THE RANGE COVERED BY A RANDOM PROCESS AND THE NEW DEFINITION OF MTIE

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Abstract

The paper is devoted to the study of the range covered by the time deviation of a clock with respect to a reference one, in a certain observation time interval. In case of a clock affected by Gaussian white phase noise, it is possible to infer the probability law of the covered range and, in particular, the probability that the time deviation exceeds a particular threshold level. The application to telecommunication as related to the Maximum Time Interval Error (MTIE) is discussed.

INTRODUCTION

In the last few years, mostly for the telecommunication community, it became necessary to estimate the possible range spanned by a random process, also without having observed and measured any realization of it. The problem may be illustrated as follows: suppose one has a clock to be used as synchronization unit somewhere in a telecommuncation network and knows that the clock signal is mostly affected by a random noise of known spectrum. What is the "time error," i.e. the phase deviation, that such a clock may accumulate in a certain time interval? Apart from deterministic trends, the answer regarding the random component may only be a probabilistic one, the nature of the process being stochastic. So the problem can be better expressed as: knowing the spectral density of phase fluctuations, what is the probability law of the range spanned by such phase fluctuations? In this paper, the case of Gaussian white phase modulation is considered and the range probability law is inferred. This study gives also an indication on the possible "maximum" range by the identification of a certain percentile in the range distribution, i.e. the range value that is not exceeded more than a certain percentage of times. The application to telecommunication is immediate because the range spanned by the phase dictates the correct dimensioning of memory buffers and it is what is contained in the quantity MTIE (Maximum Time Interval Error) largely used and discussed in the telecomm community and recently redefined as a percentile quantity. The study here presented supports such new definition and provides some materials for its understanding and its practical implementation.

THE RANGE PROBABILITY

Let's consider a Gaussian white process, i.e. a sequence of uncorrelated (white) samples whose probability law is described by a Gaussian density function $f(x)$ with zero mean and variance σ^2:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \tag{1}$$

The range spanned by a sequence of such n samples is defined as the difference between the maximum and the minimum value:

$$z = max\ value - min\ value \tag{2}$$

The study of the range thus requires the evaluation of the probability of the maximum and minimum values. The density function $g(w)$ of the maximum, i.e. the probability that the larger value falls in between w and $w + dw$ is estimated as the probability that one sample lies in between w and $w + dw$ while all the other $(n-1)$ samples lie below w, still multiplied by the n possible configurations. In case of independent samples, this is just the product of single probability, thus leading to:

$$g(w) = n[F(w)]^{n-1} f(w) dw \tag{3}$$

where $F(w)$ represents the cumulative distribution function defined as:

$$F(w) = \int_{-\infty}^{w} f(\xi) d\xi \tag{4}$$

With a similar procedure the density function of the minimum value is found. As a successive step, the density function $h(z)$ of the range z (Eq. 2) is found by means of the composition law of density functions.[1] In case of a Gaussian parent distribution, it is possible to write an analytical expression for $h(z)$[2] that becomes:

$$h(z) = \frac{n(n-1)}{2\pi\sigma^2} \left(\frac{1}{2}\right)^{n-2} \int_{-\infty}^{\infty} dy \cdot e^{\frac{(z+y)^2}{2\sigma^2}} e^{\frac{y^2}{2\sigma^2}} \left[Erf\left(\frac{z+y}{\sqrt{2}\sigma}\right) - Erf\left(\frac{y}{\sqrt{2}\sigma}\right)\right]^{n-2} \tag{5}$$

where $Erf(x)$ stands for the error function:

$$Erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \tag{6}$$

This expression of $h(z)$ is for positive range values and may be estimated, for example, by fixing n and evaluating Eq. (5) for some different values of z. Some results are illustrated in Fig. 1.

It may be noted that, as n increases, the possibility that the n observed samples extend over a larger range becomes more and more likely.

The n Gaussian samples may be interpreted as the result of sampling a clock signal affected by white phase noise. The n samples represent the n measures of the time deviation of the clock (with respect to a reference) repeated with an observation interval τ_0. At the growing of n, the observation time increases as $\tau = (n-1)\tau_0$. Fig. 1 indicates, thus, the most probable values for the time deviation of a clock detectable over an observation time τ.

PERCENTILE LEVELS AND THE MAXIMUM TIME INTERVAL ERROR (MTIE)

In the last few years, in the telecommunication community increasing attention has been paid to the development of mathematical tools suitable to characterize clock behaviors in telecommunication networks. An important information is the maximum range spanned by the phase deviation $x(t)$ of a clock because it determines the correct dimensioning of memory buffers. To this aim, the Maximum Time Interval Error (MTIE) was defined as:

$$\mathrm{MTIE}(\tau) = \max_{-\infty \leq t_0 \leq \infty} \{\max_{t_0 \leq t \leq t_0+\tau} [x(t)] - \min_{t_0 \leq t \leq t_0+\tau} [x(t)]\} \tag{7}$$

Nevertheless, this original definition gave some problems because it refers to the maximum range ever spanned, i.e. during the entire life of the clock. From the analysis of data reported in Fig. 1, it appears that, in case of pure Gaussian white phase noise, the maximum spanned range may reach infinitely large values. It is only a matter of time: the larger values are less probable, but not impossible, and after a long period of operation they may be observable. It became, thus, apparent that the original definition (Eq. 7) needed to be better refined and recently, a percentile definition to be interpreted as the maximum range, not to be exceed more than a certain percentage of times, was adopted.[3]

How to estimate the percentile of the distribution $h(z)$, i.e. a threshold level a which is expected to be exceeded only a certain percentage $(1-p)$ of times? To this aim, it is necessary to identify the range value a delimiting an area equal to p to its left, i.e.:

$$H(a) \equiv \int_0^a h(z)dz = p \tag{8}$$

The evaluation of Eq. (8), via Eq. (5), requires some numerical integrations and depends on the values of n and p. An example of some evaluations are reported in Fig. 2, where threshold levels a are individualized as a function of the number of samples n for three different percentile levels p. As the number of samples n increases, the probability that the spanned range is large increases also. Therefore, the threshold that guarantees the pth percentile of the range possibilities moves towards larger values. To be sure of covering 99 out of 100 cases, we must be ready to accept also relatively large values for the range. For example, with $10^4 \div 10^5$ data, we should also accept range values reaching ten times the standard deviation σ of the parent distribution.

ESTIMATING THE PERCENTILE MTIE

The reported study of the probability law of the spanned range is, thus, supporting the estimation of the quantity MTIE in its percentile definition. In the case of white phase noise, it is possible to assess the percentile threshold values for the spanned range only by the knowledge of the standard deviation σ of the Gaussian noise and by the aid of Eq. (8) or of Fig. 2. It appears that the percentile MTIE may be written as:

$$\mathrm{MTIE}_{p-perc}(\tau) = c(p,\tau) \cdot \sigma \tag{9}$$

where the function $c(p,\tau)$ is a suitable function of the percentile level and of the observation time and has to be estimated by the evaluation of Eq. (8) or by the aid of Fig. 2. The problem may be how to know σ. It may be estimated, for example, by the aid of the relationships between the Allan deviations and the spectral densities[4] and by exploiting the relationships among those and the standard deviation σ of the Gaussian phase noise. This leads to the following equivalent relationships:

$$\sigma(\tau_0) = \frac{\tau_0}{\sqrt{3}}\sigma_y(\tau_0) = \sqrt{\int_0^{f_h} S_x(f)df} = \sqrt{\frac{h_2}{4\pi^2}f_h} \tag{10}$$

where the introduced symbols are well known: $\sigma_y(\tau)$ is the Allan deviation, $S_x(f)$ is the spectral density of the phase deviation $x(t)$, f_h is the high-frequency cutoff, and h_2 is the constant that determines the amount of white phase noise[4] in the polynomial model of the frequency spectral density $S_y(f)$. Eq. (10) introduces, thus, a relationship between the amount of phase noise (stated as Allan deviation or spectral density level) and the percentile MTIE. Some examples of experimental validation of the study reported in this paper are presented in [5]. Some experimental measurement data were used to evaluate the original MTIE, as well as the percentile one, through the described procedure, and results are compared and discussed.

FURTHER COMMENTS AND CONCLUSION

The study here presented was devoted to the case of Gaussian white phase noise (white PM). The assumption of Gaussian noise was experimentally checked in [6]. The white PM is certainly not the only noise affecting clock signals. Nevertheless, it is an important case, particularly in the characterization of telecommunication units[7], where a behavior due to white phase noise is observed also for longer observation intervals. Moreover, the output of a clock locked to a good reference signal presents a behavior similar to the white phase even if the underlying noise may be of a different nature.[8] The extension of the above theoretical analysis to other kinds of noises usually appearing on clock signals is under development.

The study of the threshold levels reached by a random process is not only interesting for characterizing clocks or telecommunication apparatuses, but also in the determination of calibration intervals of measuring instruments of industrial interest.[9]

In conclusion, through the estimation of the probability law of the range spanned by a random process, it has been possible to determine the threshold range expected not to be exceeded more than a certain percentage of times. This led to an evaluation of the percentile definition of MTIE and of its relationship with the noise power spectrum level in the case of clocks affected by white phase noise.

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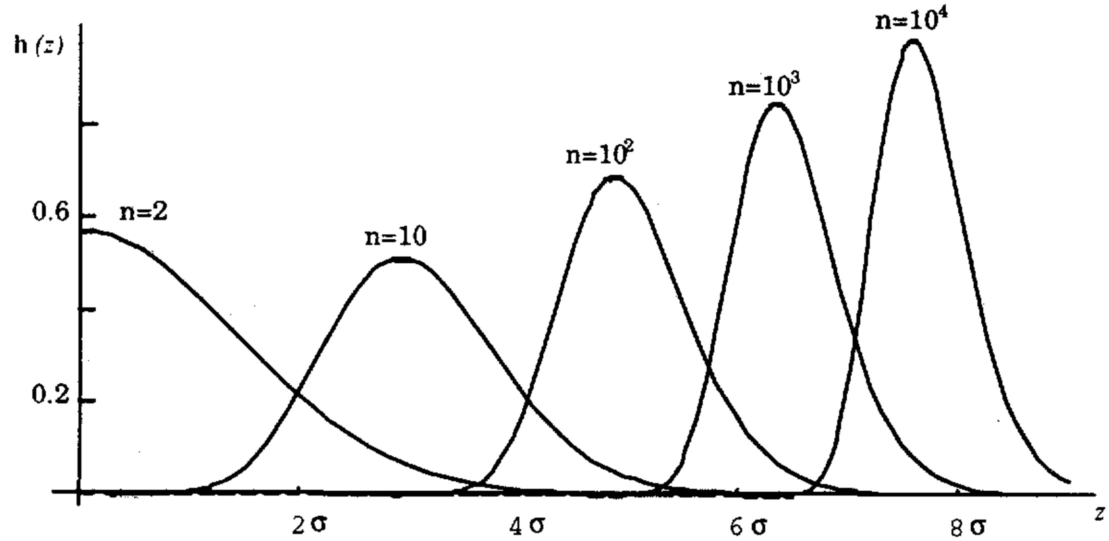

Fig. 1: Probability density fuction h(z) of the range spanned by a sequence of n uncorrelated Gaussian samples. The values of the range z are normalized to the value of the standard deviation σ of the parental distribution.

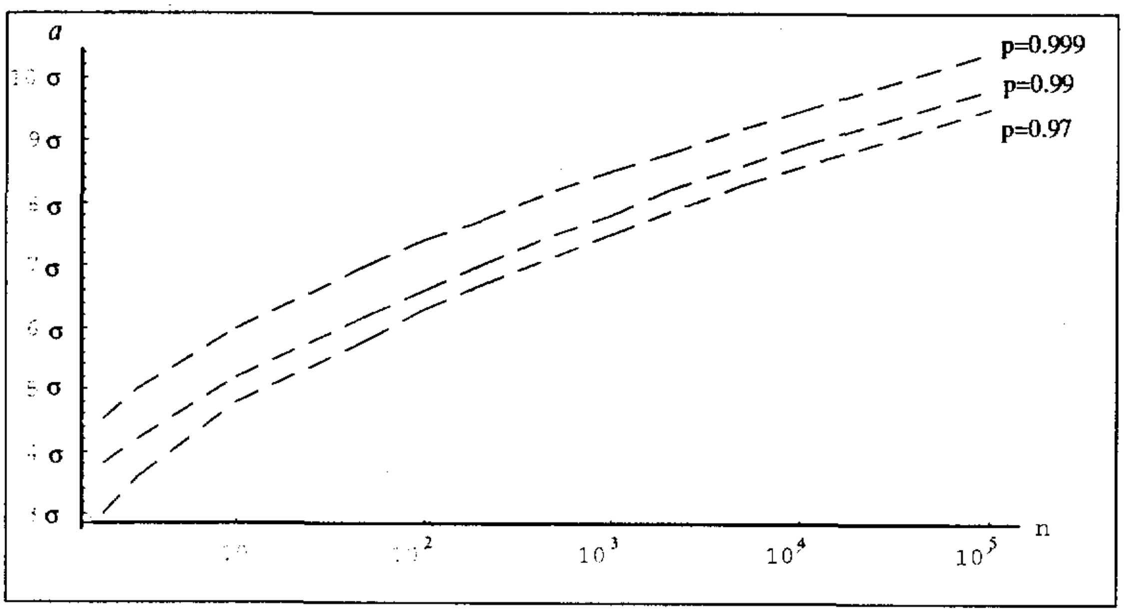

Fig. 2: Threshold levels delimiting the maximum range spanned by the phase deviation at the percentile level p, versus the number n of samples (the threshold level a is normalized to the standard deviation of the Gaussian parental distribution)