MODELING FAST MODULATION EFFECTS IN CESIUM ATOMIC CLOCKS

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Abstract

A theory for fast modulation effects in cesium frequency standards has been developed and expressions derived for frequency offsets produced by Rabi and cavity pulling with square–wave phase modulation. These are compared with equivalent expressions for sine–wave phase modulation. Ratios of the amplitudes of the frequency offsets in the two modulation schemes are calculated for a mono–velocity beam and shown to be in agreement with measurements performed on a commercial standard. This analysis allows a more complete evaluation and comparison of different modulation schemes for cesium atomic beam standards.

INTRODUCTION

A number of different modulation schemes are used to implement the servo loop in cesium atomic clocks. They include sine–wave phase (frequency) modulation, square–wave phase modulation and square–wave frequency modulation. The choice of modulation scheme is based on a number of considerations, e.g., ease of implementation, optimization of cesium flux usage and Allan variance, minimized frequency offsets. Analysis of servo loop gain has been performed for all these schemes and expressions exist to model the gain dependence upon loop parameters. However, the analysis of the effects of modulation scheme on accuracy and long term stability are less complete. Sine–wave and square–wave frequency modulation effects have been considered[1] within the context of slow modulation[2], but fast modulation transients have not been modeled because of the lack of an atomic description of those atoms that experience the step change in microwave phase or frequency. A theory for square–wave phase modulation is outlined in this paper.

In the analysis of servo loop gain, it can be assumed that the cesium atoms experience a constant microwave frequency and the effects of phase modulation only perturb the relative phase of the precessing atoms and the microwave field in the second arm of the Ramsey cavity[3]. In particular, the transient effects in square–wave phase modulation can be safely omitted, and the contribution neglected from those atoms which experience the step change in phase during their resonant interaction with the microwave field. The error incurred in the calculation of servo gain slope is only of order $\omega_m\tau$ where τ is the interaction time and ω_m is the modulation frequency. Similarly, by neglecting the frequency variation of the microwave field, small errors of order $\Delta\omega\tau$ are incurred in the frequency modulation calculations. $\pm\Delta\omega$ represents the depth of frequency modulation.

Spurious servo signals are introduced by neighboring transitions (Rabi pulling) and cavity mistuning (cavity pulling) and affect the performance of cesium frequency standards. Differences in the velocity distributions of $| +m >$ and $| -m >$ have the potential to generate a frequency offset by causing a sloping background to the $| 0 > \rightarrow | 0 >$ transition. Both σ–transitions ($\Delta m = 0$) and weak π–transitions ($\Delta m = \pm 1$) may contribute to this background. Since the background and the servo gain slope have different microwave power dependencies, the resulting offset provides a means for transforming power changes to frequency changes[4]. Cavity pulling can render cesium clocks sensitive to both microwave power level and temperature changes. Both offsets are mediated by exactly those contributions which could be safely neglected above.

Apart from transient effects in frequency modulation schemes, the unwanted signals can be calculated by allowing the atoms to experience a constant microwave frequency during resonance, but one that changes in time before succeeding atoms arrive. This preserves the conditions for standard microwave resonance. The transient conditions, on the other hand, require a different description. This paper describes an approach to include the transient contributions to both Rabi and cavity pulling. The results comparing sine–wave and square–wave phase modulation offsets are compared with experiments performed on a commercial frequency standard.

RABI PULLING

First order time–dependent perturbation theory is used to calculate the background signal resulting from transitions which are far off–resonance. This approach can be easily applied to any modulation scheme because it allows the phase modulation to be addressed by Fourier decomposition. Writing

$$< F = 4,\ m \mid \mathcal{H} \mid F = 3,\ m' >= \sum_n 2\hbar c_n \cos(\omega t + n\omega_m t + \theta_n) \tag{1}$$

for a particular $| m > \rightarrow | m' >$ transition leads to a general first order transition probability, W, for a single velocity given by

$$\begin{aligned} W &= \sum_n \sum_m c_n c_m \left\{ \frac{\sin(\Delta + n\omega_m/2)\tau \sin(\Delta + m\omega_m/2)\tau}{(\Delta + n\omega_m/2)(\Delta + m\omega_m/2)} \right\} \\ &\times \Big\{ \cos[(m-n)\omega_m[t - T(1 + D/L)] + \theta_m - \theta_n] + \cos[(m-n)\omega_m(t - TD/L) + \theta_m - \theta_n] \Big\}. \end{aligned} \tag{2}$$

The notation follows [2] where 2Δ represents how far the microwave frequency is above resonance, τ is the transit time through one arm of the Ramsey cavity, T is the transit time between arms distance L apart, and D is the detector distance from the second arm. Velocity averaging has been presumed in order to average to zero the interference term between interaction regions (there are no Ramsey oscillations in the wings of a transition!). The $\sin x/x$ terms arise from assuming perfect square–wave amplitude variation of the microwave field on passing through a cavity arm[4]. The expression can be simplified by linearizing these terms with respect to frequency over the small n,m values of interest. Then it can be applied to a particular modulation scheme to predict the fundamental Fourier signal component appropriate to narrow–band amplification and detection.

a) Sine–wave modulation: $\omega(t) = \omega + \Delta\omega \cos \omega_m t$ with $C_n = cJ_n(\frac{\Delta\omega}{\omega_m})$ and $\theta_n = 0$,

$$W \rightarrow \frac{\Delta\omega}{2} \frac{\partial}{\partial \Delta} \left\{ \frac{c^2 \sin^2 \Delta\tau}{\Delta^2} \right\} \left\{ \cos \omega_m \Big[t - T(1 + D/L) \Big] + \cos \omega_m \Big[t - TD/L \Big] \right\}. \tag{3}$$

This result has been derived previously by De Marchi[4].

b) Square–wave modulation: $\cos[\omega t + \theta(t)]$, with $\theta(t)$ alternating between 0 and θ at frequency ω_m and for which

$$c_0 = c\cos\theta/2,\ \theta_0 = \theta/2;\ C_n = (2c/n\pi)\sin\theta/2,\ \theta_n = \theta/2 \quad \text{for n odd.}$$

Only terms with $n = 0,\ \pm 1$ contribute and

$$W \rightarrow \left(\frac{\omega_m \sin\theta}{\pi}\right)\frac{\partial}{\partial\Delta}\left\{\frac{c^2\sin^2\Delta\tau}{\Delta^2}\right\}\left\{\cos\omega_m[t - T(1 + D/L)] + \cos\omega_m[t - D/L]\right\}. \tag{4}$$

$\Delta\omega$ times the phase factors in Eq.(3) may be interpreted as the instantaneous frequency in the two arms of the Ramsey cavity. $(2\omega_m/\pi)$ times the similar phase factors in Eq.(4) are the Fourier components of delta function responses from atoms residing in the two interaction regions at the time when the step changes in phase occur. These expressions for W have to be averaged over velocity distributions and summed over separate $|\, m > \rightarrow |\, m' >$ transitions to complete the calculation.

These signals are nullified by a servo loop error signal, with the resulting offset determined by the servo gain slope. The gain slopes are determined for the usual mono–velocity expressions[3] which must also be separately averaged over the velocity distribution for the $|\, 0 > \rightarrow |\, 0 >$ transition. All the averaging procedures have been suppressed for the sake of clarity in the following expressions. The procedures have been discussed elsewhere[1].

$$(\omega - \omega_c)_s = \frac{\frac{(\Delta\omega)\partial}{\partial\Delta}\left\{\frac{c^2\sin^2\Delta\tau}{\Delta^2}\right\}\cos\omega_m T/2\cos\omega_m[t - T(1/2 + D/L)]}{T\sin^2 2c\tau J_1(\beta)\cos\omega_m[t - T(1/2 + D/L)]} \tag{5}$$

$$\text{with } \beta = 2\left(\frac{\Delta\omega}{\omega_m}\right)\sin\omega_m T/2.$$

$$(\omega - \omega_c)_{sq} = \frac{\left(\frac{2\omega_m\sin\theta}{\pi}\right)\frac{\partial}{\partial\Delta}\left\{\frac{c^2\sin^2\Delta\tau}{\Delta^2}\right\}\cos\omega_m T/2\cos\omega_m[t - T(1/2 - D/L)]}{T\left(\frac{2\sin\theta}{\pi}\right)\sin^2 2c\tau\sin\omega_m T/2\cos\omega_m[t - T(1/2 + D/L)]} \tag{6}$$

In the limit of small modulation index, $\Delta\omega \rightarrow 0$ and $(\omega - \omega_c)_s = (\omega - \omega_c)_{sq}$. At finite values for $\Delta\omega$, $(\omega - \omega)_s > (\omega - \omega_c)_{sq}$. Exact values for this inequality will depend upon the velocity averages and the specific values chosen for servo parameters. However, the magnitudes of the offsets will remain very similar.

The perturbation expansion developed for W is in powers of (c/Δ). The ratio of second order to first order terms is $(c/\Delta)^2$. Noting that $c \approx 2.5$ kHz and $\Delta \approx 20$kHz, it is apparent that the series is rapidly convergent.

CAVITY PULLING

Cavity pulling is caused by the asymmetry of the microwave spectrum about its unperturbed frequency, induced by a difference between that frequency and the cavity resonant frequency. The fact that square–wave phase modulation leaves the frequency unchanged except during the step changes in phase leads to a natural modeling approach in which the atomic resonance is treated exactly before and after the step change, while the transition itself is treated by perturbation theory. Only those atoms residing in the separate arms of the Ramsey cavity at the time of the step changes are affected.

If the transition occurs infinitely fast, i.e. in a time short compared to the precession period of the atoms in the microwave field, the sudden approximation can be invoked to predict that the transient itself will not induce any atomic response. Of course, the step change in phase will result in a servo signal from those atoms, which will merely be added to the usual servo signal arising from the greater number of atoms in flight between the two arms of the cavity. However, the finite bandwidth of the microwave cavity prevents infinitely fast changes in phase from occurring and causes a breakdown of the sudden approximation. Now the transient frequency changes are finite. Moreover, the positive and negative frequency transients remain symmetric only if the microwave frequency is located at cavity resonance. It is this asymmetry which records the cavity offset and induces a cesium beam signal at the modulation frequency. This signal is derived from fast responses at the beginning and end of each half–cycle of the servo error signal and may be written as

$$-\left(\frac{2\sin\theta}{\pi}\right)\left(\frac{8Q_L^2\omega_m}{\omega_0}\right)\sin^2 2c\tau\left\{(2c\tau)/\tan(2c\tau)\right\}\cos\omega_m T/2\cos\omega_m[t-T(1/2+D/L)]\left(\frac{\omega-\omega_0}{\omega_0}\right). \quad (7)$$

When this is combined as before with the servo gain slope, the offset $(\omega-\omega_c)$ can be calculated. For a mono–velocity beam, it reduces to

$$\left(\frac{\omega-\omega_0}{\omega_0}\right)_{sq} = \frac{-8Q_L^2\omega_m}{\omega_0^2 T}\,\frac{\cos\omega_m T/2}{\sin\omega_m T/2}\left\{(2c)\tau/\tan(2c\tau)\right\}\left(\frac{\omega-\omega_0}{\omega_0}\right) \quad (8)$$

The similar result for sine–wave modulation[1] may be written

$$\left(\frac{\omega-\omega_c}{\omega_0}\right)_s = \frac{-4Q_L^2\Delta\omega}{\omega_0^2 T}\,\frac{\cos\omega_m T/2}{J_1(\beta)}\left\{1+J_0(\beta)-J_2(\beta)\right\}\left\{(2c\tau)/\tan(2c\tau)\right\}\left(\frac{\omega-\omega_0}{\omega_0}\right) \quad (9)$$

In the limit $\Delta\omega \to 0$, the two expressions become equal, and cavity pulling effects are the same for sine–wave and square–wave modulation. The sine–wave frequency offset is reduced as $\Delta\omega$ increases from zero.

EXPERIMENTAL RESULTS

Sine–wave and square–wave modulation were compared by measuring the frequency of an atomic standard for both modulation schemes, under as identical conditions as possible. This allows the differences in modulation schemes to become apparent but does not attempt to optimize performance. The square–wave modulation depth θ was held close to 90 degrees and the modulator drive signal was passed through a narrow band, unity gain filter and phase shifter to generate the sine–wave modulation. This produced a modulation index $(\Delta\omega/\omega_m)$ of unity and allowed phase adjustment to maximize the signal. The modulation frequency was close to maximizing the gain slope of the servo loop with $\omega_m T \approx \pi$ and $\beta \approx 2$.

Rabi pulling was separately identified by measuring the frequency as a function of C–field at two microwave power levels; the optimum power maximized the average beam current, and the higher power was 3 dB above optimum. Figure 1 shows the resulting fractional change in frequency with power level for both modulation schemes. Cavity pulling was determined by measuring the frequency for cavity tuning on resonance and ±12 MHz off–resonance. In the data reduction, it was assumed that the Rabi pulling was independent of cavity tuning, and the cavity pulling was independent of C–field. Three cavity settings, two power levels and two modulation schemes lead to a total of twelve curves of frequency versus C–field. The Rabi pulling could be reduced to a single curve at optimum power by the following relationships:

$$
\begin{aligned}
(\omega - \omega_c)_{sq} &= 0.94(\omega - \omega_c)_s \quad \text{for both power levels,} && (10)\\
\text{and} \quad (\omega - \omega_c)(P + 3\text{dB} &= 1.85(\omega - \omega_c)(P) \quad \text{for both modulation schemes.} && (11)
\end{aligned}
$$

This curve is shown in Fig.2 and illustrates that Rabi pulling effects are small in this standard. Zero frequency offset may be arbitrarily chosen for the highest C–field, but is truely zero only at even higher fields. The ratio 0.94 between square–wave and sine–wave amplitudes is closer to 1 than predicted from Eqs.(5) and (6). In the absence of velocity averaging, this ratio is predicted to be $J_1(2) = 0.58$. Also, the ratio between high power and optimum power is predicted to be 2.5, slightly higher than the measured value of 1.85. It is reasonable to anticipate that velocity averaging will improve the agreement between theory and experiment. The approximately linear dependence on power arises because the off–resonance transition probabilities increase linearly with power, whereas the resonant transitions are already optimized.

Table I lists the measured offset frequencies versus cavity tuning. Systematic errors of unknown origin exceed the standard deviation (2×10^{-13}) in the frequency measurements. Zero offset was arbitrarily chosen to be that for the sine–wave modulation case at optimum power and cavity tuned on resonance. Eqs.(8) and (9) would predict a mono–velocity offset ratio

$$\frac{(\omega - \omega_c)_{sq}}{(\omega - \omega_c)_s} = \frac{2J_1(2)}{1 + J_0(2) - J_2(2)} = 1.32 \qquad (12)$$

The measured ratio, obtained by taking the difference of the offsets at $\pm$ 12MHz, is 1.3 at both power levels. The power dependence of the offsets is contained in the usual expression $(2c\tau)/\tan(2c\tau)$ which is zero for $4c\tau = \pi$ and rises to -1.7 as the power is raised 3 dB. Clearly, the optimum power did not produce zero cavity pulling, but the increase by a factor 2.7 on raising the power 3 dB would imply that the nominal power was within 1.6 dB of that necessary for zero offset.

CONCLUSIONS

Our measurements show that sine–wave and square–wave phase modulation yield similar values for Rabi and cavity pulling effects in cesium frequency standards. The result is consistent with the theory presented in this paper. It predicts that, in the limit of small frequency modulation depth, sine–wave modulation produces the same frequency offsets as square–wave modulation. At higher modulation depth, however, Rabi pulling tends to be larger and cavity pulling smaller for the sine–wave modulation scheme. Exact predictions will require knowledge of specific servo parameters and averaging over beam velocities.

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TABLE I.

Fractional frequency offsets (parts in 10^{12}) determined from measurements at two microwave power levels for both sine-wave and square–wave modulation. The cavity was tuned to the cesium frequency F_0 and to $F_0 \pm$ 12MHz. The cavity pulling is normalized to the frequency for sine–wave modulation at optimum power (maximum dc signal) and cavity tuned to F_0. Standard deviation $= 3 \times 10^{-13}$.

		F_0 + 12 MHz	F_0	F_0 - 12 MHz
optimum power	sine–wave	-0.9	0	3.6
	square–wave	-3.6	-1.2	2.2
optimum + 3dB	sine–wave	-3.6	-0.2	8.5
	square–wave	-8.2	-1.0	6.9

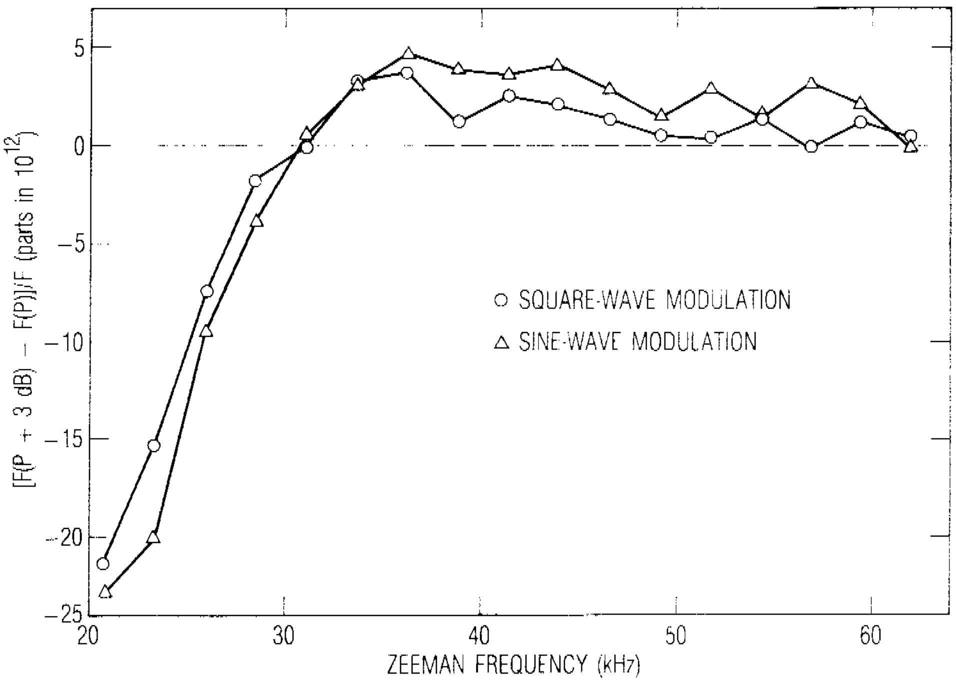

Figure 1. C-field dependence of the fractional change in frequency caused by a 3 dB increase in microwave power above optimum (maximum dc signal) for sine-wave and square-wave phase modulation.

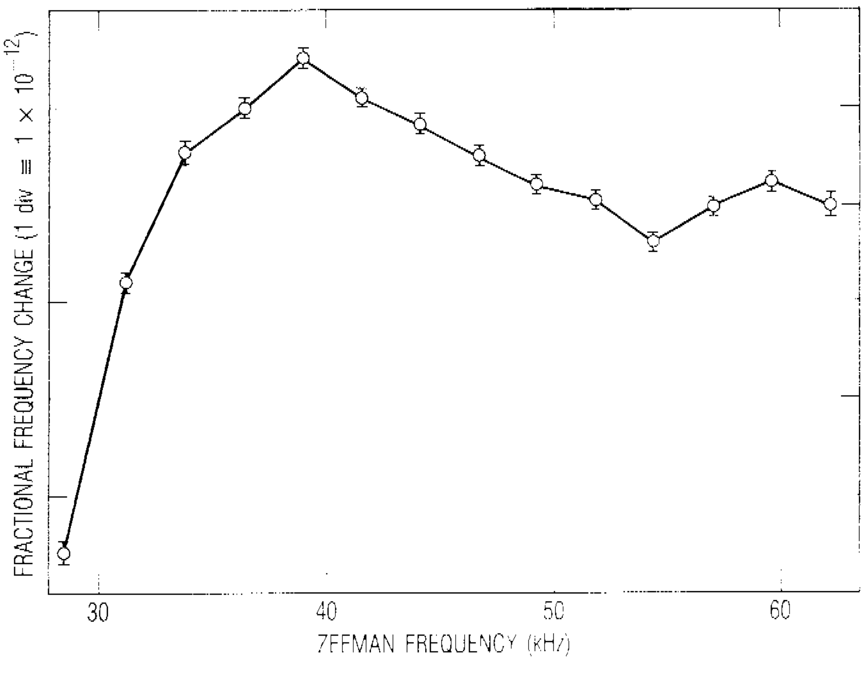

Figure 2. C-field dependence of Rabi pulling measured on a cesium frequency standard with sine-wave modulation. Rabi pulling is assumed to be those changes in frequency which depend upon C-field.