A SIMPLE ALGORITHM FOR APPROXIMATING CONFIDENCE ON THE MODIFIED ALLAN VARIANCE AND THE TIME VARIANCE*

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Abstract

An approximating algorithm for computing equivalent degrees of freedom of the modified Allan variance and its square root, the modified Allan deviation (MVAR and MDEV), and the time variance and time deviation (TVAR and TDEV) is presented, along with an algorithm for approximating the inverse chi-squared distribution. These two algorithms allow relatively simple computations of confidence intervals on MDEV and TDEV, the latter currently used as a standard in the telecommunications industry. These algorithms enable users to present variance results with confidence intervals corresponding to any useful probability for most data lengths and noise types.

1 INTRODUCTION

We present here a simplified algorithm for calculating approximate confidence intervals on the modified Allan deviation, MDEV, and therefore also on the related time deviation TDEV. The algorithm has two parts: the first gives approximate equivalent degrees of freedom, edf, for the fully overlapped estimate of MVAR; the second gives approximate values of the inverse chi-squared distribution. An algorithm for estimating edf for the other measure commonly used in time and frequency metrology, the original Allan deviation, was published previously.[1]

Confidence intervals are defined in terms of edf and the chi-squared distribution as follows. If s^2 denotes the usual sample variance of n independent and identically distributed Gaussian measurements (white noise) with actual variance σ^2, then

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$$U = \frac{s^2}{\sigma^2} \cdot \nu \tag{1}$$

has a chi-squared distribution with $\nu = n - 1$ degrees of freedom.[2] In the classical situation, the number of degrees of freedom associated with σ^2 is an integer value depending only on the number of measurements, and exact confidence limits on the measurement variance are easily calculated using percentiles of the appropriate chi-squared distribution. For example, Figure 1 shows the chi-squared distribution with 10 degrees of freedom, and also depicts the percentiles a and b that are needed to calculate uncertainty bounds on σ^2 at the 0.95 confidence level from a particular s^2 based on 11 Gaussian measurements.

A 95% confidence interval is obtained as follows. First, we find values a and b such that the probability is 0.95 that U of Eq. (1) lies between a and b. This condition is equivalent to each of the following inequalities:

$$a < \frac{s^2}{\sigma^2} \cdot \nu < b;$$

$$\frac{\nu}{b} < \frac{\sigma^2}{s^2} < \frac{\nu}{a}; \tag{2}$$

$$\frac{\nu}{b} \cdot s^2 < \sigma^2 < \frac{\nu}{a} \cdot s^2.$$

The lower and upper bounds in the final inequality are confidence limits on the unknown variance σ^2. Note that the confidence factors ν/b and ν/a needed in the calculations are independent of the actual data. They give the magnitude of the confidence interval as a function of the number of points used to compute the variance. Hence, we can compute these confidence factors for various data lengths. The factors $1 - \nu/b$ and $\nu/a - 1$ give the multipliers for the magnitude of the lower and upper confidence intervals on the variances, respectively. For deviations such as TDEV, the corresponding multipliers are $1 - \sqrt{\nu/b}$ and $\sqrt{\nu/a} - 1$.

Since the common time and frequency stability measures (AVAR, MVAR, TVAR) are calculated from data arising from non-white noise processes, the confidence limit procedure outlined above is an approximate method[3] that is based on approximating the distribution of U in Eq. (1) with the chi-squared distribution with degrees of freedom

$$\nu = \frac{2\left(\sigma^2\right)^2}{Var\left(s^2\right)}, \tag{3}$$

where σ^2 represents the appropriate stability measure, TVAR, for example, s^2 represents its corresponding estimator, and $Var\left(s^2\right)$ is the variance of the s^2 estimator. The quantity ν, which now depends on the noise type, is called the equivalent degrees of freedom, edf, since it need not be integer-valued.

In this contribution we have combined a previously published edf approximation algorithm[4] with an algorithm for approximating the inverse of the chi-squared distribution function. The latter algorithm is based on work of Barnes used in deriving tables in [5], but not published, and formulas from Abramowitz and Stegun (A&S).[6] Previously, tables for confidence of TDEV and MDEV were published in [7]. These are exact computations for edf and the associated confidence intervals for various cases in computing TDEV and MDEV. We compare values approximating the exact edf and confidence factors in tables in [7], finding a worst case disagreement of −9.7% for the edf and +10.8% for the confidence intervals. Most cases are much better than that. The confidence intervals are pessimistic if they are too large and optimistic if they are too small. In many cases here, pessimism is better than optimism, since the true value of the variance is more certain to lie in a larger range than a smaller. For the comparison with the published tables the confidence intervals are no smaller than −3.3%.

2 APPROXIMATION FOR EQUIVALENT DEGRESS OF FREEDOM

This version of the formula is restricted to the case of the usual fully overlapped estimator of MVAR or TVAR ([8], Eq. (12); [4], Eq. (6), $m_1 = 1$).

Let

$N =$ number of time residuals,

$m =$ averaging time / sample period,

$M = N - 3m + 1$, the number of terms summed in the estimate,

$q = M/m$.

Restrictions:

$N \geq 16$,

$m \leq N/5$.

The approximate edf is given by

$$edf = \frac{a_0 q}{1 - a_1/q}, \tag{4}$$

where a_0 and a_1 are given in Table I as functions of m and the noise type.

Table I. Coefficients for Approximate edf Calculation

Noise Type	$m=1$		$m=2$		$m>2$	
	a_0	a_1	a_0	a_1	a_0	a_1
White PM	0.514	0	0.935	0	1.225	0.589
Flicker PM	0.576	0	0.973	0	1.003	0.602
White FM	0.667	0	1.010	0	0.968	0.571
Flicker FM	0.811	0	1.027	0	0.947	0.416
Random-Walk FM	1.000	0	0.866	0	0.768	0.411

Under the assumptions given above, a maximum error of 11.1% in this approximation has been observed. Usually, it is much less.

3 APPROXIMATION FOR INVERSE OF CHI-SQUARED DISTRIBUTION

Let U be a chi-squared random variable with ν degrees of freedom (ν can be nonintegral). Let $0 < p < 1$. Define $x = x(p, \nu)$ as the $100p$ percentile of the distribution of U; thus p is the probability that $U < x$. The algorithm given below computes an approximation to x.

Restrictions:

$\nu \geq 1$,
$0.005 \leq p \leq 0.995$.

Maximum observed error with these restrictions: 3%

if $p \leq \frac{1}{2}$ and $\nu \leq 10$ then

! Method: truncate power series in A&S [6] 26.4.6, invert by iteration

$a = \nu/2$

! Calculation of $G = \Gamma(1+a)$ (A&S 6.1.35)

constants: $c_1 = -0.5748646$, $c_2 = 0.9512363$, $c_3 = -0.6998588$, $c_4 = 0.4245549$, $c_5 = -0.1010678$

$n =$ integer part of a

$y = a - n$

$G = 1 + c_1 y + c_2 y^2 + c_3 y^3 + c_4 y^4 + c_5 y^5$

for $k = 1$ to n ! Do nothing if $n = 0$

$G = G(y + k)$

next k

$A = pG$

$u = 0$

for $i = 1$ to 7

$$g = 1 + \frac{u}{a+1}\left(1 + \frac{u}{a+2}\left(1 + \frac{u}{a+3}\right)\right)$$
$$u = \left(\frac{Ae^u}{g}\right)^{1/a}$$

next i

$x = 2u$

else

! Method: A&S 26.4.17

$p_1 = \min(p, 1-p)$

! Calculation of $X =$ inverse of normal distribution at $1 - p_1$ (A&S 26.2.22)

constants: $a_0 = 2.30753$, $a_1 = 0.27601$, $b_1 = 0.99229$, $b_2 = 0.04481$

$t = \sqrt{-2 \ln p_1}$

$$X = t - (a_0 + a_1 t) / (1 + b_1 t + b_2 t^2)$$

$$s = signum\left(p - \tfrac{1}{2}\right) \quad ! \; signum(u) = 1 \text{ if } u > 0, \; -1 \text{ if } u < 0, \; 0 \text{ if } u = 0$$

$$b = 2/(9\nu)$$

$$x = \nu\left(1 - b + sX\sqrt{b}\right)^3$$

4 A NUMERICAL EXAMPLE

Before giving tabular results, we show by example how they are used and how they are calculated by the algorithms given above. Assume the situation of the last line of Table II below: white PM noise, 1,025 time residuals, and averaging time = 128 sample periods. Suppose that an MDEV value s is computed by a fully overlapped estimate. The tabulated 95% lower and upper factors are 33.89% and 104.1%. Therefore, a 95% confidence interval for the true MDEV is 0.661 s to 2.041 s.

The tabulated edf and confidence factors are obtained as follows: N =1,025, m = 128, M = 1025-3×128+1 = 642 (the number of summands in the estimate), $q = M/m$ =5.0156, a_0 = 1.225, a_1 = 0.589 from Table I, edf = 6.9617 from Eq. (4). For 95% confidence we need to compute the 2.5% and 97.5% chi-squared levels. The inverse chi-squared algorithm, with ν = 6.9617 and p = 0.025, gives x = 1.6720 as the 2.5% level, denoted by a in Eq. (2). Similarly, the 97.5% level is 15.928, denoted by b. The computed confidence factors are $1 - \sqrt{\nu/b}$ = 0.3389, $\sqrt{\nu/a} - 1$ = 1.0405. (Note that the values in Table II were computed from values of a_0 and a_1 having more significant digits than the ones given in Table I.)

5 RESULTS

The data in the tables are the results for white PM with fully overlapped estimates. Table II gives the approximate edf and confidence factors. Table III gives the percentage errors from the exact values as found in [7]. The errors for white PM are the largest of the various noise types.

6 REFERENCES

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Table II. Approximate edf and Confidence Factors

Noise Type: White PM

N	m	edf	lower 68%	upper 68%	lower 95%	upper 95%
17	1	7.714	17.74	39.14	32.79	94.61
17	2	5.610	19.67	50.41	36.24	128.7
33	1	15.94	13.65	23.38	25.52	52.42
33	2	13.09	14.71	26.67	27.40	60.97
33	4	7.543	17.87	39.82	33.03	96.58
65	1	32.40	10.29	14.96	19.46	31.99
65	2	28.05	10.91	16.33	20.60	35.19
65	4	17.29	13.23	22.17	24.78	49.37
65	8	7.241	18.12	41.10	33.47	100.3
129	1	65.31	7.622	9.916	14.58	20.63
129	2	57.97	8.030	10.62	15.33	22.17
129	4	36.86	9.746	13.84	18.48	29.42
129	8	16.98	13.33	22.43	24.94	50.03
129	16	7.091	18.24	41.78	33.69	102.3
257	1	131.1	5.579	6.715	10.76	13.74
257	2	117.8	5.857	7.123	11.28	14.60
257	4	76.04	7.128	9.095	13.66	18.84
257	8	36.55	9.780	13.91	18.54	29.58
257	16	16.83	13.37	22.56	25.02	50.36
257	32	7.016	18.31	42.13	33.80	103.3
513	1	262.8	4.045	4.610	7.856	9.332
513	2	237.5	4.241	4.867	8.229	9.864
513	4	154.4	5.177	6.141	10.00	12.53
513	8	75.73	7.141	9.116	13.68	18.89
513	16	36.40	9.798	13.95	18.57	29.66
513	32	16.75	13.40	22.63	25.07	50.53
513	64	6.978	18.34	42.30	33.86	103.8
1025	1	526.1	2.913	3.195	5.685	6.421
1025	2	476.9	3.052	3.363	5.954	6.766
1025	4	311.1	3.737	4.214	7.267	8.513
1025	8	154.1	5.182	6.148	10.01	12.54
1025	16	75.58	7.148	9.126	13.69	18.91
1025	32	36.32	9.806	13.97	18.59	29.70
1025	64	16.71	13.41	22.67	25.09	50.62
1025	128	6.959	18.36	42.39	33.89	104.1

Table III. Percentage Error: 100 (*Approximate* – *Correct*) /*Correct*

Noise Type: White PM

N	m	edf	lower 68%	upper 68%	lower 95%	upper 95%
17	1	–3.4	0.2	2.2	1.0	3.2
17	2	–9.7	2.2	8.4	3.0	10.8
33	1	–1.6	–0.1	0.2	0.5	1.4
33	2	–4.2	0.8	2.0	1.4	3.7
33	4	3.4	–2.1	–2.9	–1.2	–3.1
65	1	–0.8	–0.3	–0.2	0.2	0.5
65	2	–2.0	0.1	0.5	0.6	1.3
65	4	3.9	–2.2	–3.3	–1.5	–2.6
65	8	–3.6	0.3	2.4	1.0	3.4
129	1	–0.4	–0.4	–0.4	0.0	0.1
129	2	–1.0	–0.2	–0.1	0.3	0.5
129	4	4.0	–2.3	–3.0	–1.7	–2.6
129	8	–2.9	0.4	1.1	1.0	2.3
129	16	–5.3	0.9	3.9	1.6	5.2
257	1	–0.2	–0.5	–0.5	–0.1	–0.1
257	2	–0.5	–0.4	–0.3	0.0	0.1
257	4	4.1	–2.4	–2.9	–1.9	–2.5
257	8	–2.7	0.5	1.0	1.0	1.7
257	16	–4.6	1.1	2.3	1.6	3.6
257	32	–5.8	1.0	4.3	1.8	5.7
513	1	–0.1	–0.5	–0.5	–0.1	–0.1
513	2	–0.2	–0.5	–0.5	0.0	0.0
513	4	4.1	–2.4	–2.8	–1.9	–2.4
513	8	–2.6	0.6	0.9	1.0	1.4
513	16	–4.4	1.2	2.0	1.7	2.8
513	32	–5.0	1.3	2.6	1.8	4.0
513	64	–5.9	1.0	4.4	1.8	5.8
1025	1	–0.1	–0.5	–0.5	–0.1	–0.1
1025	2	–0.1	–0.5	–0.5	–0.1	–0.1
1025	4	4.2	–2.5	–2.7	–2.0	–2.4
1025	8	–2.6	0.6	0.8	1.0	1.3
1025	16	–4.3	1.3	1.9	1.7	2.5
1025	32	–4.8	1.4	2.2	1.8	3.1
1025	64	–5.2	1.3	2.6	1.9	4.1
1025	128	–5.9	1.0	4.4	1.8	5.8

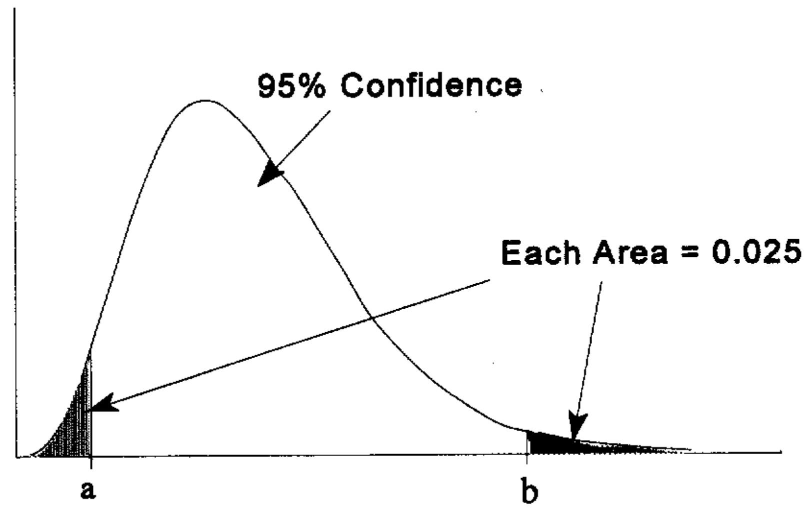

Figure 1 Finding the 95% confidence limits under the chi-squared distribution with 10 degrees of freedom.

Questions and Answers

JOHN DICK (JPL): Where does the number 68 percent come from? And what do you recommend as confidence limits in our use of error bars on Allan deviation plots?

MARC WEISS: Sixty-eight percent is a typical one sigma kind of number. When you take a standard deviation, it's typically a one sigma kind of number. Sixty-eight percent probability is one sigma on a normal distribution; that's where that came from. Ninety-five percent is two sigma. Three sigma is the 99 percent, I believe.

So that's where I chose those. Now which one to use depends on the application and how much you're willing to be wrong. If two times out of three is good enough, then one sigma's good enough. But if you want it to never fail, three sigma is not good enough.

I think that's a very important question. In our systems, we really need to think about what happens if they fail; and how much do we want to avoid that; and how much are we willing to pay for it. And when you write a spec, another big question is, "Should the confidence bars be included in the spec?" Should you write a spec that says, "The Allan variance shall not exceed this number with a 95 percent probability?" There are problems with that because if you do that, that's going to change how long you test the system in order to show it meets the spec.

ROBERT DOUGLAS (NRC, CANADA): In calculating the exact values, we presumably are using a Gaussian normal distribution so you can calculate the second moments. Is that right?

MARC WEISS: Do you want to answer that, Chuck?

CHARLES GREENHALL (JPL): Yes, we're assuming the second differences of phase are normal.

ROBERT DOUGLAS: Then my question is do you have any recommended procedures for checking that this is in fact true in our datasets?

CHARLES GREENHALL: No. There are tests for normal distribution.