ESTIMATING FREQUENCY STABILITY AND CROSS-CORRELATIONS

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Abstract

We present a method for estimating the absolute frequency stability of N clocks separate from a reference. The method introduced is a modification of the one proposed by Tavella and Premoli (1993). After developing the theory we apply the method to atomic clock data gathered from the USNO.

INTRODUCTION

The estimation of absolute frequency stability of clocks separate from a reference clock is usually required in order to produce a mean time scale. The usual method which can be employed for this task is the so-called "N-cornered-hat" method. As first presented by Gray and Allan [1], this method estimates the stability of a fixed clock among an ensemble of N clocks by forming all $(N-1)(N-2)/2$ triads of time differences which include those of the clock under test. The reason for triads is to avoid overdetermination in the estimation problem. From each of these triads one obtains stability estimates for each of the clocks in the triad under the assumption that clocks are uncorrelated – these are the "three-cornered-hat" estimates. These estimates are then weighted by a "triad uncertainty" to yield a stability estimate for each clock. Another version, developed by Barnes [2], simultaneously uses all $N-1$ time differences with a least-squares-type approach to estimate the individual stabilities. These approaches sometimes lead to stability estimates that are negative. Some attribute the negative estimate to its uncertainty [3], while others suggest the possibility that the assumption of uncorrelation is not always valid [4]; both are probably true to a significant extent.

Recently, Tavella and Premoli [4] developed an approach which avoids the problem of negative stability estimates by allowing the possibility of correlations among the ensemble of clocks. Their approach is consistent and formally equivalent to the three-cornered-hat method when clocks are uncorrelated, that is, both methods produce the same result in this situation. Their method, however, will not produce a negative variance estimate even when a three-cornered-hat approach would. Since the problem of estimating absolute frequency stability is underdetermined, they impose the condition that if correlations exist they should be "small." They propose a condition which amounts to finding the smallest correlations possible which keep a certain matrix positive-definite. This will be explained in greater detail below. Since Tavella and Premoli's work [4] was done in the framework of three clocks, a weighted triad uncertainty approach can be used to improve the estimates if there are more than three clocks in the ensemble.

In a later paper [5] Tavella and Premoli refined their analysis. They show that when the number of clocks in a comparison increases, the amount of arbitariness in determining absolute frequency stability decreases.

The goal of this paper is to generalize the methods of [4] and [5] and to estimate absolute frequency stability for N clocks ($N > 3$). The results of this analysis will be discussed and applied to atomic clock time difference data from the U. S. Naval Observatory (USNO).

NOTATIONS AND DEFINITIONS

Let $x^i = \{x^i_k : k \geq 1\}$ for $i = 1, \cdots, N-1$ be the time differences of clock i with respect to a fixed reference clock, say clock N, which is sampled every τ_0 seconds. We let $X^j = \{X^j_k : k \geq 1\}$ for $j = 1, \cdots, N$ represent the time differences with respect to a noiseless "ideal" clock, so that $x^i = X^i - X^N$.

Define the fractional frequencies:

$$y^i_k = \frac{x^i_{k+1} - x^i_k}{\tau_0}, \; k \geq 1 \tag{1}$$

and the process of averaging such fractional frequencies of order $m \geq 1$:

$$\bar{y}^i_k(m\tau_0) = \frac{y^i_{(k-1)m+1} + y^i_{(k-1)m+2} + \cdots + y^i_{km}}{m} = \frac{1}{m}\sum_{l=1}^{m} y^i_{(k-1)m+l}. \tag{2}$$

Notice that (2) reduces to (1) for $m = 1$. We define Y^i_k and $\bar{Y}^i_k(m\tau_0)$ by replacing x with X appropriately. As a measure of time domain frequency stability we use the Allan variance. The Allan variance of clock i referenced to clock N is defined as

$$s_{ii}(\tau) = \frac{1}{2}\langle(\bar{y}^i_2(\tau) - \bar{y}^i_1(\tau))^2\rangle \tag{3}$$

where $\langle\rangle$ denotes mathematical expectation. We assume here, of course, that the process of averaged fractional frequencies is stationary and ergodic so that the definition (3) is well-defined. Most authors use the notation $\sigma^2_{i,N}(\tau)$ to represent the quantity in (3), but we will use the notation set forth in the works [4] and [5]. For what follows we need to define the Allan covariance of clocks i and j referenced to clock N to be

$$s_{ij}(\tau) = \frac{1}{2}\langle(\bar{y}^i_2(\tau) - \bar{y}^i_1(\tau))(\bar{y}^j_2(\tau) - \bar{y}^j_1(\tau))\rangle. \tag{4}$$

This statistic appears in [6] with different notation. It is worth mentioning at this point that a covariance reduces to a variance for $i = j$; therefore we will refer to variances as covariances when no confusion can arise. Furthermore, it is clear from the definition (4) that $s_{ij}(\tau) = s_{ji}(\tau)$. However, all these quantities are usually never available in practice and can only be estimated through a large stretch of data (at least for $\tau = m\tau_0$ and integer $m \geq 1$) by the samples

$$\hat{s}_{ij} = \frac{1}{2(n-2m)} \sum_{k=1}^{n-2m} (\bar{y}^i_{k+1}(m\tau_0) - \bar{y}^i_k(m\tau_0))(\bar{y}^j_{k+1}(m\tau_0) - \bar{y}^j_k(m\tau_0)) \tag{5}$$

where n is the data length. The goal of this paper is estimate the dereferenced quantities:

$$r_{ii} = \frac{1}{2}\langle(\bar{Y}^i_2(\tau) - \bar{Y}^i_1(\tau))^2\rangle \tag{6}$$

and

$$r_{ij} = \frac{1}{2}\langle(\bar{Y}^i_2(\tau) - \bar{Y}^i_1(\tau))(\bar{Y}^j_2(\tau) - \bar{Y}^j_1(\tau))\rangle. \tag{7}$$

Again, some authors use the notation $\sigma^2_i(\tau)$ to represent the quantity shown in (6). The associated sample quantity is defined similar to (4):

$$\hat{r}_{ij} = \frac{1}{n-2m} \sum_{k=1}^{n-2m} (\bar{Y}^i_{k+1}(m\tau_0) - \bar{Y}^i_k(m\tau_0))(\bar{Y}^j_{k+1}(m\tau_0) - \bar{Y}^j_k(m\tau_0)) \tag{8}$$

By noting that $y^i_k = Y^i_k - Y^N_k$ we deduce from (4)-(8) the following relations:

$$s_{ij} = r_{ij} + r_{NN} - r_{iN} - r_{jN} \tag{9}$$

and

$$\hat{s}_{ij} = \hat{r}_{ij} + \hat{r}_{NN} - \hat{r}_{iN} - \hat{r}_{jN}. \tag{10}$$

If $N = 3$ and all $\hat{r}_{ij} = 0$ for $i \neq j$ then the usual three-cornered-hat estimates fall out of (10):

$$\begin{aligned} \hat{r}_{11} &= \hat{s}_{11} - \hat{s}_{12} \\ \hat{r}_{22} &= \hat{s}_{22} - \hat{s}_{12} \; . \\ \hat{r}_{33} &= \hat{s}_{12} \end{aligned} \tag{11}$$

This was noted in references [4] and [6].

THE TAVELLA-PREMOLI APPROACH

Let's first outline the approach followed by Tavella and Premoli [4]. Suppose we are interested in obtaining estimates of absolute frequency stabilities for three clocks, say clocks 1, 2, and 3. That is, we wish to estimate the matrix

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{12} & r_{22} & r_{23} \\ r_{13} & r_{23} & r_{33} \end{bmatrix}. \tag{12}$$

Notice the matrix R is inherently symmetric. Now consider the matrix

$$S = \begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix}.$$

S is symmetric and we know a priori S is also positive-definite. The relation (9) can be rewritten to give the relations:

$$\begin{aligned} r_{11} &= s_{11} - r_{33} + 2r_{13} \\ r_{22} &= s_{22} - r_{33} + 2r_{23} \\ r_{12} &= s_{12} - r_{33} + r_{13} + r_{23}. \end{aligned}$$

This means that all the entries of R can be written as functions of the elements $\mathbf{r} = (r_{13},\ r_{23})$ and the reference stability r_{33}. To find estimates for the entries of R, Tavella and Premoli proposed the following. We should find values of covariances r_{12}, r_{13} and r_{23}, which are small in some sense, but keep the matrix R positive-definite. To understand in what sense they mean small define

$$G(r_{12}, r_{13}, r_{23}) = r_{12}^2 + r_{13}^2 + r_{23}^2. \tag{13}$$

Clearly, the minimum value of G is 0 exactly when $r_{13} = r_{23} = r_{12} = 0$, and, given (9), this implies $r_{33} = s_{12}$. That is, we arrive at the three-cornered-hat estimates (11):

$$\hat{R}_{3\text{-corner}} = \begin{bmatrix} s_{11} - s_{12} & 0 & 0 \\ 0 & s_{22} - s_{12} & 0 \\ 0 & 0 & s_{12} \end{bmatrix}. \tag{14}$$

As long as

$$\begin{aligned} s_{11} &> s_{12} \\ s_{22} &> s_{12} \\ s_{12} &> 0 \end{aligned} \tag{15}$$

we have valid estimates of stability. However, if one of the diagonal elements in (14) is negative, we obtain a negative stability estimate. This can be ruled out if we insist that the estimated matrix $\hat{R}$ be positive-definite. Tavella and Premoli are able to show that the function

$$H(r_{13}, r_{23}, r_{33}) = r_{33} - (r_{13} - r_{33}, r_{23} - r_{33}) S^{-1} (r_{13} - r_{33}, r_{23} - r_{33})^T \tag{16}$$

is positive if and only if R is positive-definite. T denotes transpose here. Therefore they suggest the r_{ij} should be chosen in a way that minimizes the following expression:

$$\frac{r_{12}^2 + r_{13}^2 + r_{23}^2}{H(r_{13}, r_{23}, r_{33})} \tag{17}$$

where the minimum is taken over all values r_{13}, r_{23}, r_{33} for which H is positive (i.e., R is positive-definite). Notice that if $H(0, 0, s_{12}) > 0$ then the minimum of (17) is 0 and we again achieve the three-cornered-hat estimates. This procedure, however, will not lead to a negative stability estimate. To see this just notice that the minimization occurs over a convex region, namely the elliptic paraboloid region

$$\mathsf{P} = \{(x, y, z) : H(x, y, z) > 0\}.$$

The boundary of P, $\partial\mathsf{P}$, contains all the points where the function H identically equals 0. Since the objective function (17) is convex, it is well known that the minimizer of this problem is unique. The minimizer $(r_{13}^*, r_{23}^*, r_{33}^*)$ of (17) will always satisfy $H(r_{13}^*, r_{23}^*, r_{33}^*) > 0$ since points on the boundary of P yield undefined values in (17). Now $H > 0$ implies R is positive-definite. The result follows from the fact that positive-definite matrices have positive diagonal elements. The minimization of (17) was carried out explicitly for $N = 3$ clocks [4] and its solution was formulated through the zeroes of a sixth-degree polynomial.

The implementation of this method does show some subtle inadequacies however. As noted in the introduction, a three-cornered-hat approach may give a negative stability estimate. Although the Tavella-Premoli approach will not produce such a negative value, it may produce an optimistically small estimate of stability. Let's see why this may happen. It is easy to show that if the conditions (15) are all satisfied then $(0, 0, \hat{s}_{12}) \in \mathsf{P}$. Therefore, if $\hat{s}_{12}$ is arbitrarily small and positive, our estimate for r_{33} will also be arbitrarily small. Similarly, if $\hat{s}_{11} < \hat{s}_{22}$ and $\hat{s}_{12}$ is close to $\hat{s}_{11}$, our estimate for r_{11} may also be optimistically small. These effects have appeared in time difference data gathered at the USNO.

This optimism can be attributed to the rather large domain of admissible values in (17). The admissible region is an elliptic paraboloid in $\Re^3$. If more clocks are in our comparison then, as shown in [5], this admissible region is substantially reduced and points that would be admissible when considered through triads would be disallowed when considered in a multiple comparison. We can use this substantial reduction in admissible values to generalize the Tavella-Premoli scheme. We will take this up next.

A MODIFIED APPROACH

Suppose we have time differences from $N - 1$ clocks with a fixed reference clock N ($N > 3$) and we wish to estimate the stabilities. We could, of course, apply the method of Tavella and Premoli [4] to triads of clocks and weight appropriately. As noted earlier, this approach may produce optimistic values of stability.

Let's consider the following approach. In analogy to the method proposed by [4], we suggest an obvious modification to (17) and find the values of covariances that minimize the following Tavella-Premoli function:

$$\frac{\sum_{i<j} r_{ij}^2}{H^2(r_{1N}, \cdots, r_{NN})} \tag{18}$$

where

$$H(r_{1N}, \cdots, r_{NN}) = r_{NN} - (r_{1N} - r_{NN}, \cdots, r_{1N} - r_{NN}) S^{-1} (r_{1N} - r_{NN}, \cdots, r_{1N} - r_{NN})^T$$

the superscript T denotes transpose and the minimization is over those points r_{iN} which keep H positive. Here, the function $H > 0$ if and only if the matrix R is positive-definite [5]. Dividing by H above as mentioned earlier keeps the points r_{iN} away from the boundary of the admissible region and, consequently, will keep the matrix R positive-definite. Notice the function (18) is now a function of the N variables $(r_{1N}, \cdots, r_{NN})$ as well as, of course, the s_{ij}. Also notice that function (18) has H squared in the denominator. This squaring is suggested in order to keep the minimization problem scale-invariant. This will also be helpful in order to do the minimization from a numerical point of view. The problem of minimizing (18) is a constrained minimization problem since we are only allowing values of r_{ij} which keep the matrix R positive-definite, that is, those r_{ij} for which $H(r_{1N}, \cdots, r_{NN}) > 0$. From the same convexity considerations the minimization problem (18) has a unique solution. Usually constrained minimization problems are difficult to solve, but one can apply numerical techniques if care is taken in scaling and choice of initial data. We used a conjugate gradient method to produce the minimizer. As far as the choice of initial data we chose

$$\begin{aligned} r_{iN} &= 0 \text{ for } i < N, \\ r_{NN} &= \frac{1}{2} \cdot \frac{1}{s^*} \end{aligned}$$

$s^* \equiv (1, \cdots, 1)S^{-1}(1, \cdots, 1)^T > 0$ from the positive-definiteness of the matrix S and thus S^{-1}. The factor $\frac{1}{2}$ above is used to force the initial data to lie within the constraint (using convexity of admissible values here). This choice of initial data conforms with our belief that clocks are close to uncorrelated. Of course, from the uniqueness of the solution any reasonable point we choose initially will converge to the minimizer and this, in fact, is observed. We should mention that problems can arise if the initial datum is too close to the boundary of the admissible region. If this is the case a conjugate gradient direction may lead you outside the admissible region.

It is interesting to note that if clocks are uncorrelated ($r_{ij} = 0$ for all $i \neq j$) the minimization of (18) leads to the estimate

$$\hat{r}_{NN} = \frac{2\sum_{i<j} \hat{s}_{ij}}{(N-1)(N-2)},$$

that is, the straight average of the values that one would obtain by performing the usual Tavella-Premoli procedure on all triads of clocks.

One implicit point we have stressed in this paper is the possibility of the existence of clock noise correlations. In order to convince the reader that clock correlations are a reality we have developed a preliminary statistical test which, although heuristic, may be used to determine if clock noise correlations exist between clocks in the comparison.

A STATISTICAL TEST OF CORRELATION

Tavella and Premoli have shown [5] that if $r_{ij} = 0$ for all $i \neq j$ then the matrix S of "true" Allan covariances has the form

$$S = \begin{bmatrix} r_{11} - r_{NN} & r_{NN} & r_{NN} & \cdots & r_{NN} \\ r_{NN} & r_{22} - r_{NN} & r_{NN} & \cdots & r_{NN} \\ r_{NN} & r_{NN} & r_{33} - r_{NN} & \cdots & r_{NN} \\ \vdots & \vdots & \vdots & & \vdots \\ r_{NN} & r_{NN} & r_{NN} & \cdots & r_{N-1,N-1} - r_{NN} \end{bmatrix} \tag{19}$$

As mentioned in the previous section, the matrix S can only be estimated by the matrix $\hat{S} = (\hat{s}_{ij})$. Thus, if the clocks are uncorrelated then the $\hat{s}_{ij}$ for $i \neq j$ are all estimating the reference stability r_{NN}. If the data length n is large, one should expect $\hat{r}_{ij} \approx 0$ for $i \neq j$. If this is the case then from (10) and (19) the statistic $\hat{s}_{ij} \approx \hat{r}_{NN}$ for $i \neq j$. Now since the distribution of $\hat{r}_{NN}$ can be shown to behave as a chi-square with some number of degrees of freedom (we consider here overlapping sample estimates), which in general varies with the noise type present [3], we can test if the off-diagonal terms of $\hat{S}$ actually differ up to some statistical significance. The problem can be formulated as follows:

Consider as the null hypothesis all clocks are uncorrelated. We would like to test whether the alternative hypothesis that some pair of clocks exhibit correlations is true. So we would like to test for the simultaneous equality of the off-diagonal elements of the matrix $\hat{S}$. This is a multiple comparison problem whose analysis can be quite difficult. We instead will be content to work with pair-wise comparisons of the off-diagonal elements $\hat{s}_{ij}$. For this problem we test whether the ratio

$$f = \frac{\hat{s}_{ij}}{\hat{s}_{i'j'}}$$

for different pairs of clocks $i \neq j$ and $i' \neq j'$ is significantly different from 1. Now since the chi-squares all have the same degrees of freedom, d, for a fixed integration time τ, the test statistic f has the $F_{d,d}$-distribution, that is, the ratio of two chi-squares with d degrees of freedom each. Assume that for fixed $\tau > 0$ all the off-diagonal terms of $\hat{S}$ are positive. For 90% confidence we choose to reject the null in favor of the alternative if either

$$f > F_{d,d,.95} \quad \text{or} \quad f < F_{d,d,.05}$$

where $F_{d,d,\alpha}$ represents the α^{th} percentile of the F distribution. Since all estimates $\hat{s}_{ij}$ share the same degrees of freedom, the problem is simplified considerably since we only need to check if the statistic

$$f^* = \max \frac{\hat{s}_{ij}}{\hat{s}_{i'j'}}$$

satisfies $f^* > F_{d,d,.95}$. The above argument is only heuristic and the resulting statistical test is not entirely substantiated since significant accidental covariances for increasingly smaller sample sizes can corrupt the distributional properties of the f-statistic, that is, if $n - 2m$ is not "large" then the values $\hat{r}_{kl}$ for $k \neq l$ in (10) may not be close to zero and can bias the $\hat{s}_{ij}$-statistic away from $\hat{r}_{NN}$. However, at least for very large sample sizes n and relatively small m, the above approximation seems reasonable. Of course, a better statistical test can be achieved if one can characterize the distribution of the Allan covariance statistics.

We applied the above analysis to a group of four cesium-beam frequency standards rereferenced to a fifth cesium-beam standard (see Table 1). The time differences had data length $n = 167,513$. We made the assumption that for the integration times observed the dominant noise type was white frequency modulation and computed the appropriate degrees of freedom for the overlapping estimates:

$$d = \left[\frac{3(n-1)}{2m} - \frac{2(n-2)}{n}\right]\frac{4m^2}{4m^2+5}.$$

We found the corresponding F quantile and applied the above arguments. We noticed that although the off-diagonal entries of $\hat{S}$ may seem to be the same, at least for short integration times (from

20 seconds out to about 320 seconds) we were able to reject the null hypothesis and conclude that statistically significant correlations exist. This does not say that a correlation is "large"; indeed, $\hat{r}_{ij}$ can be relatively quite small (by orders of magnitude) when compared to the diagonal entries $\hat{r}_{ii}$ and $\hat{r}_{jj}$. However, some small covariance seems quite reasonable when we consider that some of the clocks in the above analysis were in the same environmental chamber. An exhaustive treatment of the above ideas would certainly be interesting from both a practical and theoretical point of view.

$$\hat{S}(20) = \begin{bmatrix} 7.10826E-24 & 3.81328E-24 & 3.79768E-24 & 3.79259E-24 \\ 3.81328E-24 & 7.95851E-24 & 3.82652E-24 & 3.83888E-24 \\ 3.79768E-24 & 3.82652E-24 & 7.89671E-24 & 3.82095E-24 \\ 3.79259E-24 & 3.83888E-24 & 3.82095E-24 & 6.99711E-24 \end{bmatrix}$$

$$\hat{S}(40) = \begin{bmatrix} 3.22437E-24 & 1.69300E-24 & 1.69913E-24 & 1.69097E-24 \\ 1.69300E-24 & 3.42756E-24 & 1.69388E-24 & 1.70435E-24 \\ 1.69913E-24 & 1.69388E-24 & 3.58596E-24 & 1.71195E-24 \\ 1.69097E-24 & 1.70435E-24 & 1.71195E-24 & 3.15194E-24 \end{bmatrix}$$

$$\hat{S}(320) = \begin{bmatrix} 3.55895E-25 & 1.82808E-25 & 1.85889E-25 & 1.84763E-25 \\ 1.82808E-25 & 3.65834E-25 & 1.89250E-25 & 1.85393E-25 \\ 1.85889E-25 & 1.89250E-25 & 4.04481E-25 & 1.89242E-25 \\ 1.84763E-25 & 1.85393E-25 & 1.89242E-25 & 3.55988E-25 \end{bmatrix}$$

Table 1.

Averaging Time	F-statistic	f^*-statistic	
20 sec.	1.009893	1.012205	$f^* > F$: can reject in favor of alternative.
40 sec.	1.010690	1.012407	$f^* > F$: can reject in favor of alternative.
320 sec.	1.026667	1.035239	$f^* > F$: can reject in favor of alternative.

The $\hat{S}$ matrices above were computed for τ =20, 40 and 320 seconds. Table 1 shows the results of the statistical test for these integration times.

RESULTS

We have applied the technique described in the previous section to time difference data obtained from atomic clocks at the U.S. Naval Observatory (USNO). USNO has a large ensemble of clocks consisting of both commercial cesium-beam standards and active hydrogen masers. Data from both types of clocks were considered separately.

Figures 1a and 1b show sigma-tau plots generated by a three-cornered-hat and Tavella-Premoli analysis respectively. For comparison, both analyses were carried out on cesium-beam standards labelled 229, 260, and 253 over a period of five years. The three time series were rereferenced to one of the clocks (253) to avoid the obvious problem of correlation between the time scales. We used overlapping estimates of Allan covariance over all τ, instead of just $\tau = 2^m \tau_0$ (integer $m \geq 1$) where τ_0 is the sampling time.

There are several things to notice about these plots. First, both approaches coincide where the three-cornered-hat approach produces positive stability estimates, so that the primary difference between the two is that the Tavella-Premoli scheme is able to produce an "extension" of the reference stability. Second, by plotting the data for all τ, a "pathological" behavior is apparent in both approaches. There are two "dips" in each graph: one at 60 days and the other at about 590 days. Neither of these dips are likely to be physical, since cesium-beam standards integrate down with a $1/\sqrt{\tau}$ dependence until they reach their flicker floor or begin to drift significantly. One approach to this situation might be to add another clock to the ensemble and use the additional three triads. However, even if a straight average of the resulting estimates coming from each triad were used, one would still find that the dip creates a large bias in the result.

Instead, the extra clock can be used with the modified approach outlined in the previous section. The corresponding frequency stability plot is shown in Figure 2. This approach is in close agreement with the other two until $\tau \approx 30$ days. At this point, the first spike apparent in the other approaches is smoothed out. Most importantly, the new approach now maintains the expected $1/\sqrt{\tau}$ behavior at this point in the data. The second spike is smoothed out as well.

We performed this modified approach on the data with two more clocks added in. The results are shown in Figures 3a and 3b. Again the absence of non-physical dips is apparent and agreement between this and results shown in Figure 2 is good. In all plots we noticed that one of the cesium clocks appeared to be much worse than the others. Later we were able to show that this particular clock had a significant drift component that wasn't compensated for.

We also applied the modified approach to active hydrogen maser data. We, unfortunately, were not able to extract such a long stretch of data as with the cesiums. The resulting plot is in Figure 4.

We would also like to mention that the modified approach did not differ significantly with the manufacturer specifications, and, in most cases, conformed to them closely. Although the results of this analysis are by no means complete, we believe this method shows significant promise in achieving estimates of absolute frequency stability.

CONCLUSIONS

The modified approach for estimating frequency stability generally performs well: it gives essentially the same results for short-term stabilities ($\tau < 30$ days) and non-pathological behavior for long-term stabilities ($\tau \sim 1$ year). We are encouraged by these preliminary results and are hopeful that the method can be used to give an accurate representation of long-term stability in atomic clocks.

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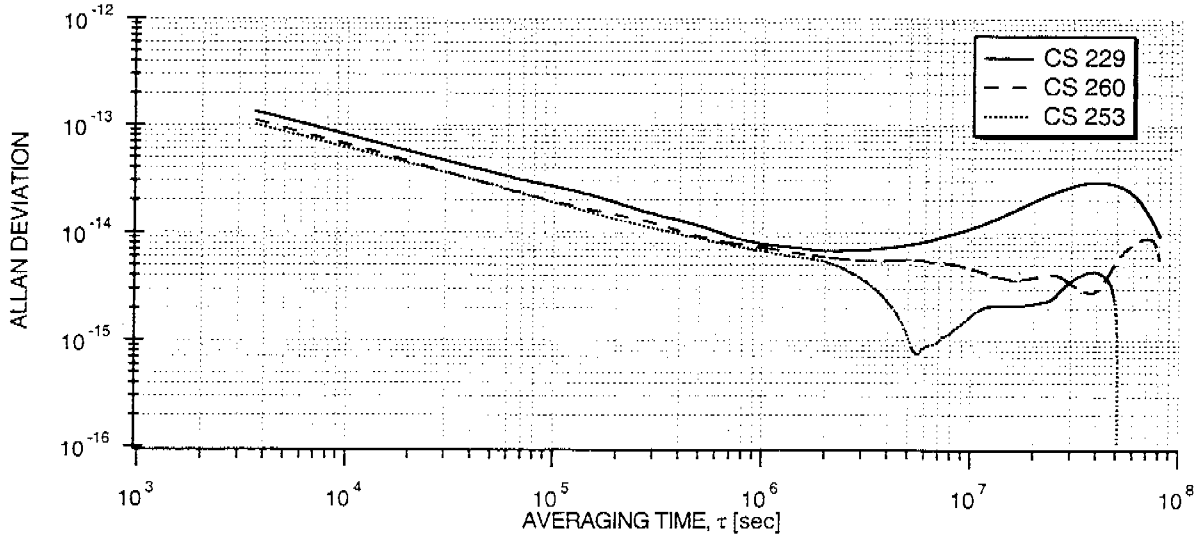

Fig. 1a: Plot of absolute frequency stability calculated using the three-cornered-hat approach.

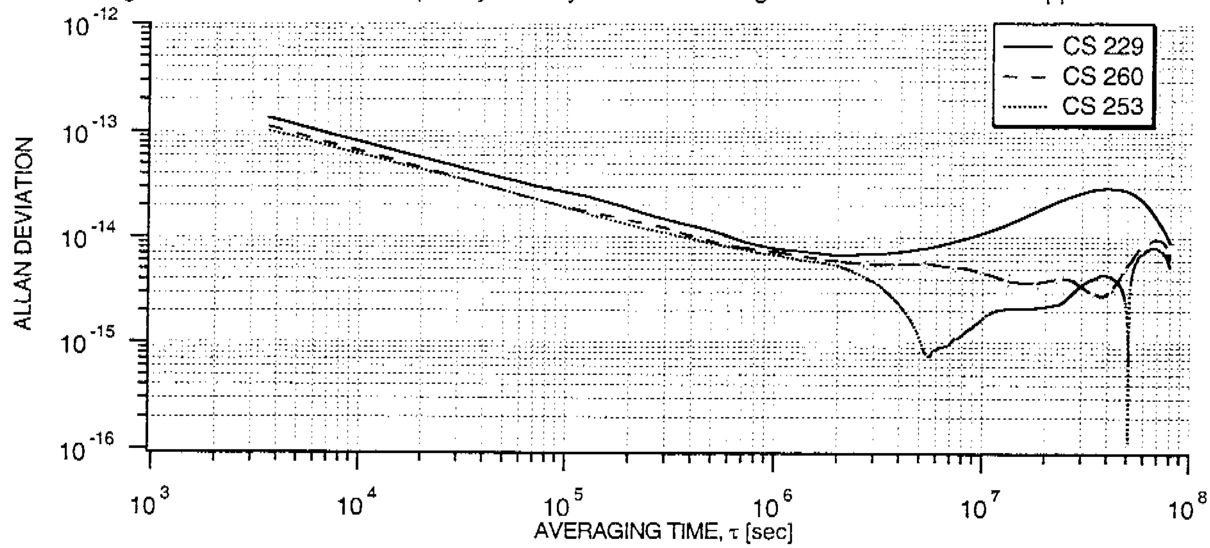

Fig. 1b: Plot of absolute frequency stability using the Tavella-Premoli approach.

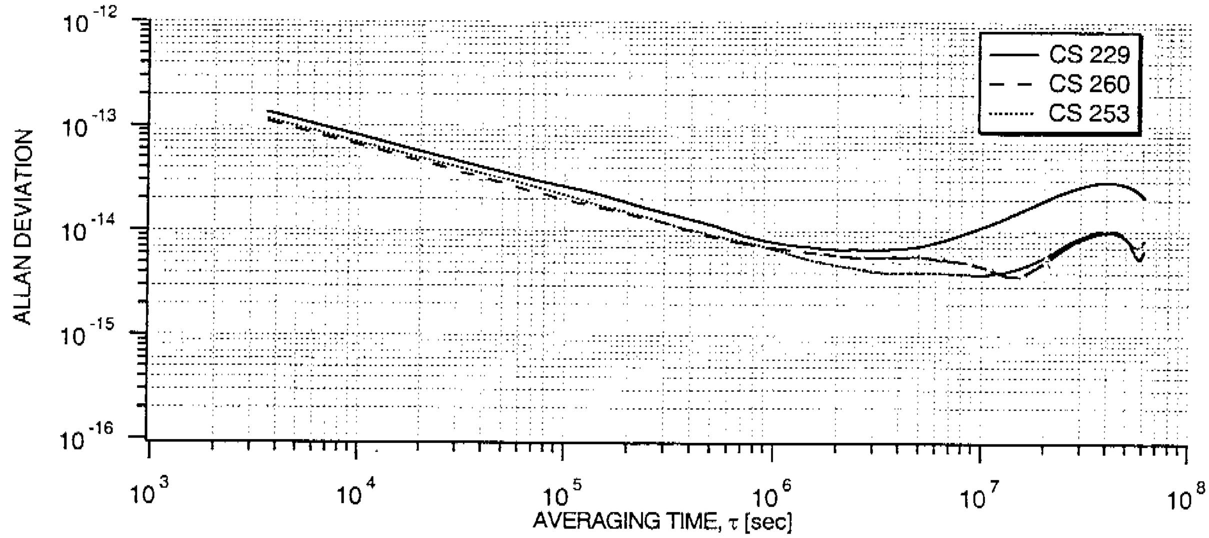

Fig. 2: Plot of absolute frequency stability using the modified approach with a fourth clock (fourth clock's stability not shown).

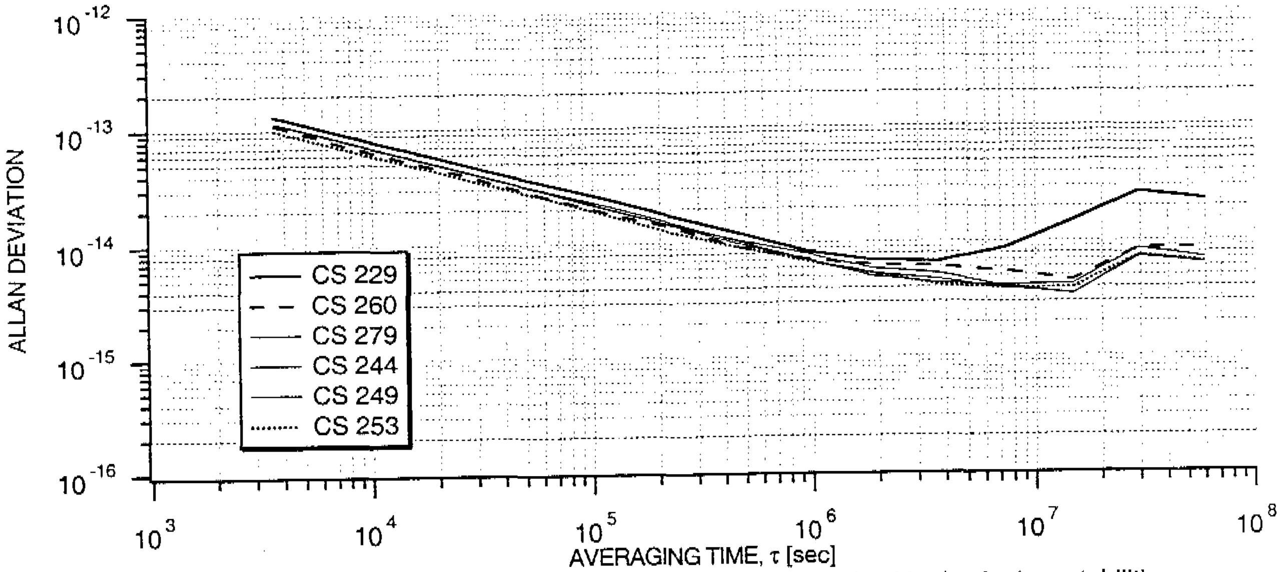

Fig. 3: Plot of absolute frequency stability using modified approach with six clocks - stabilities for all six clocks shown (cf. Fig. 2). The analysis which gave this plot was performed on dyadic averaging times only as opposed to all averaging times shown in Fig. 2.

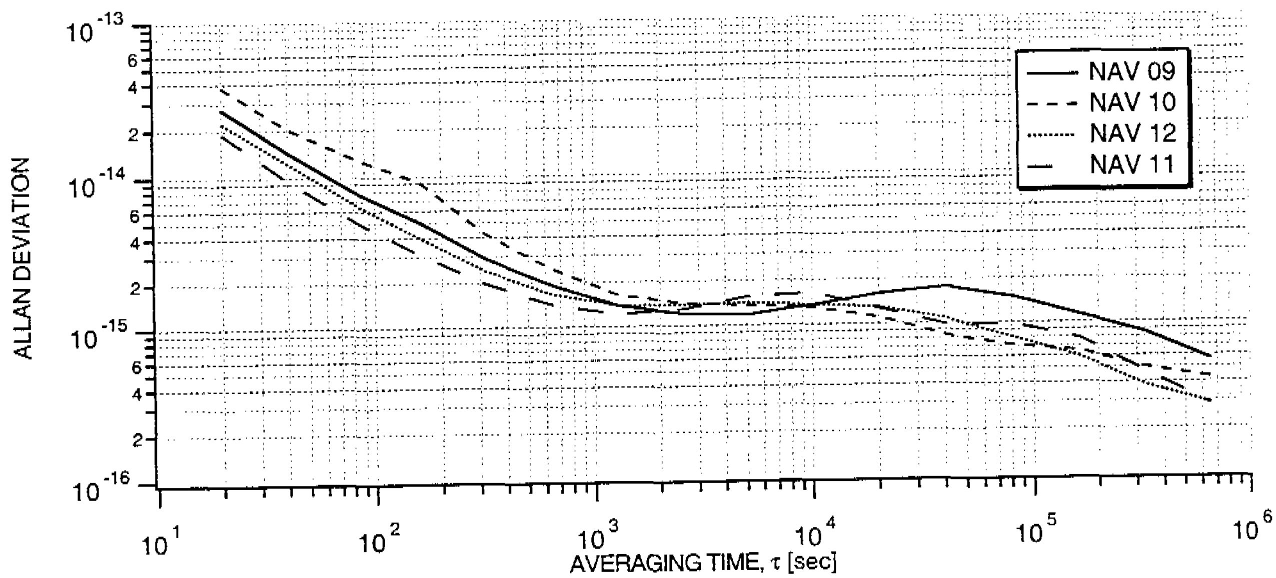

Fig. 4: Plot of absolute frequency stability using modified approach with four active hydrogen masers sampled at 20-second intervals. Analysis performed with dyadic averaging times.

Questions and Answers

PATRIZIA TAVELLA (IEN): I am happy that you are continuing with this work and you took the time to read my papers. Thank you for the quotation.

I think that maybe the optimizer function can be different in the short and long term. Maybe the long correlation is reasonable in the short term, but not in the long term. Maybe we could investigate if the possible function to be minimized should be different in long and short term.

I would also like to investigate the effect of the uncertainty of the variance evaluation on the positive definitiveness of the matrix, because since the evaluation, I am uncertain due to the limited number of measurements. Maybe the region can change.

FRED TORCASO (USNO): I think moving in that direction would be trying to get a better understanding of the distribution of the sample Allan variance for exactly those integration times. That is actually a difficult problem. There have been a lot of papers appearing which try to estimate the distribution of the sample Allan variance for integration times of, I think, eight samples. It is something ridiculously hard.

It looks like there is some evidence for large integration times in many sample estimates that a gaussian approximation is probably a good one. Maybe this would be a possible way to go.

MARC WEISS (NIST): Have you thought about how to compute uncertainty for Allan variance when you have the n-cornered-hat technique? In other words, there is an uncertainty for the Allan variance when you look at the distribution just because you have a sample variance.

FRED TORCASO: So you like to put error bars around the absolute values?

MARC WEISS: In the presence of an n-cornered hat where you are estimating a variance through the other clocks, I think there is an additional uncertainty because of that. I think the relative stability of the clocks will come in as well. In other words, if you look at one very good clock and you have ten clocks that are ten times worse, you can not see the good clock – at least in theory you should not be able to.

I do not know how to do it. I am wondering if you have looked at that?

FRED TORCASO: I actually have not looked at ways of obtaining error bars on estimates of absolute stability. I am looking into the possibility of comparing very quiet clocks, for instance, hydrogen masers, over a short time scale compared with the high-performance cesium beam clocks that we have at USNO. A similar approach to the one I described may work if we rescale this co-variance matrix that I introduced, "R", to a correlation matrix. Then all the off-diagonal terms of the matrix are of the same order. That will help in the analysis, but I have not done that.