of the numerical computations performed (see below).

Formally, the solution to this problem is given by [2]

$$\mathbf{b} = \left(B^T B\right)^{-1} B^T \mathbf{x},$$

with covariance matrix

$$V(\mathbf{b}) = s^2 \left(B^T B\right)^{-1},$$

where s^2, the root-mean-square (RMS) residual error

$$s^2 = \frac{1}{m-(n+3)} \sum_{i=1}^{m} e_i^2$$

evaluated at the solution, estimates the variance of the WPM noise process. In practice, to ensure reliability of the computed results, the polynomial $p(t, \mathbf{c})$ is expressed in terms of Chebyshev polynomial basis functions in a normalized variable [8], and the least-squares problem defined by (6) is solved using matrix factorization methods [9].

Using a 12th order (degree 11) polynomial to represent the data shown in Figure 2, the following results are obtained:

d_1 = 14.8 ns, with standard uncertainty 1.3 ns,

d_2 = −28.5 ns, with standard uncertainty 1.6 ns, and

d_3 = −20.1 ns, with standard uncertainty 1.5 ns.

In Figure 3 we show the time series of measurements together with the fitted polynomial curve following the removal of the unknown delay steps.

Although polynomials can be effective for modeling time series of TWSTFT measurements, they will not always be appropriate. The use of polynomial splines [10] provides a more general capability.

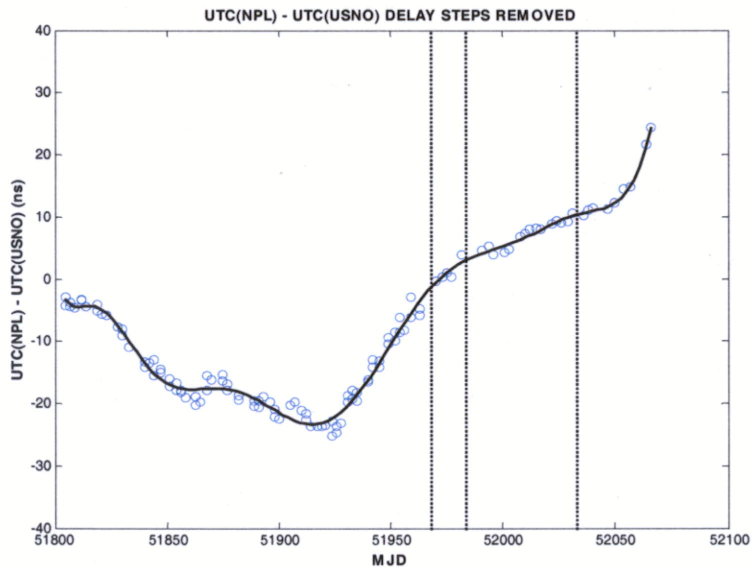

Figure 3: TWSTFT measurements corrected for unknown delay steps.

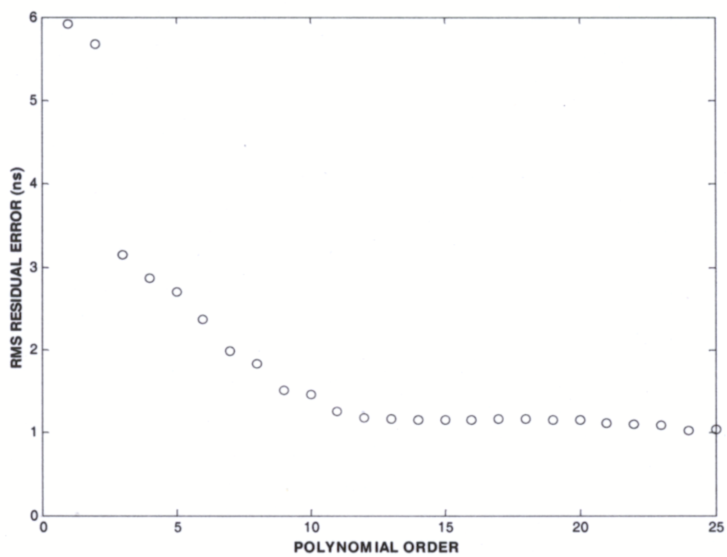

Figure 4: RMS residual error as a function of the polynomial order.

4.2 Choosing the Order of the Polynomial

An important consideration for this analysis method is the choice of polynomial order. Figure 4 shows, for the example data of Figure 2, the RMS residual error s^2 as a function of polynomial order n. The value chosen for n is the smallest value for which the RMS residual error remains essentially constant. For the example data considered here, n is chosen to be 12, for which the corresponding RMS residual error is 1.2 ns.

SUMMARY

In this paper we have considered the least-squares analysis of measurements of the phase difference between two clocks or time scales, with the aim of determining estimates at any epoch of the phase offset between the clocks or time scales. We have focused on presenting approaches developed to overcome present operational constraints of TWSTFT, i.e., the consistently but unevenly spaced nature of the time series of measurements, as well as the presence of occasional delay steps of unknown magnitude. To characterize the TWSTFT measurements, we have used second difference statistics that are calculated only in terms of the available measurements. In order to remove the unknown delay steps, we have modeled the measurements using empirical functions together with parameters representing the delay steps.

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