

CHARACTERIZATION OF FREQUENCY STANDARD INSTABILITY BY ESTIMATION OF THEIR COVARIANCE MATRIX

Patrizia Tavella
Istituto Elettrotecnico Nazionale G. Ferraris
Strada delle Cacce, 91
10135, Torino, Italy

and

Amedeo Premoli
Dip. di Elettrotecnica, Elettronica ed Informatica
Universita' di Trieste
Via A. Valerio, 10
34127, Trieste, Italy

Abstract

The popular 3-cornered hat method used for evaluating the noise contributions of individual frequency standards is revisited. This method is used in several cases, but sometimes the results are not consistent because one or more estimated clock variances turn out to be negative. Different causes of this unacceptable result have been conjectured: among them one regards the hypothesis of uncorrelated clocks, essential in this method. Since recently realistic cases of correlation between clocks, mainly due to the environmental conditions, have been observed, this paper proposes an entirely revisited version of the 3-cornered hat method which permits to evaluate the individual variances and also the possible covariances between clocks, by relaxing the hypothesis of uncorrelation. The uncertainty and the lack of contemporaneity of the measurement series are assumed to be negligible. The lack of the uncorrelation hypothesis calls for a more general mathematical model leading to an underdetermined linear system. The estimates of the (co)variances of the measurement series as well as those of the individual clocks are introduced by means of the scalar product of the related time series and arranged in the respective covariance matrices S and R . Since covariance matrix is positive definite by definition, the problem consists in estimating the unknown R , subject to the constraint of positive definiteness, from the known S . Unfortunately, this constraint is not sufficient to estimate R . Therefore a suitable optimization criterion is proposed, which assures the positive definiteness of R and, at the same time, minimizes the global correlation among clocks. Examples of frequency instability measurements processed by the "classical" 3-cornered hat method and the here-revisited method are presented showing that the solutions are identical only when the uncorrelation hypothesis doesn't violate the positive definiteness of R .

1 INTRODUCTION

The evaluation of frequency standards is performed by comparing two or more of them and measuring differences in their signals. Results depend on the simultaneous contributions of all the standards and it is often desirable to estimate the noise contributions of the single units. In the past years, this problem has been considered in several papers, which introduced the popular "3-cornered hat" method [1], successively extended to N clocks [2,3,4] and further investigated in [5,6]. *In all the above papers, a basic hypothesis consists in considering all the clocks uncorrelated.*

In the classical 3-cornered hat method, three clocks are considered, three series of time differences between all the possible pairs are measured, and their variances are estimated. Three linear equations are then written, which tie the three unknown variances of the single clocks to the known variances of the time differences. In this way, a uniquely solvable system is obtained [1]. If more than three clocks are compared, such a system becomes overdetermined because the number of possible pairs exceeds the number of clocks. In this case, it has been suggested either to deal with different triads of clocks and then combine the results in a (weighted) average [1], or to utilize the N clocks together with a least squares technique [2,3].

Independently of the number of clocks, a more crucial problem arises: the estimated clock variances can turn out to be negative. In such a case it has been suggested to use the absolute value [2] or to consider vanishing a variance that should turn negative. However these tricks are not justified by any theoretical consideration.

Several questions related to the statistical processing of these measurements and the causes of negative estimated variances are still open:

1. Uncertainty in the measured time differences.

If the noise of the measuring device is not negligible, it adds a term in the variance of the measured time differences and the linear system is not longer deterministically solvable [4,5]. To the authors' knowledge, this uncertainty is negligible in most cases, particularly in high resolution measurements. The case can be different if the clocks are compared at a distance, through a synchronization link, but the synchronization noise, usually corresponding to a white phase noise, can often be suitably modeled and filtered.

2. Lack of contemporaneity of measurements.

In this case, the contribution of each clock cannot be considered the same in each difference measurement [5,6]. However, the lack of contemporaneity of the different measurements is negligible when the integration times over which the stability is to be estimated are far longer (days) than the shift in time of the beginning of the different measurements (seconds).

3. Low number of measured samples.

In this case, a statistically significant characterization of the involved noises is not ensured [5,10]. The low number of statistical samples remains an open question because it gives a low confidence on the estimates and particular care is to be paid.

4. Correlation between clocks.

In recent years cases of correlation between clocks, mainly due to the environmental conditions, have been detected [8-18]. Different methods have been used to evidence correlation between clocks and the discussion is still lively also in understanding which is the clock component responsible for the effect but, in each case, an appreciable presence of correlation between clock data has been pointed out.

To the authors' knowledge, cases of negative estimated variances appear even when there are several measured samples and causes 1) and 2) above are certainly to be excluded. Usually, the problem appears over long integration times (months) where correlated noise can become significant but, in the same time, not easy to be modeled and previously deperated [18].

This work lifts the assumption of uncorrelation, all in considering negligible the causes indicated in points 1 and 2 above, and proposes a new method which formulates an underdetermined but consistent system of equations involving variances and covariances (jointly denoted as (co)variances) between individual clocks. In order to estimate clock (co)variances, the (co)variances of the measure series are also introduced and arranged in positive definite covariance matrices, implicitly assuring the positiveness of the variances. With 3 clocks the uncorrelation hypothesis leads to a uniquely solvable linear system, while the lack of this hypothesis leads to an underdetermined linear system of three equations in six unknowns. A method to solve this underdetermined system, subject to the constraint of positive definiteness of the clock covariance matrix, is proposed.

2 STATEMENT OF THE PROBLEM

The statistical tool useful to characterize stability is the variance estimated by means of the available measured data. Let us denote x^i the signal of the i -th clock and $x_k^i (k = 1, 2, \dots, M)$ its samples at the time instants t_1, t_2, \dots, t_M . The M samples can be represented as the vector $\bar{x}^i = [x_1^i \ x_2^i \ \dots \ x_M^i]^T$, where superscript T denotes transposition. The estimate of the expected value of x^i is

$$\bar{x}^i = \mathcal{E}[x^i] = (1/M) \sum_{k=1}^M x_k^i \quad (1)$$

which is arranged into a vector of M coincident elements $\mathbf{x}^i = [\bar{x}^i \ \bar{x}^i \ \dots \ \bar{x}^i]^T$. With these notations the estimated (co)variances r_{ij} of x^i and x^j are:

$$r_{ij} = \mathcal{E}[(x^i - \bar{x}^i)(x^j - \bar{x}^j)] = [1/(M-1)](\mathbf{x}^i - \bar{\mathbf{x}}^i)^T(\mathbf{x}^j - \bar{\mathbf{x}}^j) \quad i, j = 1, 2, 3 \quad (2)$$

When $i = j$, r_{ij} represents the variance of the i -th signal, otherwise, it represents the covariance between the i -th and j -th signals.

In the case of frequency standards, measured data are often filtered, for instance, introducing the Allan-variance. In the following, the general case of a signal x^i will be dealt with, whatever may have been its previous filtering, in order to obtain a procedure applicable in all cases.

In clock stability characterization, the physical quantities involved in \mathbf{x}^i are time deviations of the i -th clock. Since they are not directly measurable, the clock (co)variances r_{ij} play the role of the unknowns to be evaluated. The available measured quantities are differences between the signals of pairs of clocks: $y^{ij} = x^i - x^j$. When one of the three clocks, for instance clock #3, is chosen as the reference and it is compared at M different instants with clocks #1 and #2, two distinct measure vectors $\mathbf{y}^{13} = \mathbf{x}^1 - \mathbf{x}^3$ and $\mathbf{y}^{23} = \mathbf{x}^2 - \mathbf{x}^3$ are obtained.

The novelty here is that not only the variances of the signals y^{13} and y^{23} are estimated but also their covariance. This covariance was already suggested in [4,5]; however, full advantage of it could

not be taken in that work, because of the uncorrelation hypothesis. The estimates of the above measure (co)variances are:

$$s_{ij} = [1/(M - 1)][(\mathbf{y}^{i3} - \bar{\mathbf{y}}^{i3})^T(\mathbf{y}^{j3} - \bar{\mathbf{y}}^{j3})] \quad i, j = 1, 2 \quad (3)$$

where the index 3 of the reference clock has been dropped in s_{ij} . Since s_{ij} ($j = 1, 2$) and r_{ij} ($i = 1, 2, 3$) represent estimates of variances and they are sums of squares, they are necessarily positive. On the contrary, s_{ij} and r_{ij} ($i = 1, 2, 3$) may be either positive or negative, being estimates of covariances.

In case of noiseless measurements, when the covariance s_{12} is taken into account, the other possible difference measure vector $\mathbf{y}^{12} = \mathbf{x}^1 - \mathbf{x}^2$ and the related (co)variances don't add any information because they all can be obtained as linear combinations of s_{11} , s_{22} and s_{12} . In the 3-cornered hat method, the vector \mathbf{y}^{12} and the related variance is used instead of the covariance \mathbf{y}^{12} , but, in this context, the use of s_{12} is more appropriate.

The 2x2 covariance matrix \mathbf{S} and the 3x3 covariance matrix \mathbf{R} are defined as follows:

$$\begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix} \quad (4)$$

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{12} & r_{22} & r_{23} \\ r_{13} & r_{23} & r_{33} \end{bmatrix} \quad (5)$$

Substituting the definition $\mathbf{y}^{i3} = \mathbf{x}^i - \mathbf{x}^3$ ($i = 1, 2$) in (3) leads to the following relationship between \mathbf{S} and \mathbf{R} :

$$\begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix} = \begin{bmatrix} r_{11} + r_{33} - 2r_{13} & r_{12} + r_{33} - r_{13} - r_{23} \\ r_{12} + r_{33} - r_{13} - r_{23} & r_{22} + r_{33} - 2r_{23} \end{bmatrix}. \quad (6)$$

From measure vectors \mathbf{y}^{13} and \mathbf{y}^{23} , three independent estimated (co)variances, s_{11} , s_{22} and s_{12} are calculated and, according to (6), they tie the six unknowns r_{11} , r_{12} , r_{13} , r_{22} , r_{23} and r_{33} in three independent equations.

Under the hypothesis of uncorrelated clocks, (6) simplifies to:

$$\begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix} = \begin{bmatrix} r_{11} + r_{33} & r_{33} \\ r_{33} & r_{22} + r_{33} \end{bmatrix} \quad (7)$$

By inspection of (7), it can be seen that the uncorrelation hypothesis is acceptable only if matrix \mathbf{S} verifies the following conditions ensuring the positiveness of the estimated variances r_{11} , r_{22} and r_{33} .

$$\begin{aligned} s_{12} &> 0 \\ s_{12} &< s_{11} \\ s_{12} &< s_{22} \end{aligned} \quad (8)$$

In such a case, the solution of (7), formally different but equivalent to that of the classical 3-cornered hat method [1], is:

$$\begin{aligned} r_{33} &= s_{12} \\ r_{11} &= s_{11} - s_{12} \\ r_{22} &= s_{22} - s_{12} \end{aligned} \tag{9}$$

Moreover, (7) reveals that if the third (reference) clock is "quasi-ideal" ($r_{33} \ll r_{11}$ and $r_{33} \ll r_{22}$), then $s_{12} \ll s_{11}$ and $s_{12} \ll s_{22}$ and \mathbf{S} , as well as \mathbf{R} , can be considered diagonal. So, if \mathbf{S} is almost diagonal, the reference clock is of high quality and by changing reference clock we can get an idea of which of the clock is less noisy because it will result in a matrix \mathbf{S} with minimum off-diagonal terms. On the contrary, when the values of s_{11} , s_{22} and s_{12} are close, the variance r_{33} is dominant with respect to the other variances of \mathbf{R} .

Conditions (8) do not assure the uncorrelation of all the clocks, because many different matrices \mathbf{R} can be associated to the same matrix \mathbf{s} and only one of them is diagonal. In any case, (8) suggest that the uncorrelation assumption is reasonable. If one of (8) is violated, the classical 3-cornered hat method cannot be applied and the complete (non-diagonal) matrix \mathbf{R} must be taken into account.

As stated above, the matricial equation (6) is underdetermined. Some more reasonable requests are to be added in such a way as to fix the extra parameters and obtain estimates for the unknown elements of \mathbf{R} . Supposing to know somehow the three (co)variances r_{13} , r_{23} and r_{33} involving the reference clock, the other (co)variances r_{11} , r_{22} and r_{12} can be uniquely calculated. In fact, from (6), the following expressions are obtained:

$$\begin{aligned} r_{11} &= s_{11} - r_{33} + 2r_{13} \\ r_{12} &= s_{12} - r_{33} + r_{13} + r_{23} \\ r_{22} &= s_{22} - r_{33} + 2r_{23} \end{aligned} \tag{10}$$

In order to fix the values of the free parameters r_{13} , r_{23} and r_{33} an appropriate criterion ought to be formulated but there is an important constraint which bounds the solution domain and which guarantees a significant result: *the estimated covariance matrix \mathbf{R} must be positive definite.*

In fact, by means of their definitions both \mathbf{S} and \mathbf{R} as any covariance matrix are positive definite. Such a property does not depend on the number M of samples used in the estimation of the covariance matrix and it is shared by all the matrices defined as the product of a matrix times its transpose. For this reason the treatment here exposed is independent of the particular statistical tool used to estimate stability, it holds either for the variance as in (2) or for a different process as the commonly used two-sample variance. The positive definiteness of the covariance matrix implies, as a particular case, the positiveness of its diagonal elements, i.e. the variances.

The scalar conditions ensuring positive definite matrices regard the positiveness of the leading minors but, since matrices \mathbf{R} and \mathbf{S} are linked by (6), the positive definiteness of the unknown matrix \mathbf{R} is ensured by a unique scalar condition according to the following property:

Property 1: The 3x3 matrix \mathbf{R} , with arbitrary r_{13} , r_{23} , r_{33} and with r_{11} , r_{22} , r_{12} obtained from

the positive definite 2x2 matrix \mathbf{S} according to (10), is positive definite if and only if the determinant of the matrix \mathbf{R} , denoted $|\mathbf{R}|$, is positive, i.e.:

$$|\mathbf{R}| = r_{11}r_{22}r_{33} + 2r_{12}r_{23}r_{13} - r_{13}^2r_{22} - r_{12}^2r_{33} - r_{23}^2r_{11} > 0 \quad (11)$$

The proof is reported in [19]. It is also interesting to note [19] that the condition (11) would allow $r_{12} = r_{13} = r_{23} = 0$ only if the same conditions (8) above are satisfied.

The positive definiteness of matrix \mathbf{R} can be geometrically interpreted. To begin with, let's regard r_{33} as a known parameter; the necessary and sufficient condition (11) can be rearranged as:

$$s_{22}(r_{13} - r_{33})^2 - 2s_{12}(r_{13} - r_{33})(r_{23} - r_{33}) + s_{11}(r_{23} - r_{33})^2 < r_{33}|\mathbf{S}| \quad (12)$$

where the (co)variances r_{11} , r_{22} , r_{12} have been substituted by (10). This expression describes the area inside an ellipse in the plane r_{13} , r_{23} . The center is in the point of coordinates (r_{33}, r_{33}) . The direction of the principal axes depends only on \mathbf{S} and does not depend on r_{33} , because the coefficients of the quadratic terms are independent of r_{33} . The positive definiteness of \mathbf{R} is then fulfilled when the choice of the parameters (r_{13}, r_{23}) corresponds to a point inside this ellipse (for a given value of r_{33}). Fig. 1 illustrates several ellipses depending on different values of r_{33} for a given matrix \mathbf{S} . The geometrical dimensions of the ellipse grow and the position departs from the origin for increasing values of r_{33} .

3 CHOICE OF FREE (CO)VARIANCES

In the previous section it was shown that the choice of the free parameters r_{13} , r_{23} and r_{33} must always fulfil the positive definiteness of \mathbf{R} . Setting $H(r_{13}, r_{23}, r_{33}) = |\mathbf{R}|$, such a condition characterizes the domain of acceptable solutions in the space of free (co)variances r_{13} , r_{23} and r_{33} (see (12)):

$$H(r_{13}, r_{23}, r_{33}) = r_{33}|\mathbf{S}| - s_{22}(r_{13} - r_{33})^2 + 2s_{12}(r_{13} - r_{33})(r_{23} - r_{33}) - s_{11}(r_{23} - r_{33})^2 > 0 \quad (13)$$

However, this condition is not sufficient to determine a unique solution for \mathbf{R} and further requirements are therefore necessary.

The leading idea in defining an optimum choice for the free (co)variances is the hypothesis that no information is available about the possible covariances between different clocks, but they are supposed to be low. This is the same hypothesis of the "classical" method, but instead of forcing the solution of completely uncorrelated clocks, the solution of minimum correlation, compatible with the positive definiteness of \mathbf{R} , is sought. Therefore the here-proposed solution should coincide with the "classical" one (9), when the positive definiteness of \mathbf{R} (11) is safeguarded.

To this aim, the quadratic mean covariance $\sqrt{(r_{12}^2 + r_{13}^2 + r_{23}^2)/3}$ is defined as a measure of the global covariance among clocks. According to (10), it can be expressed as a function of r_{13} , r_{23} and r_{33} :

$$[G(r_{13}, r_{23}, r_{33})]^2 = (r_{12}^2 + r_{13}^2 + r_{23}^2)/3 \quad (14)$$

$$\begin{aligned}
&= [2(r_{13} - r_{33})^2 + 2(r_{13} - r_{33})^2 + 2r_{13} - r_{33}(r_{23} - r_{33}) + 2(r_{23} - r_{33})^2 \\
&\quad + 2(2r_{33} + s_{12})(r_{13} - r_{33}) + 2(2r_{33} + s_{12})(r_{23} - r_{33}) + 2r_{33}^2 \\
&\quad + (s_{12} + r_{33})^2]/3
\end{aligned}$$

From experience, it can be assumed that the global covariance can be different from zero but, on the other hand, not too high. In fact a full correlation between two clocks would imply that their signals are coincident, apart from a multiplicative factor, and this fact is to be excluded.

To combine the request of positive definiteness of the estimated matrix \mathbf{R} (13) and the minimization of the global covariance (14), let us introduce the objective function $F(r_{13}, r_{23}, r_{33})$:

$$F(r_{13}, r_{23}, r_{33}) = \frac{3\sqrt{|\mathbf{S}|}[G(r_{13}, r_{23}, r_{33})]^2}{H(r_{13}, r_{23}, r_{33})} \quad (15)$$

where the fixed factor $3\sqrt{|\mathbf{S}|}$ has been introduced for the sake of adimensionality.

In the solution domain $F(r_{13}, r_{23}, r_{33})$ represents a sort of squared global correlation and it is always positive or zero; it is zero when $G(r_{13}, r_{23}, r_{33})$ is zero, in the case of full uncorrelation. The minimization of $F(r_{13}, r_{23}, r_{33})$ leads to a solution of minimum global correlation safeguarding the positive definiteness of the resulting matrix \mathbf{R} . The quantity $H(r_{13}, r_{23}, r_{33})$, in the denominator of the objective function (15), prevents the choice of the free (co)variances from falling on the boundary of the feasible domain defined by (13). Such occurrence would yield a matrix \mathbf{R} only positive semidefinite with a disequilibrium in the estimated covariance terms. Since no information is supposed to be available about the possible covariance between clocks, the solution with estimated covariance terms of similar amount is here preferred.

Such features have led to the choice of this objective function among the several ones investigated at the early stages of this work.

One and only one global minimum of $F(r_{13}, r_{23}, r_{33})$ exists inside the solution domain, while $F(r_{13}, r_{23}, r_{33})$ goes to infinity on its boundary. In fact, three-dimensional surfaces $F(r_{13}, r_{23}, r_{33}) = f$ (with f a positive constant) are associated to decreasing values of F going inward from the surface $H(r_{13}, r_{23}, r_{33}) = 0$ (corresponding to $f = \infty$, until they collapse to a single point corresponding to the global minimum. By the study of these surfaces [19], the minimization of $F(r_{13}, r_{23}, r_{33})$ can be performed in an analytical way supplying, as a result, the coordinates r_{13}^{\min} , r_{23}^{\min} and r_{33}^{\min} of the minimum. The provided solution coincides with the "classical" one (see (9)) of uncorrelated clocks, when conditions (8) are verified.

As a final remark it should be added that this definition of the objective function $f(r_{13}, r_{23}, r_{33})$ can be useful when the clocks are to be considered of the same quality level and when there is no information about their possible correlation. Otherwise $F(r_{13}, r_{23}, r_{33})$ could be defined by introducing a weighting factor for each covariance term in (14), if some reasons for two clocks to be less correlated than the others were known. The less correlated pair can have a larger weight factor multiplying its covariance term in (14), so that the search of the minimum will attribute a smaller correlation coefficient to that pair of clocks. Similarly, if the clock variances are expected to be different, (for instance, when clocks of different types are compared), also weights for the variances r_{ii} can be introduced in the minimandum function (15).

4 EXPERIMENTAL RESULTS

In order to illustrate the effective capabilities of the method here-proposed, the data of three commercial cesium beam frequency standards maintained at IEN, Torino, Italy during the whole year 1987 are considered. The three clocks, designated by the serial numbers 12 303, 14 1230, 14 893, are considered as the first, second and third clock hereafter. The time difference of the clock signals are measured once a day and arranged in vectors y^{13} and y^{23} . The measured samples are processed according to the Allan variance with overlapping samples for the integration times 1, 2, 5, 10, 30, 60, 100 days.

For each integration time the matrix S is calculated (second column of Table 1). The corresponding matrix R , evaluated according to the here-revisited method, is reported in the third column and the (necessarily diagonal) matrix R calculated according to the classical 3-cornered hat method, is reported in the last column.

For short integration times (1, 2, 5, and 10 days) the results supplied by both methods coincide. In fact the matrix S doesn't violate conditions (8) allowing the uncorrelated solution and the minimization of the proposed function leads to the minimum allowed global correlation.

For longer integration times, the uncorrelated solution is not allowed and the matrix R estimated by the new method is not yet diagonal but gives information also about the covariance between clocks. The application of the classical method to these cases results in one negative estimated variance. By definition, the proposed minimand function (15) leads to a solution with covariance terms of similar amount because no weight are inserted in (15). This is the simplest hypothesis when there is no information about the different clocks and their noises.

5 CONCLUSIONS

This paper reports a revisited version of the popular 3-cornered hat method suitable for estimating the individual clock variances and covariances, by lifting the too restrictive hypothesis of uncorrelated clocks. This formulation requires the introduction of covariances of measured data and of clocks arranged in positive definite covariance matrices and leads to a underdetermined system of equation. The underdeterminess has been resolved by considering a suitable objective function, whose minimization supplies an unique solution. Examples of the application of the proposed method to data of clocks maintained at IEN, Torino, Italy are presented: the obtained results show that, in this case, for long integration times the uncorrelation hypothesis doesn't hold and the revisited 3-cornered hat method provides a consistent solution of minimum allowed global correlation.

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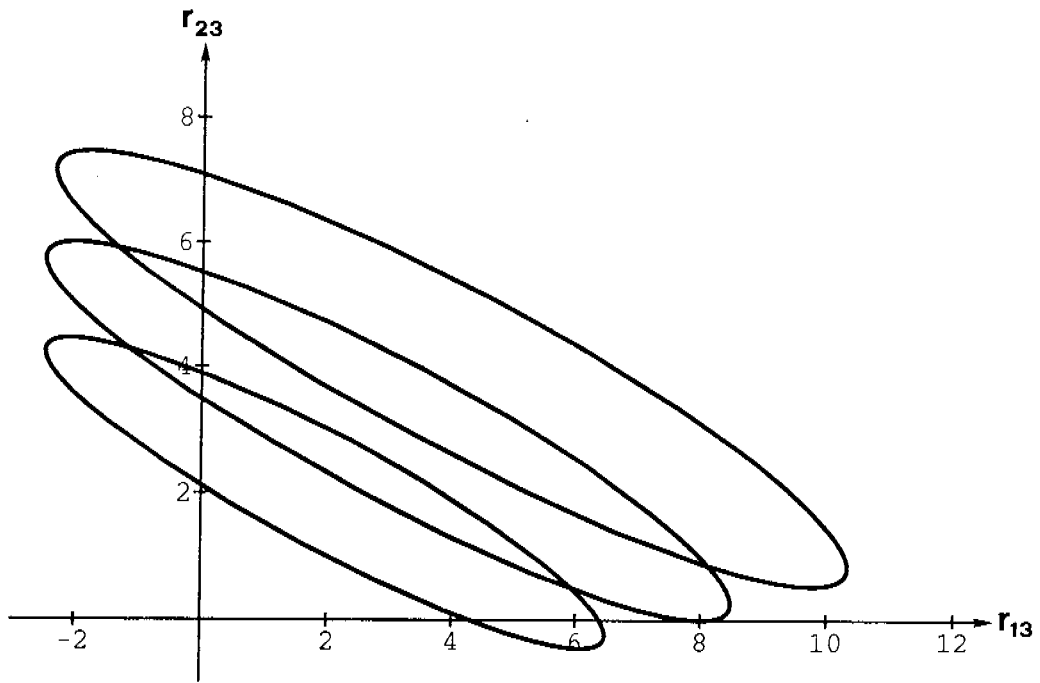


Fig. 1: Elliptical regions yielding the positive definiteness of \mathbf{R} on the plane (r_{13}, r_{23}) for a given matrix \mathbf{S} ($s_{11} = 10, s_{22} = 3, s_{12} = -5$) and different values of r_{33} ($r_{33} = 2, 3, 4$).

τ [days]	measured difference covariance matrix $\begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix}$	clock covariance matrix by the here-revisited method $\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{12} & r_{22} & r_{23} \\ r_{13} & r_{23} & r_{33} \end{bmatrix}$	clock variances by the classical method $\begin{bmatrix} r_{11} & & \\ & r_{22} & \\ & & r_{33} \end{bmatrix}$
1	$\begin{bmatrix} 412 & 128 \\ 128 & 161 \end{bmatrix}$	$\begin{bmatrix} 284 & 0 & 0 \\ 0 & 33 & 0 \\ 0 & 0 & 128 \end{bmatrix}$	$\begin{bmatrix} 284 & & \\ & 33 & \\ & & 128 \end{bmatrix}$
2	$\begin{bmatrix} 247 & 101 \\ 101 & 106 \end{bmatrix}$	$\begin{bmatrix} 146 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 101 \end{bmatrix}$	$\begin{bmatrix} 146 & & \\ & 5 & \\ & & 101 \end{bmatrix}$
5	$\begin{bmatrix} 115 & 48.6 \\ 48.6 & 53.3 \end{bmatrix}$	$\begin{bmatrix} 66.4 & 0 & 0 \\ 0 & 4.7 & 0 \\ 0 & 0 & 48.6 \end{bmatrix}$	$\begin{bmatrix} 66.4 & & \\ & 4.7 & \\ & & 48.6 \end{bmatrix}$
10	$\begin{bmatrix} 80.6 & 30.5 \\ 30.5 & 56 \end{bmatrix}$	$\begin{bmatrix} 50.1 & 0 & 0 \\ 0 & 25.5 & 0 \\ 0 & 0 & 30.5 \end{bmatrix}$	$\begin{bmatrix} 50.1 & & \\ & 25.5 & \\ & & 30.5 \end{bmatrix}$
30	$\begin{bmatrix} 39.5 & -28.3 \\ -28.3 & 109 \end{bmatrix}$	$\begin{bmatrix} 55.21 & -14.05 & 17.78 \\ -14.05 & 121.8 & 16.32 \\ 17.78 & 16.32 & 19.85 \end{bmatrix}$	$\begin{bmatrix} 67.8 & & \\ & 137.3 & \\ & & -28.3 \end{bmatrix}$
60	$\begin{bmatrix} 55.2 & -99.1 \\ -99.1 & 211 \end{bmatrix}$	$\begin{bmatrix} 117.0 & -43.23 & 60.64 \\ -43.23 & 260.9 & 54.67 \\ 60.64 & 54.67 & 59.44 \end{bmatrix}$	$\begin{bmatrix} 154.3 & & \\ & 310.1 & \\ & & -99.1 \end{bmatrix}$
100	$\begin{bmatrix} 72.3 & -102 \\ -102 & 181 \end{bmatrix}$	$\begin{bmatrix} 134.2 & -44.39 & 61.46 \\ -44.39 & 234.3 & 57.15 \\ 61.46 & 57.15 & 60.99 \end{bmatrix}$	$\begin{bmatrix} 174.3 & & \\ & 283 & \\ & & -102 \end{bmatrix}$

Table 1: Estimated variances and covariances for different integration times of three clocks maintained at IEN, Torino, Italy during 1987. The matrix elements are in unit of 10^{-28} .