STEERING OF FREQUENCY STANDARDS BY THE USE OF LINEAR QUADRATIC GAUSSIAN CONTROL THEORY

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Abstract

Linear quadratic Gaussian control is a technique that uses Kalman filtering to estimate a state vector used for input into a control calculation. A control correction is calculated by minimizing a quadratic cost function that is dependent on both the state vector and the control amount. Different penalties, chosen by the designer, are assessed by the controller as the state vector and control amount vary from given optimal values. With this feature controllers can be designed to force the phase and frequency differences between two standards to zero either more or less aggressively depending on the application. Data will be used to show how using different parameters in the cost function analysis affects the steering and the stability of the frequency standards.

INTRODUCTION

The steering of frequency standards (atomic clocks) is a very important procedure in the timing community. Steering is used to synchronize remote clocks using very accurate time transfer methods such as two-way time transfer and the Global Positioning System. Also in time scale applications a standard is steered to a paper, or calculated, clock in order to give the time scale a physically realizable output. This paper will discuss how the linear quadratic Gaussian (LQG) technique applies to the designing of control systems to steer frequency standards.

In any real world application, a control system must deal with some amount of uncertainty, whether it comes in the form of sensor noise, process modeling error, or any other noise sources. The LQG technique is used for designing optimal control systems for uncertain physical processes. An important feature of this technique is that the stability of the control system is assured if system parameters have the properties of observability and controllability. Kalman filtering is used in order to estimate the actual state variables from measurements made of the stochastic system.

TWO-STATE LQG THEORY FOR FREQUENCY STANDARDS

In the LQG theory[1,2,3] the state equation is assumed to be given as a linear function of a state vector and a control vector:

$$x(k+1) = \Phi x(k) + B u(k) + w(k), \tag{1}$$

where

$$\mathbf{x}(k) = \text{state vector} = \begin{vmatrix} x_1(k) \\ x_2(k) \end{vmatrix},$$

$x_1(k)$ is the phase difference, and $x_2(k)$ is the fractional frequency difference, between the reference and the steered standard,

$\mathbf{u}(k) =$ control vector which is a scalar in this case corresponding to the fractional frequency change of the synthesizer controlling the steered standard,

$\Phi =$ transition matrix $= \begin{vmatrix} 1 & \tau \\ 0 & 1 \end{vmatrix}$, τ is the time interval between measurements,

$\mathbf{w}(k) =$ white noise characterized by covariance Q_k,

$$Q_k = \begin{vmatrix} \frac{1}{2}h_0\tau + 2h_{-1}\tau^2 + \frac{2}{3}\pi^2h_{-2}\tau^3 & \pi^2h_{-2}\tau^2 \\ \pi^2h_{-2}\tau^2 & 2\pi^2h_{-2}\tau \end{vmatrix},$$ and the h's are calculated from the two-sample

(Allan) variance of the comparison between two standards[4,5].

The noisy measurement $z(k)$ is related to the state vector by

$$z(k) = \mathbf{H}\mathbf{x}(k) + \mathbf{v}(k) \tag{2}$$

where

$z(k) =$ measurement, in our case a scalar phase difference,

$\mathbf{H} =$ connection matrix $= |\ 1 \quad 0\ |$, and

$\mathbf{v}(k) =$ white noise characterized by covariance $R_k =$ measurement noise.

The linear state equation (1) is an approximation to the state modeling equation that includes higher order terms. In order to help give the linear approximation validity, the control vector $\mathbf{u}(k)$ is chosen such that the quadratic cost function

$$J = \sum_k [\hat{\mathbf{x}}(k)^T\mathbf{W}_Q\hat{\mathbf{x}}(k) + \mathbf{u}(k)^T\mathbf{W}_R\mathbf{u}(k)] \tag{3}$$

is minimized.

$\mathbf{W}_Q$ and $\mathbf{W}_R$ are matrices that are chosen by the designer in order to set the relative penalties assessed to the state vector estimate $\hat{\mathbf{x}}(k)$ and control vector $\mathbf{u}(k)$ as they vary from zero. In general, if $\mathbf{W}_R$ is large compared to $\mathbf{W}_Q$, the penalty is large for the system attempting to drive the state vector toward zero too rapidly. Conversely, if $\mathbf{W}_Q$ is large compared to $\mathbf{W}_R$, the system faces a smaller penalty for large control effort and the system is driven toward zero more quickly.

Due to the noisy measurement of $\mathbf{x}(k)$, we are faced with a compound problem of optimal control and estimation. A very useful theorem from control theory known as the separation principle allows us to solve the optimal control and the estimation problem independently. Kalman filtering is the technique used to estimate the true state $\mathbf{x}(k)$ from the noise. The Kalman filter is calculated as usual[3] with the exception that the update estimate must now include control terms:

$$\hat{\mathbf{x}}(k+1) = \Phi\hat{\mathbf{x}}(k) + \mathbf{B}\mathbf{u}(k) + \mathbf{K}_g[z(k+1) - \mathbf{H}(\Phi\hat{\mathbf{x}}(k) + \mathbf{B}\mathbf{u}(k))] \tag{4}$$

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where K_g is the Kalman filter gain, $\hat{x}(k)$ is the state estimation, and $B = \begin{vmatrix} \tau \\ 1 \end{vmatrix}$ for the two-state model with a frequency synthesizer as the control mechanism. The optimal control for the given cost equation is

$$u(k) = -\hat{G}_0 \hat{x}(k) \tag{5}$$

where

$$\hat{G}_0 = (B^T \hat{K}_0 B + W_R)^{-1} B^T \hat{K}_0 \Phi \tag{6}$$

and $\hat{K}_0$ is a solution to the steady state Ricatti equation

$$\hat{K}_0 = \Phi^T \hat{K}_0 \Phi + W_Q - \Phi^T \hat{K}_0 B (B^T \hat{K}_0 B + W_R)^{-1} B^T \hat{K}_0 \Phi. \tag{7}$$

This gives us a statistically optimal control $u(k)$ for the given cost function with the designer specified parameters W_Q and W_R. Now that the control is optimized, we need to be concerned with the stability of the control design. Stability is assured if the pair (W_Q, Φ) are observable, the pair (Φ, B) are controllable, the Kalman filter is stable, and the model is reasonably good (see [2]). Controllability is the ability to steer the system from an initial state to another state in a finite amount of time, and observability is the ability to determine the state at any time from a finite number of measurements.

SIMULATIONS

Actual data measured from frequency standards at the United States Naval Observatory (USNO) were used in the simulations. An LQG control was applied to the data as if one of the standards frequency was being adjusted by a frequency synthesizer. Thus, the only assumption in the simulations is that the synthesizer works ideally.

The hydrogen maser NAV8 was chosen to be steered to the USNO Mean[6]. The Mean is a paper clock that is calculated using an ensemble of hydrogen masers and cesium frequency standards. Maser NAV8 has excellent short-term stability, but due to the poor environment that it was in during the data collection, its long-term stability suffered. One of the best performing standards at USNO is maser NAV4. Figure 1 shows the performance differences between NAV8 and the Mean versus NAV4. As can be seen, we face an interesting problem of steering maser NAV8 to the Mean in phase and frequency while attempting to preserve the short-term stability of the maser and gain the long-term stability of the USNO Mean. The phase difference between NAV8 and the Mean is given in Figure 2. In order to minimize initial offsets, a frequency offset was removed from the data, and a constant was subtracted out giving the initial phase difference point to be near zero.

In trial 1 we set $W_R = 10^5$ and $W_Q = \begin{vmatrix} .001 & 0 \\ 0 & .001 \end{vmatrix}$, which gives

$$\hat{G}_0 = |\ 6.50277 \times 10^{-5} \quad .57714\ |$$

after solving equations (5) and (6). Figure 3 shows the phase difference between the Mean and NAV8 after steering NAV8 using the above solution. The phase difference is kept very small with the difference having a standard deviation of 140 picoseconds.

In trial 2 we set $W_R = 10^{12}$ and $W_Q = \begin{vmatrix} .001 & 0 \\ 0 & .001 \end{vmatrix}$ which gives

$$\hat{G}_0 = \begin{vmatrix} 3.3185 \times 10^{-8} & .014976 \end{vmatrix}$$

after solving equations (5) and (6). Figure 4 shows the phase difference between the Mean and NAV8 after steering NAV8 using the trial 2 parameters. The phase differences after the initial settling have a standard deviation of 691 picoseconds.

A plot of the two-sample deviation of NAV4 versus the steered NAV8 for both trial 1 and trail 2 parameters is shown in Figure 5. This plot shows that the short-term stability of the maser in trial 1 has been perturbed by the fairly aggressive steering. While for trial 2, the stability exhibits the short-term stability of the maser and excellent performance in the long term.

Another application of the LQG technique is the steering of remote clocks to UTC (USNO) via GPS. Figure 6 shows data obtained between a keyed GPS receiver and Hewlett-Packard HP5071 cesium standard #249. The initial data had 50 nanosecond phase and 4.0×10^{-14} frequency offsets. Also shown in Figure 6 is the phase difference after a simulation run with the LQG control using the parameters of trial 1. We assume that there are two remote clocks being compared by a noisy GPS measurement system. The stability plot in Figure 7 shows how the cesium performed during the steers compared to a hydrogen maser after the initial settling of the controlled system. The solid line on the plot shows the performance specification for the 5071 cesium. The slightly worse stability near 10 hours is most likely due to modelling errors incurred from assuming whiteness of the GPS data.

The parameters chosen for the LQG depend on the systems being used, the desired outcome, and the individual designer. This can be seen in the differences between the results in trial 1 and 2. In trial 1 the short-term stability is sacrificed slightly for a tight control in the differences between the standards. The stability is still good, but if this does not meet the frequency stability needs for a system then the parameters of trial 2 could be used, or any other parameter set that gives the desired results as determined through simulation.

EXPERIMENTAL RESULTS

One of the great concerns in designing a controller is whether or not it will be stable and robust. This was tested by offsetting an external synthesizer, called an Auxiliary Output Generator manufactured by Sigma Tau Standards Corporation, driven by maser NAV2. The phase offset made was approximately 8 milliseconds compared to maser NAV4. Figure 8 shows how the controller with parameters given in trial 1 of the simulations reacted to this phase step that was nearly 7 orders of magnitude greater than would be expected in practice. The system remained stable and brought the signals within 300 picoseconds in approximately 6 days.

Figure 9 shows experimental data of maser NAV2 being steered to the USNO Master Clock using an Auxiliary Output Generator that received its input from a distribution amplifier driven by NAV2. The several hundred picosecond humps in the data are caused by temperature changes in the testing lab where the 5 MHz distribution amplifier with a poor temperature coefficient resides. Temperature control of the lab was poor during the installation of a back-up air conditioning system.

CONCLUSION

The LQG design philosophy is a robust, statistically optimal method for steering frequency standards. Simulations can be run without undue difficulty in order for the designer to characterize how different parameters will affect system responses. This technique could also be used to steer one standard very tightly to another, thus creating an independent back-up that is in phase and on frequency with its reference. Testing is now under way for implementing the LQG technique to synchronize remote systems using the Global Positioning System and two-way satellite time transfer methods.

REFERENCES AND NOTES

[1] M. Athans 1971, *"The Role and Use of the Stochastic Linear Quadratic Gaussian Problem in Control System Design,"* IEEE Trans. Auto. Control, pp. 529-552.

[2] K. Ogata 1987, Discrete-Time Control Systems (Prentice Hall, Englewood Cliffs, New Jersey).

[3] R. Brown, and P. Hwang 1992, Introduction to Random Signals and Applied Kalman Filtering (John Wiley and Sons, New York).

[4] I. Ahn, and R. Brown 1987, *"Assessing the Validity of Suboptimal 2-State Clock Models,"* Proceedings of the 18th Annual Precise Time and Time Interval (PTTI) Applications and Planning Meeting, 2-4 December 1986, Washington, DC, pp. 421-436.

[5] Ahn gives several Q matrix models. For the data used in this paper, we did not find a significant difference in LQG results based on the different models. Simulations should be run in order to determine the best modeling for different applications.

[6] The USNO Mean is presently composed of an ensemble average of 58 frequency standards (10 hydrogen masers and 48 cesium standards).
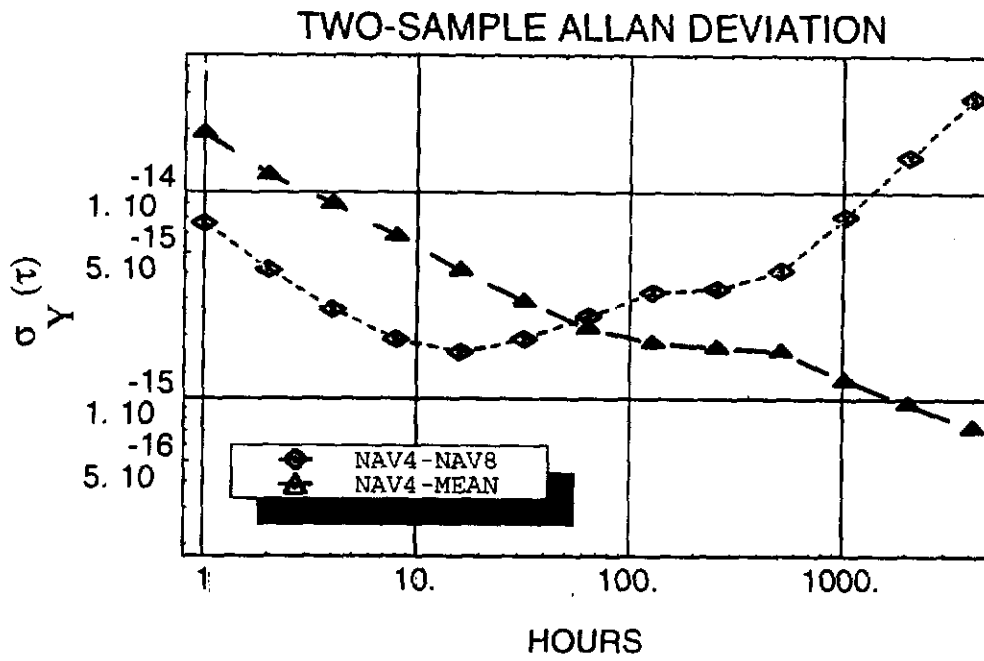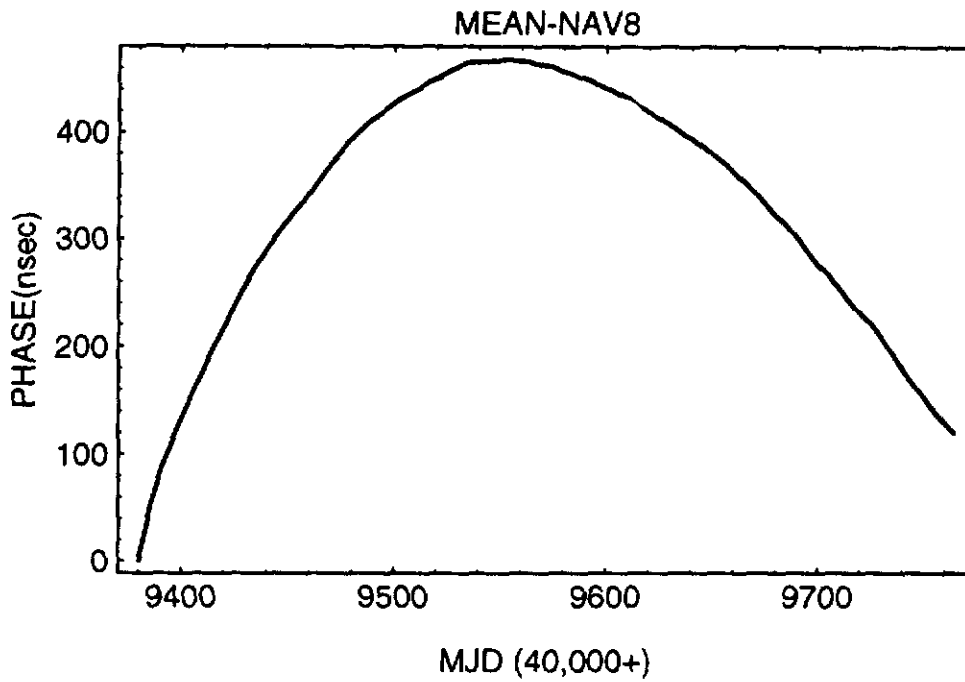
TWO-SAMPLE ALLAN DEVIATION

Figure 1

MEAN-NAV8

Figure 2

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TRIAL 1: MEAN-NAV8(STEERED)

MJD (40,000+)
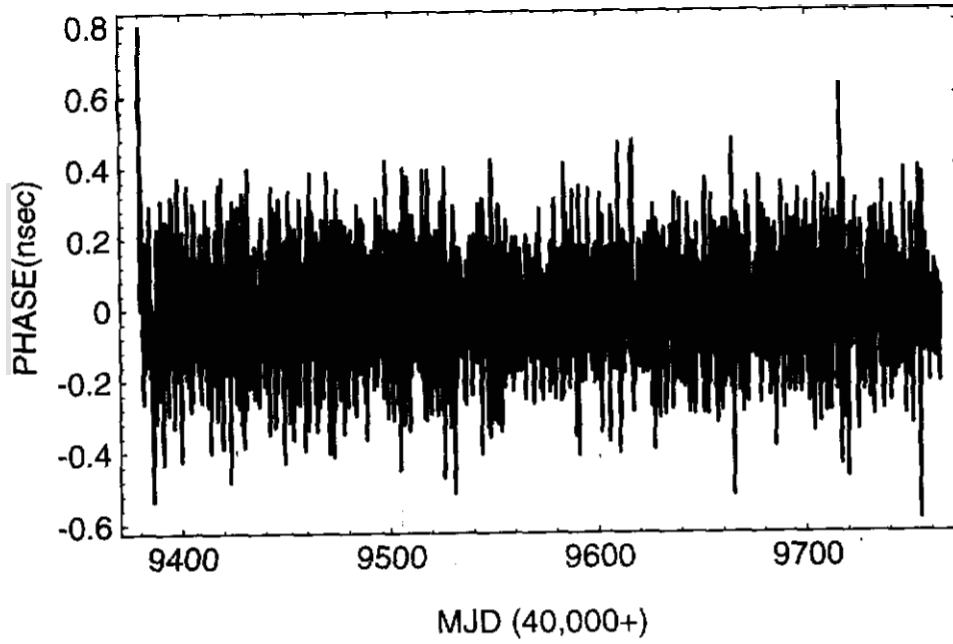
Figure 3

TRIAL 2: MEAN-NAV8(STEERED)

MJD (40,000+)

Figure 4

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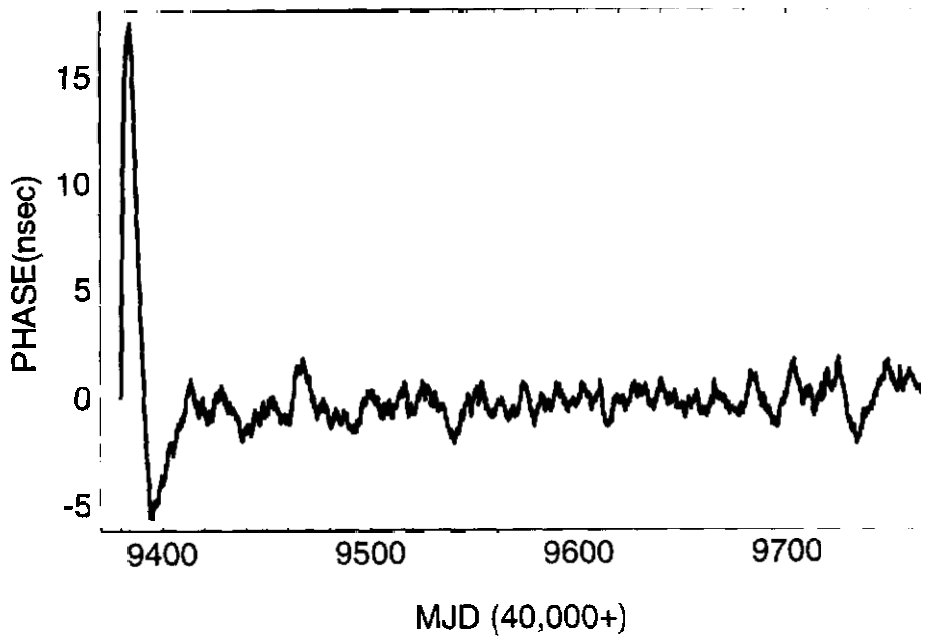
TWO-SAMPLE ALLAN DEVIATION
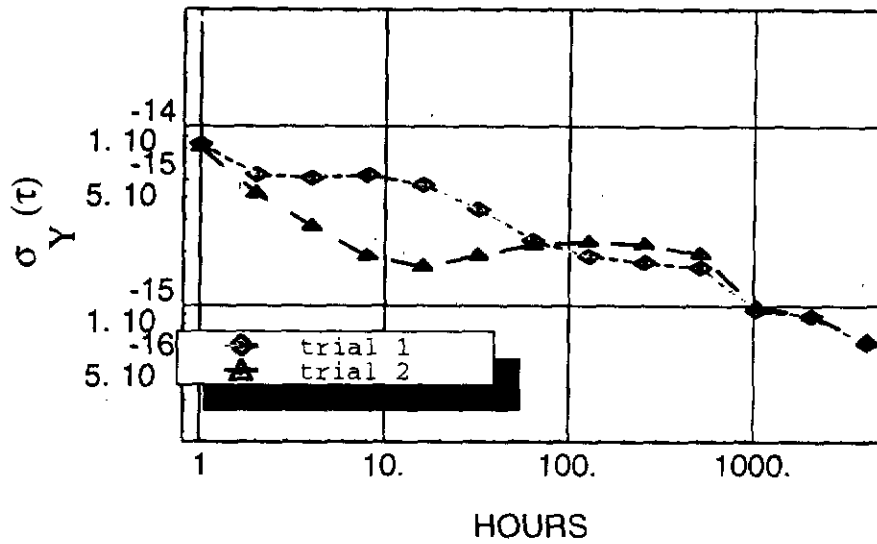## NAV4 - NAV8(STEERED)

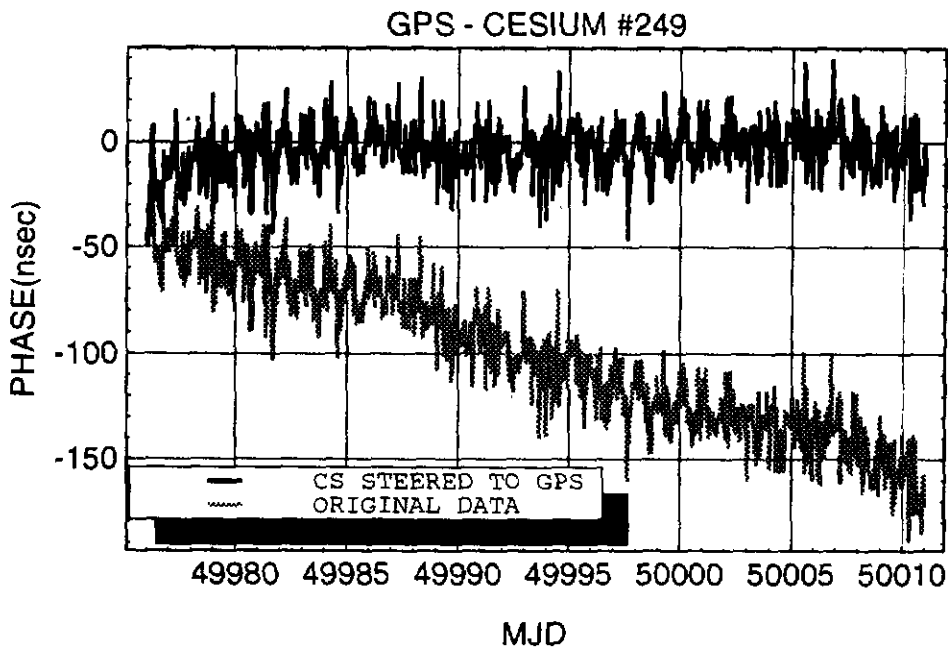

Figure 5

GPS - CESIUM #249

Figure 6

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TWO-SAMPLE ALLAN DEVIATION
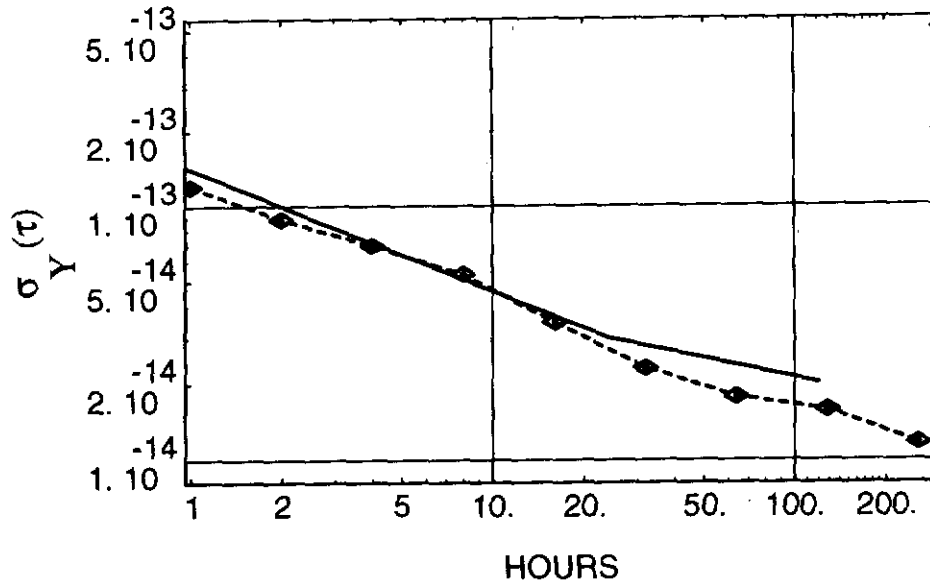## Cs #249(STEERED) VS. HYDROGEN MASER(MC2)

Figure 7

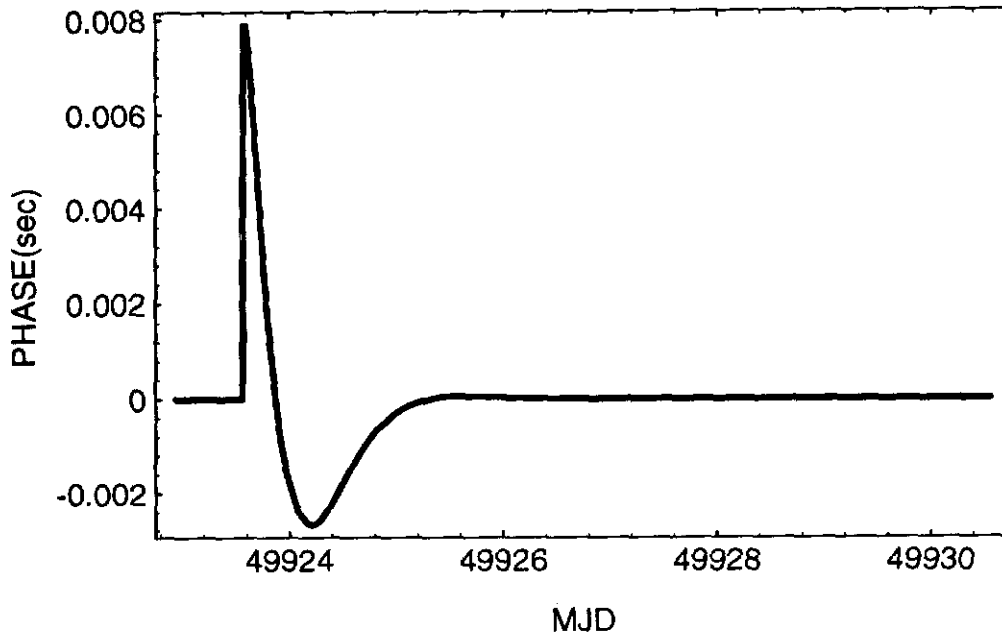## NAV4 - NAV2(STEERED TO NAV4)
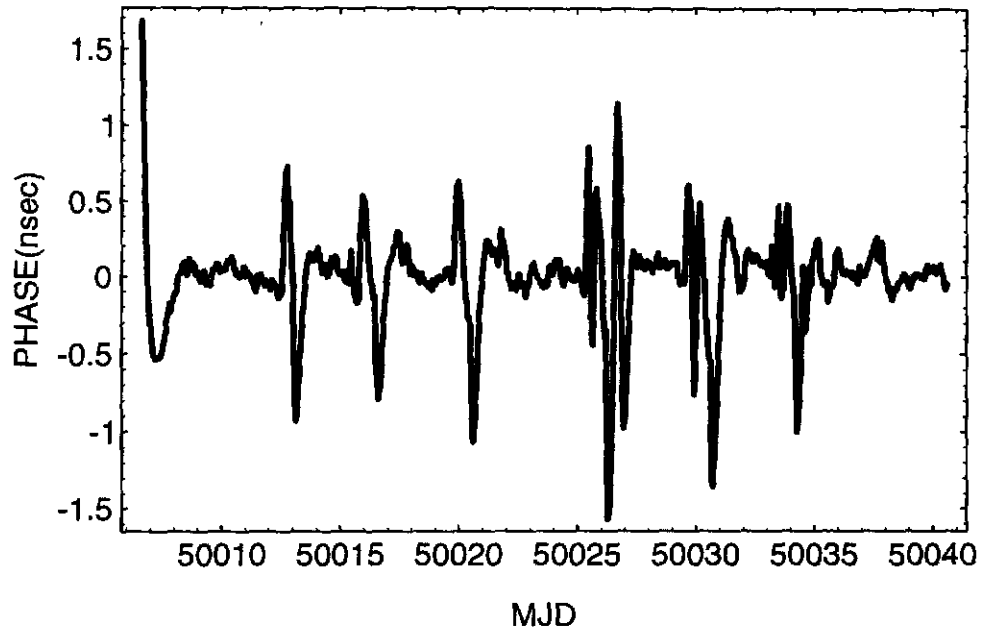

Figure 8

MASTER CLOCK - NAV2(STEERED)

Figure 9

Questions and Answers

ALBERT KIRK (JPL): I have actually three questions. The first one, what are the temperature variations in your laboratory?

PAUL KOPPANG (USNO): They were approximately five to six degrees C.

ALBERT KIRK (JPL): I see. The second question is: What is the smallest step you can use on your synthesizer to correct the frequency?

PAUL KOPPANG (USNO): 10^{-19}

ALBERT KIRK (JPL): The final question is: How do you determine, or how does your system determine, the loop time constant for each maser in response to — you know, to steer the maser to some average that you mentioned here?

PAUL KOPPANG (USNO): That's done by the Kalman filtering; it would steer to a Kalman filter value.

ALBERT KIRK (JPL): Can you select that, then, for each particular maser, depending on its characteristics?

PAUL KOPPANG (USNO): Yes.