Relativistic theory for picosecond time transfer in the vicinity of the Earth

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Abstract. *The problem of light propagation is treated in a geocentric reference system with the goal of ensuring picosecond accuracy for time transfer techniques using electromagnetic signals in the vicinity of the Earth. We give an explicit formula for a one way time transfer, to be applied when the spatial coordinates of the time transfer stations are known in a geocentric reference system rotating with the Earth. This expression is extended, at the same accuracy level of one picosecond, to the special cases of two way and LASSO time transfers via geostationary satellites.*

1. Introduction

It is well known that in relativity the notion of simultaneity is not defined a priori so that a conventional choice of a definition has to be made. This choice will then lead to a corresponding definition of clock synchronization as synchronised clocks must simultaneously produce the same time markers. A widely used definition is that of coordinate simultaneity and corresponding coordinate synchronization, as given, for example, by Klioner (1992):

"Two events fixed in some reference system by the values of their coordinates (t_1,x_1,y_1,z_1) and (t_2,x_2,y_2,z_2) are considered to be simultaneous with respect to this reference system, if the values of time coordinate corresponding to them are equal: $t_1 = t_2$. In the following this definition of simultaneity (and corresponding definition of synchronization) we shall call coordinate simultaneity (and coordinate synchronization)."

Clearly, the synchronization of two clocks by this definition is entirely dependent on the chosen reference system and is thus relative in nature, rather than absolute.

In practice, coordinate synchronization between two distant clocks can be achieved by the exchange of an electromagnetic signal. From the knowledge of the positions of the clocks at emission and reception of the signal in the reference system of synchronization and the laws of light propagation in the same reference system, the coordinate time elapsed during transmission T_t can be calculated.

For the construction and dissemination of international reference time scales, coordinate synchronization in a geocentric, non-rotating (oriented with respect to fixed celestial objects) reference system is required. We choose the geocentric non-rotating reference system as defined by the resolution A4 of the IAU (1992), with asymptotically flat spatial coordinates, but with Terrestial Time TT being the coordinate time. TT is an ideal form of the International Atomic Time TAI, which is the basis of the measurement of time on the Earth. By its definition, TT differs from the coordinate time of the IAU by a constant rate.

The clocks that are to be synchronized are usually fixed on the Earth, and have their spatial positions given in a rotating reference frame. Using the metric equation of the non-rotating system and taking into account the displacement of the clocks (in the non-rotating system) resulting from the relative movement of the two systems during signal propagation, the transmission coordinate time T_t can be calculated.

Recently, the precision of clock synchronization between remote clocks on the surface of the earth has reached the sub-nanosecond level (Hetzel & Soring 1993; Veillet et al. 1992; Veillet & Fridelance 1993) with further improvements expected in the near future. For these applications it seems sensible to develop the theory to the picosecond accuracy level. Recent theoretical studies in this field claim an accuracy of 0.1 nanosecond (Klioner 1992), and in some cases (CCIR 1990, CCDS 1980) the provided formulae are expressed in terms of path-integrals making them more difficult to use than explicit expressions. In this article we provide explicit equations for synchronization in a geocentric non-rotating system of two clocks that have their positions given in the rotating system. All terms that in the vicinity of the Earth (within a geocentric sphere of 200000 km radius) are greater than one picosecond are included. Outside this sphere terms due to the potential of the Moon may amount to more than 1 ps and need to be accounted for separately. We also present formulae (to the same accuracy) for the special cases of two way time transfer (section 3) and LASSO (LAser Synchronization from Stationary Orbit, section 4) time transfers via a geostationary satellite. Here a possible small residual velocity of the satellite (< 1 m/s) results in further terms contributing some tens of picoseconds for two way- and LASSO time transfers.

We will assume that all clocks are rate corrected for the gravitational potential at their positions, and their velocities in the reference system of synchronization, and hence run at the rate of TT.

2. Formula for a one way transfer

We consider a rotating frame $(t,\bar{x}_r)$ which rotates at a constant angular velocity ω with respect to a fixed star oriented one $(t,\bar{x})$ with two clocks a and b, at $\bar{x}_{ra}$ and $\bar{x}_{rb}$ at time t_0 when the two frames coincide. The two clocks are to be coordinate synchronized by the transmission of an electromagnetic signal from a (emission at t_0) to b (reception at t_1).

To this end the coordinate time interval $T_t = t_1 - t_0$ elapsed between emission and reception of the signal needs to be calculated.

The metric of a geocentric non-rotating system in the first post-Newtonian approximation with TT as coordinate time and asymptotically flat spatial coordinates (for $r \to \infty$ the components of the spatial metric $g_{ij} = \delta_{ij}$) is:

$$ds^2 = -(1 - 2U/c^2)(1 + L_g)^2c^2dt^2 \\ +(1 + 2U/c^2)(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2) \quad (1)$$

where: ds is the relativistic line element.
t is the coordinate time TT.
r, θ (colatitude), ϕ (longitude) are spherical coordinates in a non-rotating geocentric coordinate system.
U is the gravitational potential of the Earth (positive sign).
$L_g = 6.969291 \times 10^{-10}$.

The scaling factor $(1 + L_g)$ results from the choice of TT as coordinate time. L_g is equal to U_g/c^2, where U_g is the value of the gravitational potential on the geoid including the centrifugal potential due to the rotation of the Earth.

Introduction of post-post-Newtonian terms and terms due to the tidal potentials of the moon, sun and planets into the metric leads to a correction to the propagation time of a light signal in the vicinity of the Earth of less than one picosecond. Hence (1) is sufficient for our purposes.

Transforming to the rotating frame by

$$d\phi = \omega dt + d\phi_r \quad (2)$$

then setting $ds^2 = 0$ for a light signal and solving the resulting quadratic for dt provides an expression for the transmission coordinate time T_t:

$$T_t = \int \{du/c - U_g du/c^3 + \omega r^2 \sin^2\theta d\phi_r/c^2 + [1+r^2\sin^2\theta(d\phi_r/du)^2]\omega^2 r^2 \sin^2\theta du/2c^3 + 2U du/c^3\} + O(c^{-4}), \quad (3)$$

where du is the increment of coordinate length along the transmission path and the integral is to be taken from a to b along the transmission path in the rotating frame.

Evaluating the above expression (for a detailed derivation see Petit & Wolf (1993)) gives an explicit formula for the transmission time:

$$T_t = R_0/c + \delta$$

$$= R_0/c - U_g R_0/c^3 + \bar{R}_0.\bar{v}_b/c^2 + (v_b^{\,2} + \bar{R}_0.\bar{a}_b + (\bar{R}_0.\bar{v}_b)^2/R_0^{\,2})R_0/2c^3 + 2GM_E \ln\{[x_{rb} + n.\bar{x}_{rb}]/[x_{ra} + n.\bar{x}_{ra}]\}/c^3 \quad (4)$$

where:

δ is the total relativistic correction.
$\bar{R}_0 = \bar{x}_{rb} - \bar{x}_{ra}$
$\bar{v}_b = \bar{\omega} \times \bar{x}_{rb} + \bar{v}_{rb}$
$\bar{a}_b = \bar{\omega} \times (\bar{\omega} \times \bar{x}_{rb}) + \bar{\omega} \times \bar{v}_{rb} + \bar{a}_{rb}$
$n = \bar{R}_0/R_0$ is the unit vector along the transmission path
$\bar{v}_{rb}$ is the satellite velocity in the rotating frame
$\bar{a}_{rb}$ is the satellite acceleration in the rotating frame
and the two frames coincide at $t = t_0$.

The above expression provides the coordinate transmission time for a light signal travelling from station a to station b in the vicinity of the Earth (within a geocentric sphere of 200000 km radius) with the coordinates of the two stations given in an Earth fixed rotating frame. All terms that are greater than one picosecond are included. Note however, that atmospheric delays which can amount to several tens of nanoseconds are not considered and need to be taken into account separately.

3. Two way time transfer

We consider a two way time transfer between two stations c and d, fixed on the surface of the Earth, via a geostationary satellite s (as shown in Fig. 1).

Two signals are transmitted in opposite directions leaving c and d at t_0 and $t_0+\Delta t$ respectively. They reach the satellite at t_1 and t_3, where they are immediately retransmitted, and arrive at the opposite stations at t_2 and t_4. From the clocks two coordinate time intervals are obtained (assuming that the clocks are rate corrected as mentioned in section 1):

$$t_c = t_4 - t_0$$
$$t_d = t_2 - t_0 - \Delta t \qquad (5)$$

For synchronization the interval Δt is required. We shall assume that the clocks have been synchronized previously to within 0.1 s, a typical station to satellite transmission time (which can be achieved without difficulty in practice), and that the satellite has a residual velocity v_r smaller than 1 m/s and a residual acceleration in the rotating frame of less than 10^{-5} m/s^2. These values have been chosen as typical after consultation of the EUTELSAT satellite control centre.

The defining equations for the transmission times are:

$$T_1 = t_1 - t_0$$
$$T_2 = t_2 - t_1$$
$$T_3 = t_3 - t_0 - \Delta t$$
$$T_4 = t_4 - t_3 \qquad (6)$$

and solving for Δt yields:

$$\Delta t = (t_c - t_d)/2 + \delta$$

$$\delta = (T_1 + T_2 - T_3 - T_4)/2 \qquad (7)$$

The relativistic correction δ arises from the motion of the stations and the satellite in the frame of synchronization and the gravitational delays for the individual transmissions T_1 to T_4.

Using equation (4) to calculate T_1 to T_4 and substituting the results into (7) gives an expression for the relativistic correction (Petit & Wolf (1993)):

$$\delta = \{\bar{R}_{cd}.(\bar{\omega} \times \bar{x}_{rs}) + [(R_{cs} - R_{ds} - c\Delta t)(R_{ds}\bar{R}_{cs} + R_{cs}\bar{R}_{ds}).\bar{v}_r] / (2R_{cs}R_{ds})\}/c^2 + O((v/c)(v_r/c)\Delta t) \qquad (8)$$

where:

$$\bar{R}_{cs} = \bar{x}_{rs} - \bar{x}_{rc}$$
$$\bar{R}_{ds} = \bar{x}_{rs} - \bar{x}_{rd}$$
$$\bar{R}_{cd} = \bar{x}_{rd} - \bar{x}_{rc}$$

The first term is equivalent to $2\omega A_E/c^2$ with A_E being the equatorial projection of the area of the quadrangle whose vertices are the centre of the Earth and the positions of the satellite and the stations in the rotating frame.

The second term of (8) varies with v_r and Δt, and can amount to several hundred picoseconds. If $\Delta t \sim 0$, it can amount to several tens of picoseconds, depending on the residual velocity which is in general not well known. However, one can compensate for it by *intentionally introducing a desynchronisation in order to drive this term towards zero,* which is the case when the two signals arrive at S at about the same time (ie. $t_1 \approx t_3$).

4. LASSO

In this method laser pulses emitted from the stations c and d at t_0 and $t_0+\Delta t$ respectively are reflected by the geostationary satellite and return to the stations (as shown in fig. 2).

The satellite is equipped with a clock which measures the time interval between arrival of the signals. Hence three coordinate time intervals (after rate correction of the clocks) are obtained:

$$\begin{aligned} t_c &= t_2 - t_0 \\ t_d &= t_4 - t_0 - \Delta t \\ t_s &= t_3 - t_1 \end{aligned} \tag{9}$$

For synchronization Δt is required. Similarly to the two way case, the defining equations (6) for T_1 to T_4 yield:

$$\Delta t = (t_c - t_d)/2 + t_s + \delta$$

$$\delta = (T_1 - T_2 - T_3 + T_4)/2 \tag{10}$$

Using (4) to calculate the individual transmission times T_1 to T_4 gives for the relativistic correction (Petit & Wolf (1993)):

$$\begin{aligned} \delta = \quad & [\bar{R}_{cd}.(\bar{\omega} \times \bar{x}_{rs}) + \Delta t(\bar{\omega} \times \bar{v}_r).\bar{x}_{rd}]/c^2 \\ & + O((v/c)(vr/c)(R_0/c)) \end{aligned} \tag{11}$$

As in (8) the first term is equivalent to $2\omega A_E/c^2$.

The second term varies with $\bar{v}_r$ and Δt. This term is smaller than 10^{-2} ps for $\bar{v}_r \sim 1$ m/s and $\Delta t \sim 0.1$ s, which is the case for a two way transfer and hence it does not appear in (8). However, for LASSO Δt can amount to several minutes in practice (Veillet et al. 1992; Veillet & Fridelance 1993) and therefore the second term in (11) can contribute up to 10 ps.

Note also that while the second term of (8) can be minimised by an appropriate choice of Δt, this is not the case in (11).

5. Constraints for practical applications

For picosecond accuracy, the relativistic correction δ contains terms in c^{-2} and in c^{-3} in the case of one-way time transfers (4), and terms in c^{-2} only in the case of two-way (8) and LASSO (11) transfers.

The term in c^{-2} can amount to a few hundred nanoseconds, depending on the relative positions of the transmission and reception points. For example, between the Earth and a geostationary orbit, the maximum value is about 200 ns for the one way- and 400 ns for the two way case. In order to compute this term with picosecond accuracy, it is sufficient for all quantities in the term in c^{-2} to be known with a relative uncertainty of one or two parts in 10^6. This requires coordinates known to within 6-12 m for the Earth stations, including uncertainties in the realization of the reference frame which are below ~1 m for e.g. WGS84 and ITRF. This is generally the case for time laboratories. The satellite position should be known to within some tens of metres, depending on its orbit, and this is generally not the case a priori for a satellite without geodesic objectives. In addition the velocity of the satellite should be known to the same relative uncertainty of one or two parts in 10^6, which is also not the case in general. Typically the position of a geostationary satellite is known to an accuracy of ~1 km which results in an error in the computation of the c^{-2} term of ~10 ps. Similar arguments can be made to set constraints in the case of higher orbits or satellite to satellite time transfers.

In the real case of a non-perfect geostationary orbit, the constraint on the knowledge of the velocity of the satellite is transferred to the residual velocity v_r. For the one way and two way techniques, this constraint is about 1 cm/s for picosecond accuracy but in the two way technique it can be completely relaxed by an intentional desynchronisation of the emission of the signals at the two stations, as mentioned in section 3. For LASSO, the constraint on v_r is about 10 cm/s if one wishes to use laser pulses from the two stations separated by Δt of several minutes. The constraint on v_r can be relaxed by severing that on Δt.

When one of the stations is on the Earth, propagation through the atmosphere is the major problem for one way time transfer. It leads to delays that can reach several tens of nanoseconds and can certainly not be calibrated to picosecond accuracy. This problem is not considered in this study. However the effects cancel to the picosecond level in the two way (provided the up and down frequencies are close enough) and LASSO techniques.

6. Conclusion

We have derived the relativistic correction for a one way time transfer between two stations that have their position given in a geocentric reference frame rotating with the Earth (equation (4)) including all terms in c^{-3} and larger. For time transfer with a geostationary satellite the terms in c^{-3} can amount to around 10 ps for the Sagnac correction and 80 ps for the gravitational delay. At present, one way time transfers are not accurate enough to necessitate the consideration of these terms. However, with accuracy expected to increase in the near future, and in view of possible satellite to satellite transfers (which would eliminate uncertainties due to atmospheric delays) these terms might well become significant.

We also provided expressions for the relativistic corrections that need to be applied to two way and LASSO techniques. We have shown that the main errors in computing these corrections are due to the uncertainties in the position and the residual velocity of the satellite. The uncertainty in the position leads to an error in the computation of $2\omega A_E/c^2$ of the order of 10 ps for both techniques. The uncertainty in the residual velocity affects the two techniques differently. For LASSO the second term in (11) is typically of the order of 10 ps, hence reducing the overall uncertainty for LASSO requires better knowledge of the satellite position as well as consideration of the additional term. For two way time transfers, on the other hand, the second term in (8) can reach 80 ps (for $\Delta t = 0$). Hence reducing this term by an appropriate choice of Δt will improve the overall accuracy of the two way time transfer even in the case where $\bar{v}_r$ is unknown.

In both techniques, the precision of experiments repeated over periods of several weeks could be affected by the variation of the residual velocity of the satellite, if the corresponding terms are not accounted for.

This shows that the time community is rapidly approaching levels of precision and accuracy that will necessitate a more exact development of the theory. We consider the present paper a step in that direction.

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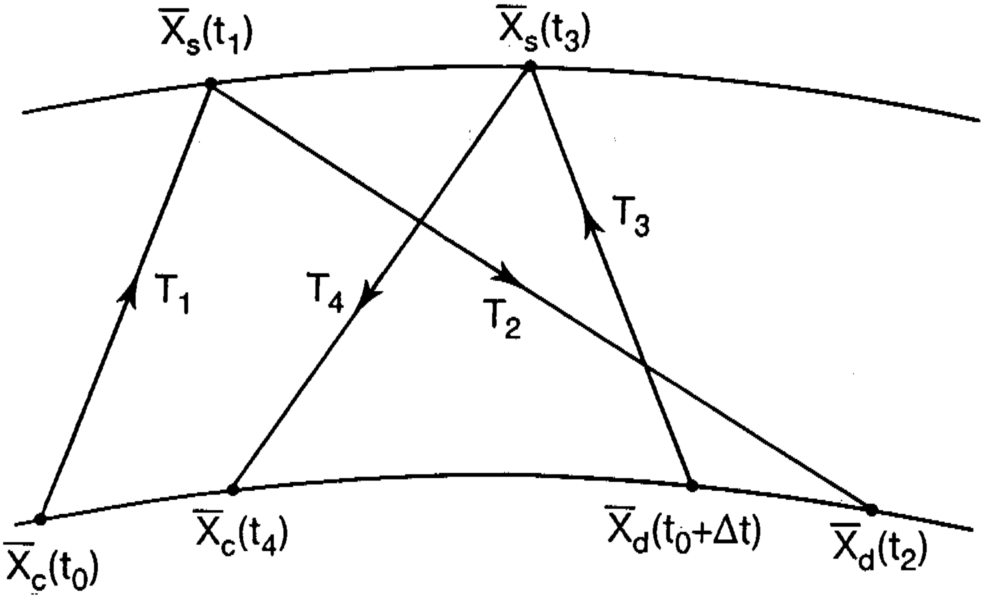

Fig. 1. Two way time transfer in the non-rotating frame. Two signals are transmitted in opposite directions leaving c and d at t_0 and $t_0+\Delta t$ respectively. They reach the satellite at t_1 and t_3, where they are immediately retransmitted, and arrive at the opposite stations at t_2 and t_4.

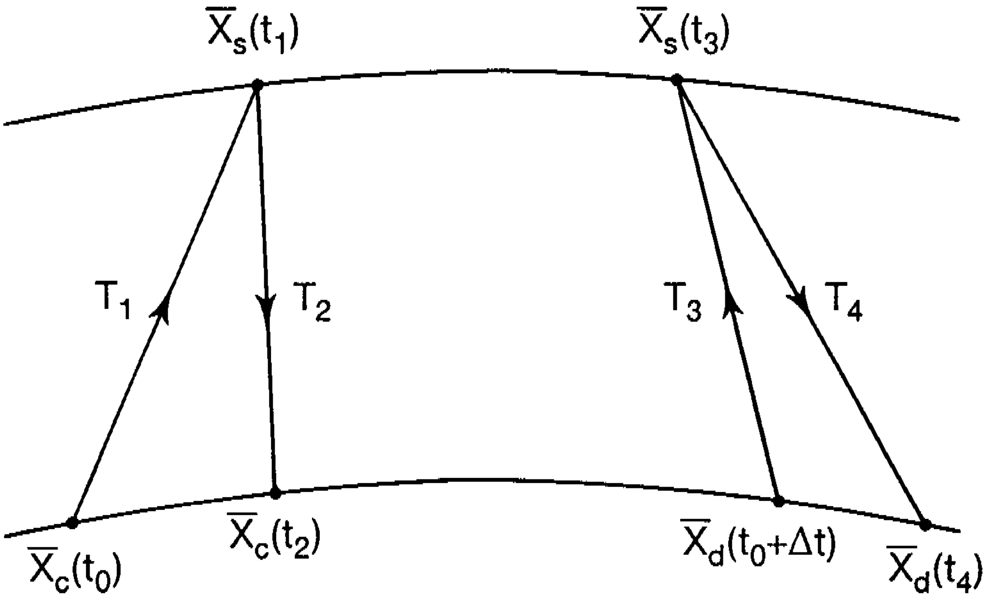

Fig. 2. LASSO time transfer in the non-rotating frame. Laser pulses emitted from the stations c and d at t_0 and $t_0+\Delta t$ respectively are reflected by the geostationary satellite and return to the stations. A clock on board the satellite measures the time interval between arrival of the pulses.

QUESTIONS AND ANSWERS

Dieter Kirchner, TUG: One comment to the microsecond issue, from the practical point of view it is not easy to hold this 10 microseconds. Because a well kept geostationary satellite has a range rate of many tens of microseconds. So you would have to change your offset for each measurement. And this is not very convenient to do.

Peter Wolf: What was that first bit? What varies several microseconds per day?

Dieter Kirchner: The range of a geostationary satellite with respect to a station. If you measure the range to a geostationary satellite, this range changes.

Peter Wolf: That is quite right. That is the problem that you don't know its position exactly all the time. It kind of moves, which simply changes the distance.

Dieter Kirchner: If you measure the range to the satellite, you have a figure which changes and may be 100 to 150 microseconds.

Peter Wolf: It is a trade–off. You have to do it one way or the other if you want to be more precise. Either you manage to change your offset for every measurement so you can get rid of the additional term, or you get some knowledge on the velocity of the satellite; and then you can calculate the additional things. However the velocity of the satellite – as far as I know, what they do with geostationary satellites, they have them sort of wandering about a kind of observation window; as soon as it approaches the edge, they give it a boost to go back where it belongs. So it sort of varies quite a bit. I'm not sure. If you know the velocity, you are fine. If you don't, you have to get around it.

David Allan, Allan's Time: Regarding GPS time with the laser retro–reflectors, then we can also calibrate that path which should help in the uncertainty, at the sub–ns level I believe.

Peter Wolf: Of course, yes.