RELATIONSHIPS BETWEEN DRIFT COEFFICIENT UNCERTAINTIES AND NOISE LEVELS : APPLICATION TO TIME ERROR PREDICTION

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Abstract

Oscillators are affected by drifts (linear phase drift, linear frequency drift, i.e. quadratic phase drift) and different types of noise according to the power law model of power spectral density (from f^{-2} to f^{+2} frequency noise, i.e. f^{-4} to f^{0} phase noise). Generally, for long-term instability characterization (duration greater than one hour), drift coefficients are estimated by using least squares whereas noise levels are obtained from the residuals by using variances (AVAR, MVAR, TVAR, ...).

However, the low frequency noises, such as random walk FM, induce very long term fluctuations which may be confused with deterministic drifts. This effect, due to the non-stationarity of these noises, depends on the low cut-off frequency which must be introduced in order to ensure power convergence for low frequencies. We calculate the standard deviation of "artificial" drifts due to long-term random fluctuations, versus the noise levels.

The first interest of these results concerns the estimation of the measurement uncertainty of drift coefficients : knowing the noise levels of an oscillator we calculate the standard deviation of the artificial drift coefficient due to these noises; thus, if a "real" deterministic drift is identified in the signal, its coefficients are determined plus or minus the artificial drift coefficients. The standard deviation of the artificial drift coefficients may be considered as the measurement uncertainty of the deterministic drift coefficient.

The second interest concerns the predictability of an oscillator affected by a deterministic drift. Thus, the knowledge of the drift coefficient uncertainties yields a criterion for quantifying the reliability of a time error prediction.

1 INTRODUCTION

We consider a sequence of frequency deviation samples composed of a deterministic part, i.e. a linear frequency drift, and a random part:

$$y(t_k) = C_1 t_k + C_0 + \epsilon_k. \tag{1}$$

An estimation by least squares yields estimates $\hat{C}_0$ and $\hat{C}_1$ of the real coefficients C_0 and C_1. Denoting the interpolated samples by $\hat{y}(t)$, we obtain:

$$\hat{y}(t_k) = \hat{C}_1 t_k + \hat{C}_0. \tag{2}$$

The residuals are defined as:

$$e_k = y(t_k) - \hat{y}(t_k) \tag{3}$$

1.1 Random Fluctuations and Deterministic Drifts

The instantaneous frequency is defined from the nominal frequency and the frequency deviation samples by:

$$\nu_k = \nu_0 \,(1 + y_k) \tag{4}$$

If the sequence y_k is not centered, there are two possibilities:

- the real nominal frequency is different from the assumed nominal frequency: this is a problem of inaccuracy of the oscillator;
- there are long-term random fluctuations (with period much longer than the duration of the sequence) which are seen as constant over the sequence[1].

The same problem may occur with linear frequency drift.
It is impossible to distinguish a "true" deterministic drift from a "false" random drift.

1.2 Statement of the Problem

The Power Spectral Density (PSD) may be modelled as:

$$S_y(f) = \sum_{\alpha=-2}^{+2} h_\alpha . f^\alpha \tag{5}$$

- If no deterministic drift exists, what are the relationships between the noise levels h_α and the estimated drift coefficients $\hat{C}_0$ and $\hat{C}_1$?
- If a deterministic drift exists, what are the uncertainties of the estimated drift coefficients $\hat{C}_0$ and $\hat{C}_1$?
- In both cases, what is the Time Interval Error (TIE) due to an extrapolation of the linear frequency drift?

2 LINEAR REGRESSION

2.1 Coefficient Calculation

We consider N measurements (t_i, y_i): $\{(t_0, y_0), \cdots, (t_{N-1}, y_{N-1})\}$, regularly spaced with a sampling period τ_0:

$$t_k = t_0 + k.\tau_0 \tag{6}$$

We need to know the coefficient of the linear model:

$$y_k = \hat{C}_1 t_k + \hat{C}_0 + e_k \tag{7}$$

The most probable coefficient values, in the sense of the least squares, are given by:

$$\hat{C}_0 = \frac{2(2N-1)}{N(N+1)} \sum_{i=0}^{N-1} y_i + \frac{-6}{N(N+1)\tau_0} \sum_i t_i.y_i \tag{8}$$

$$\hat{C}_1 = \frac{-6}{N(N+1)\tau_0} \sum_i y_i + \frac{12}{N(N-1)(N+1)\tau_0{}^2} \sum_i t_i.y_i \tag{9}$$

2.2 Estimation of the Uncertainties

From (8) and (9), it is possible to calculate $\sigma^2(C_0)$ and $\sigma^2(C_1)$:

$$\begin{aligned}\sigma^2(C_0) &= \frac{4(2N-1)^2}{N^2(N+1)^2}\sigma^2\left(\sum y_i\right) + \frac{36}{N^2(N+1)^2{\tau_0}^2}\sigma^2\left(\sum t_i.y_i\right) \\ &-\frac{24(2N-1)}{N^2(N+1)^2\tau_0}Cov\left(\sum y_i, \sum t_i.y_i\right) \end{aligned} \tag{10}$$

$$\begin{aligned}\sigma^2(C_1) &= \frac{36}{N^2(N+1)^2{\tau_0}^2}\sigma^2\left(\sum y_i\right) + \frac{144}{N^2(N-1)^2(N+1)^2{\tau_0}^4}\sigma^2\left(\sum t_i.y_i\right) \\ &-\frac{144}{(N-1)N^2(N+1)^2{\tau_0}^3}Cov\left(\sum y_i, \sum t_i.y_i\right) \end{aligned} \tag{11}$$

with

$$\sigma^2\left(\sum y_i\right) = \sum_i \sum_j \langle y_i.y_j \rangle \tag{12}$$

$$\sigma^2\left(\sum t_i.y_i\right) = {\tau_0}^2 \sum_i \sum_j i.j.\langle y_i.y_j \rangle \tag{13}$$

$$Cov\left(\sum y_i, \sum t_j.y_j\right) = \tau_0 \sum_i \sum_j i.\langle y_i.y_j \rangle \tag{14}$$

where $\langle\ \rangle$ denotes an average over an infinite number of identical processes (ensemble average).

2.3 Correlation of the Samples

$S_y(f)$	R_{ij} (with $i \neq j$)	R_{ii}
$h_{-2}.f^{-2}$	$h_{-2}\left[\frac{1}{f_l} + \pi^2 \lvert t_j - t_i \rvert\right]$	$\frac{h_{-2}}{f_l}$
$h_{-1}.f^{-1}$	$-h_{-1}\left[C + \ln(2\pi f_l) + ln\lvert t_j - t_i \rvert\right]$	$-h_{-1}\ln(2\tau_0 f_l)$
$h_0.f^0$	0	$h_0 f_h$
$h_{+1}.f^{+1}$	$h_{+1}\frac{(-1)^{(j-i)} - 1}{4\pi^2(t_j - t_i)^2}$	$h_{+1}\frac{{f_h}^2}{2}$
$h_{+2}.f^{+2}$	$h_{+2}\frac{f_h \cos[2\pi f_h(t_j - t_i)]}{2\pi^2(t_j - t_i)^2}$	$h_{+2}\frac{{f_h}^3}{3}$

Table 1: Correlations of the y_k samples versus the noise levels h_α. C is the Euler constant: $C \approx 0,5772$. Assuming a sampling satisfying the Shannon rule, the high cut-off frequency is $f_h = \frac{1}{2\tau_0}$. f_l is the low cut-off frequency.

The PSD $S_y(f)$ is the Fourier Transform of the autocorrelation function. Thus, if no real drift exists in the sequence, we have:

$$\begin{aligned}\langle y_i . y_j \rangle &= \int_{-\infty}^{+\infty} S_y^{2S}(f) e^{+j2\pi f(t_j - t_i)} df \\ &= \int_0^{+\infty} S_y(f) \cos[2\pi f(t_j - t_i)]\, df = R_{ij} \end{aligned} \tag{15}$$

which leads to the results given in Table 1.

2.4 Mean Value Subtraction

Table 1 shows that, for low frequency noises (f^{-2} and f^{-1} FM), the correlations of the samples depend on the low cut-off frequency f_l. This cut-off frequency must be introduced in order to ensure the power convergence.
If the inverse of the low cut-off frequency is much larger than the duration of the sequence $[t_0, t_N]$, the very long term fluctuations (period $\approx \frac{1}{f_l}$) are seen as a constant[1] (see Figure 1).
On the other hand, the subtraction of the mean value of the sequence cancels the dependence on f_l.
Denoting the mean value of the sequence by $\bar{y}$ and the centered sequence samples by y'_k:

$$\bar{y} = \frac{1}{N} \sum_{j=0}^{N-1} y_j \tag{16}$$

$$y'_k = y_k - \bar{y} \tag{17}$$

The subtraction of the mean value is equivalent to a correction of the nominal frequency by a factor $(1 + \bar{y})$:

$$\nu_k \approx \nu_0 (1 + \bar{y})(1 + y'_k) \tag{18}$$

After subtraction of the mean value, it follows that:

$$\sum_{i=0}^{N-1} y'_i = 0 \tag{19}$$

$$\sum_{i=0}^{N-1} t_i . y'_i = \tau_0 \left[\sum_i i . y_i - \frac{N+1}{2} \sum_j y_j \right] \tag{20}$$

Thus:

$$\sigma^2 \left(\sum t_i . y'_i \right) = \tau_0^2 \left[\sum_i \sum_j i . j . R_{ij} - (N+1) \sum_i \sum_j i . R_{ij} + \frac{N(N+1)}{4} \sum_i \sum_j R_{ij} \right] \tag{21}$$

Considering the new linear frequency drift model:

$$y'_k = \hat{C}'_1 t_k + \hat{C}'_0 + e'_k \tag{22}$$

we have:

$$\sigma^2(C'_0) = \frac{36}{N^2 (N+1)^2 \tau_0{}^2} \sigma^2 \left(\sum t_i . y'_i \right) \tag{23}$$

$$\sigma^2(C'_1) = \frac{144}{N^2 (N-1)^2 (N+1)^2 \tau_0{}^4} \sigma^2 \left(\sum t_i . y'_i \right) \tag{24}$$

It may be demonstrated that $\hat{C}'_1 = \hat{C}_1$ and then $\sigma^2(C'_1) = \sigma^2(C_1)$.

2.5 Estimation of the Residuals

The differences between the estimated drift and the y_k samples are:

$$e_k = y_k - \hat{C}_1 t_k - \hat{C}_0 \tag{25}$$

The variance of the residuals is given by:

$$\begin{aligned}\sigma^2(e) &= \sigma^2(y) + \frac{(N-1)(2N-1){\tau_0}^2}{6}\sigma^2(C_1) + \sigma^2(C_0) \\ &\quad - \frac{2}{N}Cov\left(\sum y_k, C_0\right) - \frac{2}{N}Cov\left(\sum t_k.y_k, C_1\right) + (N-1)\tau_0 Cov\left(C_0, C_1\right)\end{aligned} \tag{26}$$

The residuals don't depend on the subtraction of the mean value:

$$e_k = y_k - \hat{C}_1 t_k - \hat{C}_0 = y'_k - \hat{C}'_1 t_k - \hat{C}'_0 \tag{27}$$

3 RESULTS

$S_y(f)$	$\sigma(C_0)$	$\sigma(C'_0)$	$\sigma(C_1)$	$\sigma(e)$
$h_{-2}.f^{-2}$	$\sqrt{\frac{h_{-2}}{f_l}}$	$\sqrt{\frac{3\pi^2\tau h_{-2}}{5}}$	$\sqrt{\frac{12\pi^2 h_{-2}}{5\tau}}$	$\sqrt{\frac{2\pi^2\tau}{15}h_{-2}}$
$h_{-1}.f^{-1}$	$\sqrt{\left[\frac{3}{2} - \frac{1}{4}\ln\left(2f_l\tau\right)\right] h_{-1}}$	$\frac{3\sqrt{h_{-1}}}{2}$	$\frac{3\sqrt{h_{-1}}}{\tau}$	$\sqrt{[C + \ln(\pi\tau)]\, h_{-1}}$
$h_0.f^0$	$\sqrt{\frac{2h_0}{\tau}}$	$\sqrt{\frac{3h_0}{2\tau}}$	$\sqrt{\frac{6h_0}{\tau^3}}$	$\sqrt{\frac{h_0}{2\tau_0}} = \sqrt{f_h h_0}$
$h_{+1}.f^{+1}$	$\sqrt{\frac{5\left[1.37 + \ln(2f_h\tau)\right] h_{+1}}{\pi^2\tau^2}}$	$\sqrt{\frac{9\left[1.27 + \ln(2f_h\tau)\right] h_{+1}}{2\pi^2\tau^2}}$	$\sqrt{\frac{18\left[1.27 + \ln(2f_h\tau)\right] h_{+1}}{\pi^2\tau^4}}$	$\sqrt{\frac{{f_h}^2}{2}h_{+1}}$
$h_{+2}.f^{+2}$	$\sqrt{\frac{10 f_h \ln(2) h_{+2}}{\pi^2\tau^2}}$	$\sqrt{\frac{9 f_h \ln(2) h_{+2}}{\pi^2\tau^2}}$	$\sqrt{\frac{36 f_h \ln(2) h_{+2}}{\pi^2\tau^4}}$	$\sqrt{\frac{{f_h}^3}{3}h_{+2}}$

Table 2: Standard deviation of the drift coefficients and of the residuals versus the noise level h_α and the duration of the sequence τ. The high cut-off frequency is $f_h = \frac{1}{2\tau_0}$ and the low cut-off frequency is f_l.

Thus, after measuring the h_α noise levels, we may estimate the uncertainties $\sigma(C_0)$ and $\sigma(C_1)$ by using Table 2.
This table shows that the subtraction of the mean value cancels the dependence of C_0 on f_h. For high frequency noises, $\sigma(C_0)$ remains very close to $\sigma(C'_0)$. Moreover, neither $\sigma(C_1)$ nor $\sigma(e)$ are modified by this subtraction.

3.1 Measurement Uncertainties of Drift Coefficients

If no real deterministic drift exists, the determination of the drift coefficients yields:

$$-2\sigma(C_0) < \hat{C}_0 < 2\sigma(C_0) \qquad \text{with 95.5\% confidence}$$
$$-2\sigma(C_1) < \hat{C}_1 < 2\sigma(C_1) \qquad \text{with 95.5\% confidence}$$

Thus, measuring a drift coefficient C within the interval $[-2\sigma(C), +2\sigma(C)]$ is compatible with a null drift hypothesis (with a risk of the second kind of 4.5%).
On the other hand, if a real deterministic drift exists, the estimates $\hat{C}_0$ and $\hat{C}_1$ converge toward the real coefficients C_0 and C_1:

$$\langle \hat{C}_0 \rangle = C_0 \quad \text{and} \quad \langle \hat{C}_1 \rangle = C_1.$$

The uncertainty domains of the coefficients C_0 and C_1 are:

$$C_0 = \hat{C}_0 \pm 2\sigma(C_0) \quad \text{with 95.5\% confidence}$$

$$C_1 = \hat{C}_1 \pm 2\sigma(C_1) \quad \text{with 95.5\% confidence}$$

3.2 Frequency and Time Error Prediction

3.2.1 Frequency error prediction

If $\hat{C}_0$ and $\hat{C}_1$ are estimated over a sequence of N samples (duration $\tau = N\tau_0$), what error results from an extrapolation of the linear model to $t_N + T$?

$$\hat{y}(t_N + T) = \hat{C}_1.(t_N + T) + \hat{C}_0 \tag{28}$$

The Total Frequency Error (TFE) may be defined as:

$$TFE(T) = y(t_N + T) - \hat{y}(t_N + T) \tag{29}$$

The TFE is composed of a Deterministic Frequency Error (DFE):

$$DFE(T) = (C_1 - \hat{C}_1)(t_N + T) - (C_0 - \hat{C}_0) \tag{30}$$

plus a random error (see Figure 2):

$$TFE(T) = DFE(T) + y_r(t_N + T) \tag{31}$$

$y_r(t_i)$ is a centered random variable without drift, with a variance $\sigma^2(y_r) = R_{ii}$.
Thus, denoting $t' = t_N + T$, we obtain:

$$\begin{aligned} \langle TFE^2(T) \rangle \;=\; & \sigma^2(C_0) + \sigma^2(C_1).t'^2 + \sigma^2(y_r) \\ & -2Cov\,(C_0, y_r(t')) - 2Cov\,(C_1, y_r(t'))\,.t' + 2Cov\,(C_0, C_1)\,.t' \end{aligned} \tag{32}$$

$Cov\,(C_0, y_r(t'))$ is the covariance between the parameter C_0 estimated over the sequence $[t_0, t_N]$ and the random sample y_r at the date $t' = t_N + T$.

3.2.2 Time error prediction

If a sequence of $x(t_k)$ is known over a duration τ (from t_0 to $t_N = t_0 + \tau$), the Time Interval Error (TIE) at $t_N + T$ may be defined as [2, 3]:

$$TIE(T) = x(t_N + T) - x(t_N) - T\bar{\hat{y}}_{t_N,T} \tag{33}$$

with

$$\bar{\hat{y}}_{t_N,T} = \frac{1}{T}\int_{t_N}^{t_N+T} \hat{y}(t)dt \approx \frac{1}{M}\sum_{i=N}^{N+M-1} \hat{y}_i \tag{34}$$

where $\hat{y}_i$ is the extrapolated frequency deviation at t_i and M is defined as $T = M\tau_0$.

$$TIE(T) = \tau_0 \sum_{i=N}^{N+M-1} (y_i - \hat{y}_i) \tag{35}$$

Thus, denoting $M' = N + M - 1$, we obtain:

$$\begin{aligned} \langle TIE^2(T)\rangle &= {\tau_0}^2 \sum_{i=N}^{M'}\sum_{j=N}^{M'} R_{ij} + {t_M}^2\sigma^2(C_0) + {t_M}^2\left(t_N + \frac{T}{2}\right)^2\sigma^2(C1) \\ &+2{t_M}^2\left(t_N + \frac{T}{2}\right)Cov(C_0, C_1) \\ &-2{\tau_0}^2\sum_{i=N}^{M'}\sum_{j=N}^{M'} Cov(y_i, C_0) - 2{\tau_0}^2\sum_{i=N}^{M'}\sum_{j=N}^{M'} t_j Cov(y_i, C_1) \end{aligned} \tag{36}$$

3.2.3 Example of f^{-2} frequency noise

In order to use (32) the covariances $Cov(C_0, y_r(t'))$ and $Cov(C_1, y_r(t'))$ must be calculated:

$$\begin{aligned} \langle C_0, y_{rM'}\rangle &= \frac{2(2N-1)}{N(N+1)}\sum_{i=0}^{N-1}\langle y_i.y_{rM'}\rangle - \frac{6}{N(N+1)}\sum_{i=0}^{N-1} i\langle y_i.y_{rM'}\rangle \\ &= \frac{2(2N-1)}{N(N+1)}\sum_{i=0}^{N-1} R_{iM'} - \frac{6}{N(N+1)}\sum_{i=0}^{N-1} i.R_{iM'} \\ &= h_{-2}\left[\frac{1}{f_l} - \pi^2(t_N + T)\right] \end{aligned} \tag{37}$$

For $Cov(C_1, y_r(t'))$, we obtain:

$$Cov(C_1, y_r(t')) = \frac{h_{-2}\pi^2}{\tau_0} \tag{38}$$

Therefore, for an f^{-2} frequency noise, the standard deviation of the TFE is:

$$\sqrt{\langle TFE^2(T)\rangle} = \sqrt{h_{-2}\left[\frac{4\pi^2 t_N}{15} + \frac{12\pi^2 T}{5t_N}(t_N + T)\right]} \tag{39}$$

It is interesting to notice that the DFE and the Random Frequency Error are fully separated:

$$\sqrt{\langle TFE^2(T)\rangle} = \sqrt{2\sigma^2(e) + \sigma^2(C_1)(t_N + T)} \tag{40}$$

Thus, if $T = 0$ (interpolation), the standard deviation of the TFE is $\sqrt{2}$ times the standard deviation of the residuals, i.e. it is the standard deviation between two residuals.
Concerning the TIE, from (36), (37) and (38), we obtain:

$$\sqrt{\langle TIE^2(T)\rangle} = \sqrt{\frac{\pi^2 h_{-2} T^2}{15 t_N}\left(9{t_N}^2 + 13 t_N T + 4T^2\right)} \qquad (41)$$

4 CONCLUSION: CHOICE OF THE FREQUENCY MODEL

What is the physical meaning of the low cut-off frequency of an oscillator? Is it a real feature of low frequency noises or a mathematical trick? In practice, it is possible to avoid its use.
For an f^{-2} frequency noise, the derivative of the frequency deviation, the ageing $z(t)$, is a white noise:

$$y(t) = \int_{t_0}^{t} z(\theta)d\theta \qquad (42)$$

where t_0 is the switch-on date of the oscillator. In this case, we have assumed that the oscillator was syntonized and synchronized at t_0. f_l is no longer necessary, $y(t)$ is a centered random variable whose standard deviation increases with Θ:

$$f_l \equiv \frac{1}{\Theta} \qquad (43)$$

What is the "real" frequency of the oscillator over $T \ll \Theta$: its nominal frequency or its mean frequency over T?
The answer depends on the frequency model:

- the use of the power law PSD model implies that the nominal frequency and the h_α noise levels are time-independent: they are the constants of this model. This model is suitable for free-running oscillators, e.g. frequency standards involved in the TAI computation;
- the determination of the nominal frequency as the mean frequency over a sequence of finite duration implies that the nominal frequency is time-dependent: this nominal frequency is only valid over this whole sequence but neither over a part of this sequence nor over another sequence. This model is suitable for oscillators used for an experiment of well-defined duration.

References

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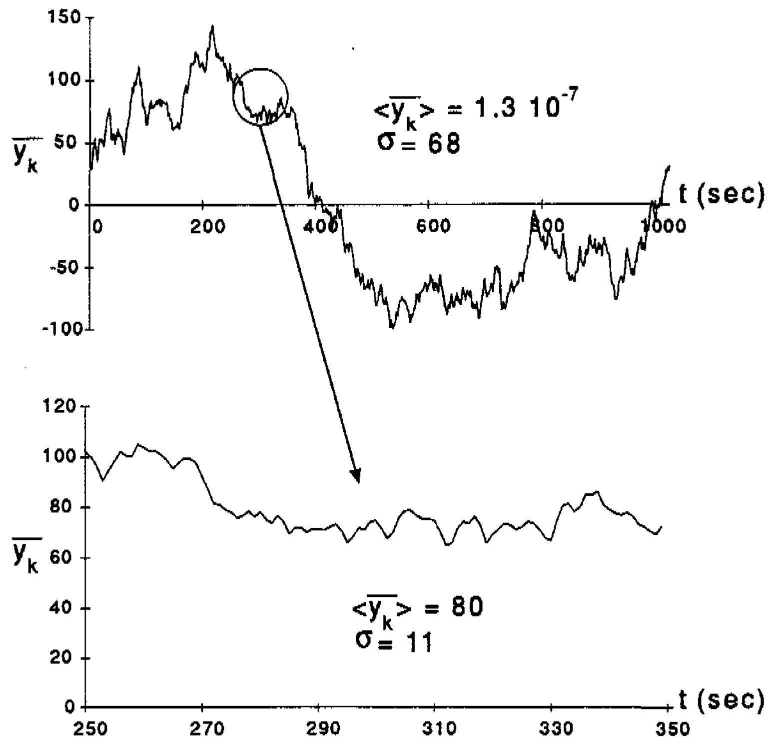

Figure 1: Sequence of frequency deviation samples for an f^{-2} FM noise. Above, the duration of the sequence is about the inverse of the low cut-off frequency. Below is an enlargement of a part of this graph: the inverse of the low cut-off frequency is far larger than the duration of the sequence, and the samples are no longer centered

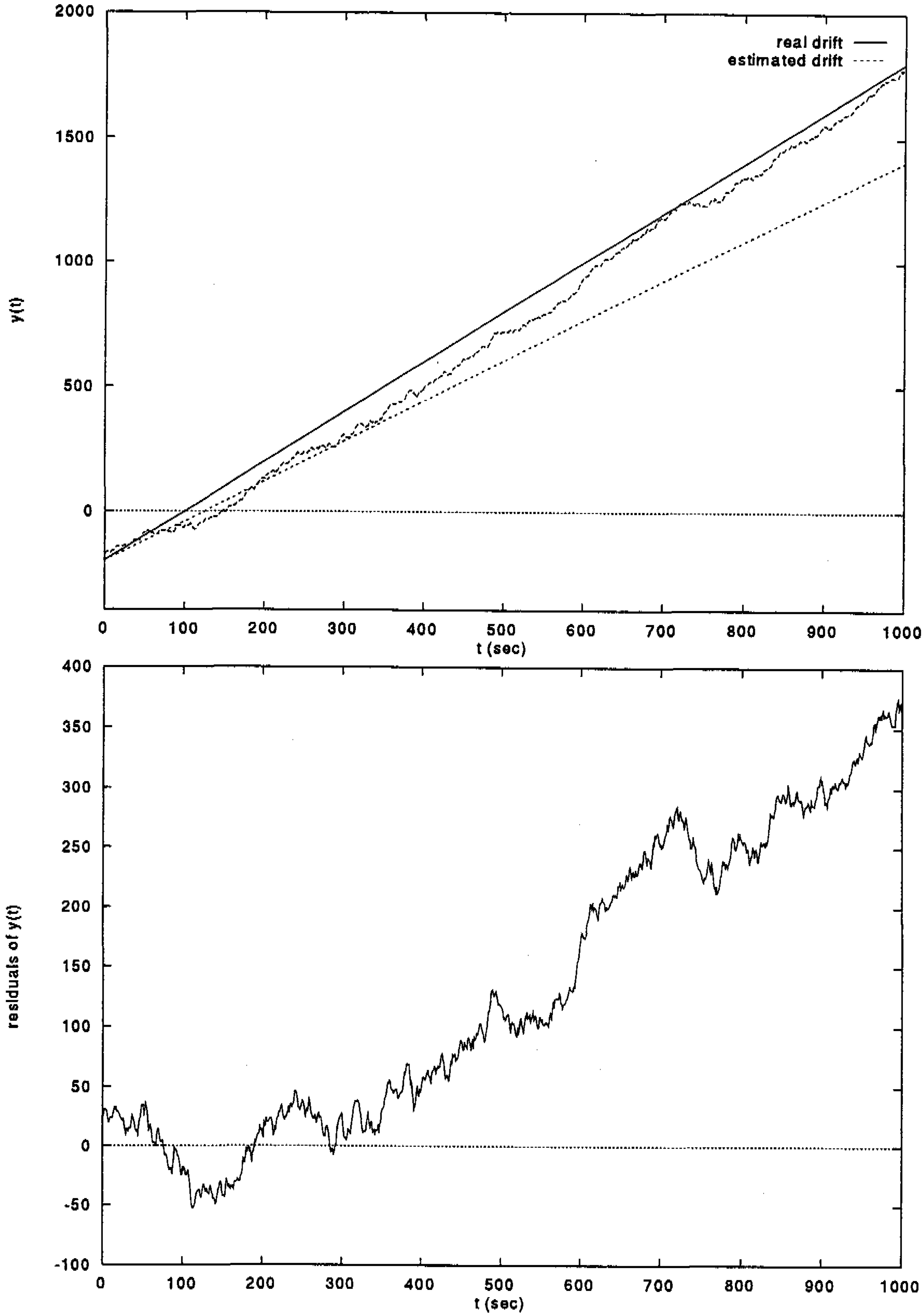

Figure 2: Estimation of the drift over a sequence of frequency deviation altered by f^{-2} FM noise (above). The drift was estimated over the first 256 samples (256 sec). After this time, the sequence moves away from the estimated drift. This effect is more obvious in the residuals (below).