# A DERIVATION OF THE DICK EFFECT FROM CONTROL-LOOP MODELS FOR PERIODICALLY INTERROGATED PASSIVE FREQUENCY STANDARDS*

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**Abstract**

***The phase of a frequency standard that uses periodic interrogation and control of a local oscillator (LO) is degraded by a long-term random-walk component induced by downconversion of LO noise into the loop passband. The Dick formula for the noise level of this degradation can be derived from explicit solutions of two LO control-loop models. A summary of the derivations is given here.***

## INTRODUCTION

In 1987, following a suggestion of L. Cutler, G.J. Dick[1] described a source of long-term instability for a class of passive frequency standards that includes ion traps and atomic fountains. In these standards, the frequency of a local oscillator (LO) is controlled by a feedback loop whose detection and control operations are periodic with some period $T_c$. For each cycle, the output of the detector is a weighted average of the LO frequency error over the cycle. The weighting function $g(t)$, derived from quantum-mechanical calculations, depends on the method by which the atoms are interrogated by the RF field generated by upconversion of the LO signal to the atomic transition frequency.[1,2,3] In general, $g(t)$ can be zero over a considerable portion of the cycle. The LO control signal over each cycle is a function of the detector outputs from previous cycles.

The purpose of a frequency-control loop is to attenuate the frequency fluctuations of the LO inside the loop passband, while tolerating them outside the passband. As Dick saw, though, the periodic interrogation causes out-of-band LO noise power, near the cycle frequency $f_c = 1/T_c$ and its harmonics, to be downconverted into the loop passband, thus injecting random false information about the current average LO frequency into the control signal. This random false frequency correction causes a component of white FM, or random walk of phase, to persist in the output of the locked LO (LLO) over the long term. Dick gave a formula for the white-FM noise level contributed by this effect, namely:

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$$S_y^{LLO}(0) = 2\sum_{k=1}^{\infty} \frac{g_k^2}{g_0^2} S_y^{LO}(kf_c), \tag{1}$$

where $S_y^{LLO}(f)$ is the spectral density of the Dick-effect portion of the normalized frequency noise of the LLO, $S_y^{LO}(f)$ is the spectral density of the normalized frequency noise of the free-running LO, and $g_k$ is the Fourier coefficient

$$g_k = \frac{1}{T_c}\int_0^{T_c} g(t)\cos(2\pi k f_c t)\,dt, \tag{2}$$

where $g(t)$ is assumed to be symmetric about $T_c/2$. This level of white FM near Fourier frequency zero contributes an asymptotic component of Allan variance given by:

$$\sigma_y^2(\tau) \sim \frac{S_y^{LLO}(0)}{2\tau} \qquad (f_c\tau \to \infty). \tag{3}$$

My purpose here is to supplement previous derivations[1,2,3,4] of the Dick formula (Eq. 1) by an approach that uses explicit time-domain solutions of simple LO control loop models with a general detection weighting function $g(t)$. Careful interpretation of these solutions yields formulas for the LLO spectral density, and conditions for the validity of the Dick formula. These models are not represented to be realistic models of actual frequency standards. By exhibiting the presence of the Dick effect in models of transparent simplicity, I intend to remove any remaining doubt of its existence and to isolate its essential nature, in the hope of aiding efforts to reduce it.

This paper gives only a summary of the solution method; details will be submitted elsewhere.

## CONTROL-LOOP MODELS

Figure 1 shows two models for an LO control loop. Model 1 is intended to correspond to Dick's models.[1,2] Model 2 extends the model of Lo Presti, Patanè, Rovera, and De Marchi[4] to a general weighting function. A unified treatment of the two models is presented at the expense of a conflict of notation between this paper and [4]: because the model of Lo Presti et al. includes the effect of alternate interrogation of the two sides of a Ramsey fringe, the cycle period $T_c$ used here corresponds to the sample period $T_s = 2T_c$ in [4], and the $g(t)$ used in Model 2 really consists of two periods of the $g(t)$ used in Model 1.

In Model 1, the box $G_1$ represents a linear time-invariant filter with impulse response $g(t)/(T_c g_0)$ for $0 < t < T_c$, and zero elsewhere. It is important to observe that $G_1$ has unity gain at DC. Its transfer function is

$$G_1(f) = \frac{1}{T_c g_0}\int_0^{T_c} g(t) e^{-i2\pi f t}\,dt. \tag{4}$$

The output of the box $G_1$ at time $t$ is

$$G_1 y_{LLO}(t) = \frac{1}{T_c g_0}\int_0^{T_c} g(u)\, y_{LLO}(t-u)\,du,$$

which is fictitious unless $t$ is a multiple of $T_c$. The output of the sampler at time $nT_c$ is the normalized interrogation report

$$G_1 y_{LLO}(nT_c) = \frac{1}{T_c g_0} \int_0^{T_c} g(u)\, y_{LLO}(nT_c - u)\, du. \tag{5}$$

for the $n$th cycle. (Recall the symmetry of $g(t)$ about the midpoint of the cycle.) The detection noise term $v_n$ can represent photon-count fluctuations in frequency standards with optical detection, for example. The cumulative sum of the error signals, multiplied by a gain factor $\lambda$ between 0 and 1, is the frequency correction $y_n$ that is applied to the LO during the next cycle. Except for initial conditions, Model 1 is specified completely by (Eq. 5) and the equations

$$y_n - y_{n-1} = \lambda \left( G_1 y_{LLO}(nT_c) + v_n \right), \tag{6}$$

$$y_{LLO}(t) = y_{LO}(t) - y_{n-1}, \quad (n-1) T_c < t \le nT_c, \tag{7}$$

in which it is convenient to suppose that $n$ runs through all integers.

In Model 2, the hold and integration operations emit a delayed linear interpolation of the cumulative sum of the input to the hold, modulo a constant of integration. Let $y_n$ be $\lambda$ times that cumulative sum. Then $y_n$ again satisfies Eq. (6). In place of Eq. (7) we have

$$y_{LLO}(t) = y_{LO}(t) - \left( \frac{t}{T_c} - n + 1 \right) y_{n-1} - \left( n - \frac{t}{T_c} \right) y_{n-2}, \quad (n-1) T_c < t \le nT_c. \tag{8}$$

In Model 1, the frequency correction during a cycle is constant; here, it is a ramp.

## SUMMARY OF SOLUTION METHOD

The derivation of the LLO frequency spectrum from these model equations is carried out by the following steps.

First, by isolating the digital aspects of the models, one can solve for $y_n$. In Model 1, substitution of Eq. (7) into Eq. (6) gives a first-order difference equation for $y_n$ in terms of the quantity

$$w_n = G_1 y_{LO}(nT_c) + v_n. \tag{9}$$

The solution of this difference equation has the form $y_n = H_{d1} w_n$, where $H_{d1}$ is a unity-gain lowpass digital single-pole filter with transfer function

$$H_{d1}(z) = \frac{\lambda}{1 - (1 - \lambda) z^{-1}}.$$

The time constant is approximately $T_c/\lambda$ for $\lambda \ll 1$. The transient component of the solution is neglected. Model 2 gives a second-order difference equation that is solved by the two-pole filter

$$H_{d2}(z) = \frac{\lambda}{1 - \phi_1 z^{-1} - \phi_2 z^{-2}},$$

whose coefficients depend in a simple way on $g(t)$ and the gain factor $\lambda$. Under a reasonable assumption on $g(t)$, one can adjust the gain to make the filter overdamped, critically damped, or underdamped.

Second, with $y_n$ known, it is evident from Eq. (7) or Eq. (8) that the LLO frequency is a known function of time on each cycle. Because of the piecewise nature of the solution, we need to use care in its interpretation to obtain a well-defined spectrum. Let us agree that the low-frequency spectral behavior of the LLO phase is adequately known if we can determine the discrete-time spectrum of the phase when sampled with period $T_c$. In turn, we know the sampled-phase spectrum if we know the discrete-time spectrum of the sequence of average LLO frequencies. Let $\bar{y}_{LLO}(nT_c)$ be the average value of LLO frequency over the cycle ending at $nT_c$. Knowing the LLO frequency as a function of time over this cycle, we can generate an explicit formula for the average frequency. This formula is shown as a block diagram in Fig. 2, which applies to both loop models. The block labeled "average" is the continuous-time moving-average filter for period $T_c$; the following sampler gives the sequence of LO frequencies averaged over successive cycles. The only component that depends explicitly on the model is the block labeled $H_e$, a unity-gain lowpass digital filter with transfer function $z^{-1}H_{d1}(z)$ for Model 1, $\frac{1}{2}(z^{-1}+z^{-2})H_{d2}(z)$ for Model 2.

Third, the two-sided LLO frequency spectrum can be deduced from the block diagram of Fig. 2 by observing that the diagram is equivalent to a certain continuous-time operation followed by a single sampler. In terms of the two-sided LO frequency spectrum $S_y^{LO}(f)$ and detection-noise spectrum $S_v(f)$, the LLO spectrum can be written as follows:

$$S_{\bar{y}}^{LLO}(f) = S_{\bar{y}}^{0}(f) + S_{\bar{y}}^{1}(f), \quad |f| \le f_c/2,$$

where

$$S_{\bar{y}}^{0}(f) = \left|\frac{1-z^{-1}}{i2\pi fT_c} - H_e(z)G_1(f)\right|^2 S_y^{LO}(f) + |H_e(z)|^2 S_v(f), \tag{10}$$

the main spectrum, so to speak, and

$$S_{\bar{y}}^{1}(f) = \sum_{k\neq 0}\left|\frac{1-z^{-1}}{i2\pi(fT_c+k)} - H_e(z)G_1(f+kf_c)\right|^2 S_y^{LO}(f+kf_c), \tag{11}$$

the aliased spectrum. In these formulas, $z = e^{i2\pi fT_c}$. The sum includes both positive and negative $k$.

## MAIN AND ALIASED SPECTRA

Consider the main part (Eq. 10) of the LLO frequency spectrum. The LO spectrum is multiplied by a factor that is $O(f^2)$ as $f \to 0$. This is the basic action of the first-order frequency control loop, which attenuates the excursions of the LO inside the loop bandwidth. For example, flicker FM in the LO is reduced to flicker PM in the LLO, and random walk FM is reduced to white FM. In addition, there is a lowpass-filtered white detection noise in the LLO frequency. We can regard $H_e(z)G_1(f)$ as the closed-loop transfer function from LO frequency noise to LO correction signal.

The Dick effect is supposed to come from a long-term white-FM component in the aliased spectrum. There is such a contribution if the aliased spectrum (Eq. 11) is continuous and positive at $f = 0$. Under reasonable conditions, this is so, and we may set $f = 0$ ($z = 1$) in Eq. (11). Because $H_e(1) = 1$, we have

$$S_y^1(0) = 2\sum_{k=1}^{\infty} |G_1(kf_c)|^2 S_y^{LO}(kf_c), \qquad (12)$$

where we have now used the symmetry of the summands about zero frequency. This formula holds for one-sided spectral densities also.

The numbers $|G_1(kf_c)|^2$ are invariant to cyclic translations of the function $g(t)$ in time. It follows that the result (Eq. 12) is invariant to shifts in the time origin, i.e., if the LLO phase is sampled on any time grid with spacing $T_c$, then the samples will include a white-FM component with spectral density (Eq. 12) at zero frequency. If $g(t)$ is symmetric about $T_c/2$ for our time origin, then

$$G_1(kf_c) = \frac{g_k}{g_0},$$

where $g_k$ is given by Eq. (2). Thus, Eq. (12) extends the Dick formula (Eq. 1) to asymmetric weighting functions.

The Dick formula, which gives the limiting value of spectral density at zero Fourier frequency, is exact for both models, even though the LLO spectrum at nonzero frequencies is different for the two models. A simple approximation for the aliased spectrum (Eq. 11) holds if the gain constant $\lambda$ is much less than 1. Then the loop bandwidth is much less than $f_c$ (time constant much greater than $T_c$). Assume also that $G_1(kf_c + f)$ and $S_y^{LO}(kf_c + f)$ can be regarded as approximately constant for nonzero $k$ and for $f$ within the loop bandwidth. Then, for such $f$, the aliased spectrum has approximately the same shape as the frequency response of the digital filter $H_e$, with value at 0 given by the Dick formula. For both models, this shape is approximately Lorentzian. Thus, the Dick-effect Allan variance component takes the asymptotic white-FM form (Eq. 3) only for averaging times $\tau$ greater than roughly twice the loop time constant. In this approximation, the Dick-effect and detection noises appear inside the loop bandwidth, the non-aliased LO noise outside.

## REMARKS

Although I have not considered any other models, the Dick effect appears to be an inherent property of periodic local-oscillator control loops. For the two models treated here, this was shown by a careful interpretation of explicit solutions for the output frequency as function of time.

I have now come full circle on this topic. My involvement began in 1987 when John Dick asked me to derive the spectrum of $G_1 y_{LO}$ after sampling. I did not understand: in Fig. 1, $G_1$ is applied to $y_{LLO}$, not to $y_{LO}$. Nevertheless, I did the calculation, thereby contributing the factor 2 in Dick's formula. Now, from the block diagram in Fig. 2, we see how the sampled $G_1 y_{LO}$ fits into the picture. Could the Dick effect be cancelled by replacing the averaging filter by a $G_1$ filter? Alas—this block diagram is merely a graphical representation of a mathematical formula; it has no physical existence.

# REFERENCES

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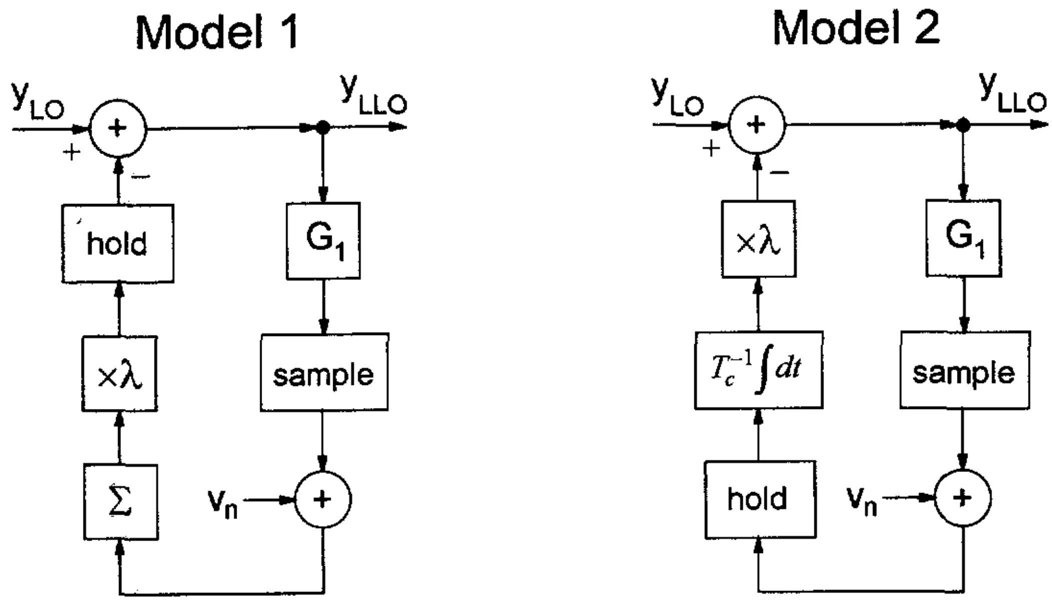
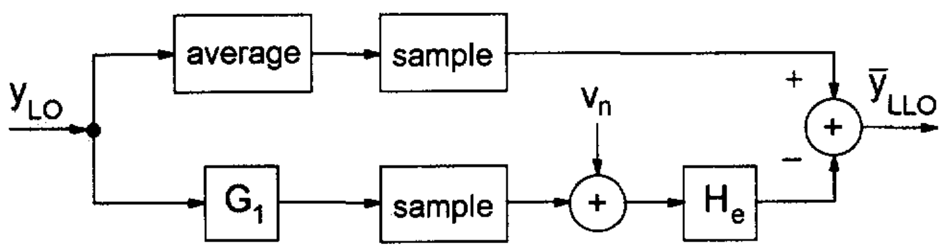



Fig. 1. Simplified models of local-oscillator control loops




Model 1: $H_e(z) = \dfrac{\lambda z^{-1}}{1-(1-\lambda)\,z^{-1}}$

Model 2: $H_e(z) = \dfrac{\frac{\lambda}{2}(z^{-1}+z^{-2})}{1-\phi_1\,z^{-1}-\phi_2\,z^{-2}}$

Fig. 2. Solution of both models for LLO frequency averages