A REVIEW OF METHODS OF ANALYZING FREQUENCY STABILITY

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ABSTRACT

Extensive research over the past years has provided a model for the description of frequency instabilities of clocks and oscillators. This model consists of the superposition of three distinct parts: (1) random, nondeterministic fluctuations described as noise; (2) long-term, systematic trends or aging; and (3) fluctuations induced by environmental sensitivities of the oscillator or clock. The random part of the model includes noises which have presented certain mathematical problems. These mathematical problems are partly responsible for the creation of numerous techniques of analysis, but these techniques have neither produced substantively new models nor have they added insight into the physical origins of the random fluctuations--some of which remain obscure. The purpose of the measurement process is to estimate the levels and kinds of instabilities present in a given device--that is, to quantify the model. The mathematical analysis used is merely a means toward this end, and it is important to retain this perspective. Fortunately, there are relatively simple means of analysis which are also commonly used--the two-sample variance and the power spectral density.

Crucial to any measurement are the intended uses of the result. This includes the levels of accuracy and precision needed, as well as the intended application. For example, one may wish only a relative comparison between two oscillators; and, thus, absolute accuracy (as opposed to precision) is of no interest. The specific application intended for the measurement will often influence the form in which the final quantified model is reported.

I. INTRODUCTION

It appears to be a custom, nowadays, to have a paper reviewing the concepts of the measurement of frequency stability at conferences dealing with time and frequency. Consistent with this trend, this paper has been prepared at the request of the program committee.

Perhaps one of the most useful aspects of review papers is to provide an entrance into the literature of the field being reviewed. In the case of frequency stability, there is really a great deal published; and it is with some difficulty, now, that anything new can be added--especially in review papers. In order to provide entrance into the literature, four specific papers are cited here. The first two reasonably cover the technical subject prior to about 1973. The next two papers are excellent review papers on the subject and are more current:

[1] Barnes et al., "Characterization of Frequency Stability," IEEE Trans. on I&M, Vol. IM-20, No. 2, May 1971, pp. 105-120.

[2] Lesage and Audoin, "Estimation of the Two-Sample Variance with a Limited Number of Data," Proceedings of the 31st Annual Symposium on Frequency Control, 1977, pp. 311-318.

[3] Rutman, "Oscillator Specifications: A Review of Classical and New Ideas," Proceedings of the 31st Annual Symposium on Frequency Control, June 1977, pp. 291-301.

[4] Winkler, "A Brief Review of Frequency Stability Measures," Proceedings of the 8th Annual Precise Time and Time Interval (PTTI) Applications and Planning Meeting, Nov. 1976, pp. 489-527.

Not only are the last two excellent review papers, but they also provide rather extensive bibliographies covering the more important publications in the field.

The present paper attempts (in a rather tutorial vein) to review the subject of frequency stability mainly from the viewpoint of Operations Research (OR), with the hope of developing an intuitive understanding of the concepts and operations. Those who prefer the more mathematical approach are referred to the above-cited references. Thus, it is by intent that detailed mathematical calculations are avoided as far as possible in this paper.

II. MEASUREMENT AS AN OPERATION

Figure 1 is a somewhat simplified diagram of measurement operations. In any measurement, one begins with some idea about what one wants to measure. That is, one has a <u>model</u> in mind.

In the case of a measurement of the acceleration due to gravity, for example, one assumes Newton's laws and assumes that g is reasonably constant. In the case of a voltage measurement, as another example, one assumes that a volt meter will not unduly load the source and that the meter covers the range of expected voltages. All of these underlying assumptions are part of the modeling process, and it is important not only to make the assumptions but also to document them.

In the case of frequency fluctuations of clocks and oscillators, rather comprehensive models already exist and are discussed below to some significant extent. Thus, it is not necessary to invent new models to begin the measurement process--they are already well documented. On the other hand, a researcher might be able to improve our understanding substantially with new models, but research is often a separate field from straightforward measurements. The emphasis of this paper is to review present measurements and not to discuss the opportunities for research to produce or evaluate new models.

Based on the model assumed and the equipment available, one designs an experiment to evaluate the model. If the experimenter is wise, he will also design his analysis of the data at the beginning.

If one is performing a measurement, there is some aspect of the model which is unknown. In the above-mentioned example, concerning the measurement of the acceleration due to gravity, the parameter g of the equation $S = 1/2\ gt^2$ was to be determined by experiment. It was assumed that the object accelerated uniformly, and only the constant g was to be determined. That is, g was an undetermined parameter of the model.

In the case of the measurement of frequency fluctuations, the typical models consist (among other things) of the superposition of several different noises. The levels of each of these noises are not normally known in advance. That is, the noise levels are undetermined parameters to be estimated in the measurement operation.

This leads us to what I think is an important definition: A measurement is an operation designed to estimate the numerical value of a model parameter. Thus, without a model, no meaningful measurment is possible. Given a model, many measurements often suggest themselves. For example, when Newton suggested that every particle in the universe attracts every other particle in the universe with a force that is proportional to etc., etc., one can immediately set about measuring that constant of proportionality, big-G. Without the model, the experiments don't make sense.

Based on the experimental design, one next actually performs the experiment and obtains data. The data are subjected to the analysis routines which had been previously designed, and the unknown parameters of the model are "fitted" to the results. One typically designs test experiments to verify the functioning of the equipment and to ensure that one is actually fitting parameters to the oscillator being measured and not the measuring system. For example, if the noise level of the measuring system is higher than that of the oscillators under test, one might get good model fits to the data; however, the results will not be applicable to the oscillator but rather to the measurement system.

It is at this point that the researcher and the engineer part company. The researcher into frequency fluctuations will be interested in perfecting and refining the models or gaining insight into more fundamental models. The specific levels of noise might not be of particular interest to him, but rather the adequacy of the model in reflecting "reality might be more important. For example, he might be interested in whether other models might provide a better fit. On the other hand, the engineer who just wanted to know how bad the oscillator was in order to decide whether or not it was useful in his particular application has his measurement, and he can leave the operations of Figure 1.

Fortunately, researchers have produced some rather comprehensive and useful models which seem quite adequate to describe present-day oscillators rather completely. This is a sign of a mature field of study. Even though the physical origins of some of the noise components of oscillator models remain obscure, one really doesn't expect significant revisions in the models themselves. We may gain added insight into their origins or get small refinements; but, overall, no great changes are expected in the mathematical form of the models.

III. MODELS OF FREQUENCY FLUCTUATIONS

The conventional models [1] of an oscillator begin by assuming that the output signal, V(t), is approximately sinusoidal and can be represented by the equation

$$V(t) = [V_0 + \varepsilon(t)] \sin \Phi(t) \qquad (1)$$

where V_0 is the nominal (constant) amplitude; $\varepsilon(t)$ represents the fluctuations in amplitude; and $\Phi(t)$ is the instantaneous phase of the oscillator. In particular, the phase is assumed to be represented in the form

$$\Phi(t) = 2\pi\nu_0 t + \phi(t), \qquad (2)$$

where ν_0 is the nominal frequency of the oscillator and $\phi(t)$ represents the instantaneous phase deviation from the nominal phase, $2\pi\nu_0 t$.

For the sake of completeness and utility, it should be noted that there are certain restrictions on eq. (1) and (2). Stated in a qualitative way, these restrictions are (a) that the fluctuations in amplitude, $\varepsilon(t)$, are small compared to the nominal amplitude, V_0; and (b) that the fluctuations in instantaneous frequency, $\frac{1}{2\pi}\frac{d\phi}{dt}$, are small compared to the nominal frequency, ν_0. Mathematically, these can be expressed in the form

$$\varepsilon(t) \ll V_0,$$

$$\frac{1}{2\pi}\frac{d\phi(t)}{dt} \ll \nu_0.$$

Normally, high-quality oscillators easily meet these conditions.

It is of value, also, to introduce certain terms which are commonly used throughout the literature on frequency stability. The instantaneous (angular) frequency of an oscillator is defined to be the time rate of change of the phase. Expressed as a cycle frequency, $\nu(t)$,

$$\nu(t) \equiv \frac{1}{2\pi}\frac{d}{dt}\phi(t), \tag{4}$$

where $\nu(r)$ is expressed in Hertz.

Also, the instantaneous, fractional frequency fluctuations about the nominal are commonly defined by the expression [1]

$$y(t) \equiv \frac{1}{2\pi\nu_0}\frac{d\phi(t)}{dt}; \tag{5}$$

which, with the aid of equations (2) and (4) can be expressed in the form

$$y(t) = \frac{\nu(t) - \nu_0}{\nu_0}. \tag{6}$$

Most models are concerned with the quantity $y(t)$. Occasionally, however, one sees in the literature the time error, $x(t)$, being used. This quantity, $x(t)$ is defined by the relation

$$x(t) \equiv \frac{1}{2\pi\nu_0}\phi(t), \tag{7}$$

and clearly satisfies the relation

$$y(t) = \frac{dx(t)}{dt} . \tag{8}$$

In words, $x(t)$ is the instantaneous time error of a clock run from the oscillator. It is thus expected that frequency stability measurements will be concerned with various statistical functions of both $x(t)$ and $y(t)$ such as power spectral densities, e.g., $S_y(f)$.

The conventional model used to describe frequency fluctuations, $y(t)$, has three main subdivisions (see table 1): (1) random, non-deterministic fluctuations described as noise; (2) long-term, systematic trends or aging; and (3) fluctuations induced by environmental sensitivities of the oscillator or clock.

The treatment of environmental sensitivities is a rather special case and (although extremely important) will not be covered here. In general, one wants to minimize environmental sensitivities. The separation of environmentally induced fluctuations from intrinsic fluctuations is normally done by correlation techniques. Toward this end the transfer function models of Box & Jenkins [5] are of value.

The treatment of systematic trends is often given short shrift. In principle, a linear drift in frequency of an oscillator with time can be measured with arbitrarily high accuracy. In practice, however, one might not have the time to evaluate the model parameter with sufficient accuracy to relegate it to the status of an adequately well-known constant. This is especially true when one realizes that the random parts of the model disturb the accuracy with which the systematic parts can be determined, and that an error in the frequency drift term, for example, becomes a quadratic error with running time in the indicated time (or phase) of a clock. Indeed, when one is attempting to predict clock performance for the future, errors in estimating the systematic terms almost always predominate for very long prediction intervals (months to years). Percival [15] has proposed using prediction errors directly as a measure of frequency stability because the important contribution of the systematic terms is automatically incorporated.

The random, non-deterministic parts of the model have received a great deal of attention with primary emphasis on the continuous, Gaussian elements.

Generally, experiments have revealed five different noise types that might be needed to model an oscillator. Typically, only two or three of the terms are necessary to describe the noise elements of an individual oscillator over a large range of time intervals. The five terms of the Gaussian noise elements listed in table 1 have names and are listed in table 2.

The sporadic elements constitute a special problem. The objective documentation of their existence is limited [7,8], and no clear consensus exists on how to handle them [4,6,8]. However, it does appear, in general, that no great errors are committed if one simply ignores these elements entirely. The reasons behind this are twofold. First, the step sizes are normally small (of the order of other noises); and second, they tend to be infrequent (perhaps less than once per day). For the remainder of this paper, the sporadic elements will be ignored--a subject more appropriate to research on oscillator models than to practical measurements of frequency stability. It is acknowledged, however, that in some applications the sporadic parts could be quite important.

III. EXPERIMENT AND ANALYSIS

To this point in the paper everything has been reasonably straightforward and without much controversy. Indeed, there is amazing consensus on the model elements. In the areas of analysis and (to a lesser extent) experimentation, there is not such close agreement. One can find numerous analytical techniques designed to fit the model parameters (noise levels, drift rate, etc.) to the data.

Each of the analytical techniques has its own advantages. Typically, these advantages either favor a certain method of analysis of the data as derived from special equipment (e.g., frequency counters or spectrum analyzers), or they have the value of being useful for richer models than those considered here. (For example, if one were to consider a model with a frequency spectral density ($S_y(f)$) varying as $|f|^\alpha$ for $\alpha < -3$, then very special analysis techniques would be needed. However, such models are almost never needed in practice.)

There are two general considerations which should guide the design of the experiments and the analysis. Basically, these two considerations are (a) the resources available and (b) the intended use of the results.

The resources available which limit the measurements include equipment, mathematical and computational sophistication, and time. In the absence of elaborate equipment it may well be necessary to fill in with more mathematical and computational skills. Time constraints limit both the precision and range of the results and hence the range over which a model can be verified.

The intended use of the results is equally important. For example, if an experimenter has a requirement for an oscillator to remain stable in frequency to, say, a part in 10^6 for sample times from 1 to 10 seconds, and he discovers that his oscillator is, say, about 100 times better than this, then that experimenter would be foolish to pursue his measurements to a 1 percent accuracy tolerance--he can be sure that it's adequate with a very rough experiment. Also, of course, one has greater confidence in results which require less mathematical manipulation. Thus, one gains in designing experiments which come closest to reflecting the intended use of the oscillator or clock when that use is known.

In the absence of overriding reasons to choose one analytical technique or data-acquisition system over another, it clearly makes sense to choose the simplest and easiest. Of course, with the simpler techniques there are fewer opportunities for errors, and one can obtain reasonable confidence in a short period of time. In keeping with the above, this paper is confined to two analytical techniques as recommended in [1]: the power spectral density of the fractional frequency fluctuations, $S_y(f)$, and the two-sample variance, $\sigma_y^2(\tau)$. (The two-sample variance is sometimes referred to as the "Allan variance." Certainly there are values in other techniques, but they are adequately covered in the literature and will not be covered here. In particular, Rutman [3] provides an especially lucid comparison of the various analytical techniques.

Power Spectral Density of $y(t)$

Since the noise models suggested above were expressed directly in terms of the spectral density, $S_y(f)$, of the fractional frequency fluctuations, $y(t)$, an obvious approach is to estimate directly $S_y(f)$. This is typically done either with analog techniques or by sampling $y(t)$ at regular intervals and converting to a spectral density with the aid of a computer [9]. Systems also exist which perform the sampling automatically and convert to spectral estimates without the need to transfer the data to a computer.

Of course, the spectral estimates will not automatically be provided in the same form as the model elements given in Tables 1 and 2. One must "fit" the parameters or noise levels to, say, a graphical representation of the spectral density. In such a display, the periodic elements of the model are revealed in an especially lucid form. The presence of the other systematic elements, however, is not so obvious and will probably require special treatment to resolve.

With any of the measurement schemes there is often a problem in obtaining reliable values of $y(t)$ or an analog signal of $y(t)$ not contaminated by the noise of associated circuitry. For very high-quality signal sources this is a major problem. Often one takes two or more comparable oscillators and "beats" their signals together to obtain the difference frequency between the two oscillators. This difference signal can be amplified and analyzed by fairly conventional techniques provided this difference signal also satisfies the constraints of (3), above. However, the data are representative of <u>both</u> oscillators, and one must make some model assumptions about how to <u>divide</u> the results between the two oscillators. If three oscillators are intercompared in all possible combinations, one can make a statistical separation of the results if one makes use of a model assumption that the fluctuations of each oscillator are statistically uncorrelated with the others [10]. Various techniques for acquiring data on very stable oscillators have been devised and can be found in the literature [11,12,13].

Two-Sample Variance, $\sigma_y^2(\tau)$

One of the most common ways of acquiring frequency data is by means of a frequency counter which totals the number of cycles, n, of the signal in a specified interval of time, τ. The sample frequency is then just n/τ, and large volumes of data are easily generated.

The quantity n/τ can be related to y(t) by some straightforward mathematical manipulation. One can define the average fractional frequency, $\bar{y}_k$, over the interval t_k to $t_k + \tau$ by the relation

$$\bar{y}_k \equiv \frac{1}{\tau} \int_{t_k}^{t_k+\tau} y(t)dt. \qquad (9)$$

where $t_{k+1} = t_k + T$, and T is the interval between the beginnings of successive averages. With the aid of (4) and (6) this can be written in the form

$$y_k = \frac{1}{\tau} \int_{t_k}^{t_k+\tau} \frac{\nu(t) - \nu_0}{\nu_0} \, dt$$

$$= \frac{1}{2\pi\nu_0\tau} \int_{t_k}^{t_k+\tau} \frac{d\Phi}{dt} \, dt - 1 \qquad (10)$$

$$= \frac{1}{\nu_0\tau} \left[\frac{\Phi(t_k+\tau) - \Phi(t_k)}{2\pi}\right] - 1.$$

However, the quantity in the brackets is just the accumulated phase (in radians) during the interval, divided by 2π. That is, it is the number of cycles, n_k. Thus, (10) becomes

$$\bar{y}_k = \frac{1}{\nu_0} \cdot \frac{n_k}{\tau} - 1, \qquad (11)$$

and n_k is just the k-th counter reading.

At this point one might be tempted to do some simple and conventional statistics on the $\bar{y}_k$. For example, one might compute a mean and a variance for the set or perform a linear regression. However, this is fraught with difficulty. The problem arises from the model elements of $S_y(f)$ which are proportional to $|f|^{-1}$ and $|f|^{-2}$; that is, flicker frequency noise and random walk frequency noise. These types of noise do not have convergent variances. That is, as more and more data are taken and the variance is estimated with more and more data, the variance itself grows, seemingly without bound.*

Clearly, some other statistical tool is necessary to analyze the data for practical applications. Since the problem arises with adding too much data to the variance estimate, the obvious solution is administratively to limit the data used for each estimate and then average all of the individual estimates. By convention the data limit has been taken at only two values of $\bar{y}_k$ for each variance estimate [1]. In order to gain confidence in this estimate, one averages many estimates of the variance. Lesage and Audoin [2] have derived expressions for the confidence intervals for the estimates of the two-sample variance estimated in this way for the model elements listed in Table 2.

It is of value to explicitly state the computations involved with the two-sample variance. Normally, statisticians estimate a mean of a set of N values of a random variable, u_i, by the equation

$$\bar{u} = \frac{1}{N} \sum_{i=1}^{N} u_i . \tag{12}$$

The variance, σ^2, is estimated by the equation

$$\sigma^2 = \frac{1}{N-1} \sum_{i=1}^{N} (u_i - \bar{u})^2 . \tag{13}$$

For the case N = 2, equations (12) and (13) can be combined in the simple form

$$\sigma^2(N = 2) = \frac{(u_2 - u_1)^2}{2} . \tag{14}$$

*One is not suggesting here that the frequency fluctuations of real devices are actually unbounded. Rather, in a practical sense one is unlikely to have the time (perhaps many, many years) to take enough data to see convergence in fact. See ref. [8].

At this point, we can define the two-sample variance, $\sigma_y^2(\tau)$, as the (infinite) average of estimates of the variance for two samples. That is, mathematically,

$$\sigma_y^2(\tau) = \lim_{m \to \infty} \frac{1}{m-1} \sum_{k=1}^{m-1} \frac{(\bar{y}_{k+1} - \bar{y}_k)^2}{2} . \qquad (15)$$

In practice, of course, one estimates $\sigma_y^2(\tau)$ from a finite set of m values of $\bar{y}_k$ and cannot pass to the limit.

An additional restriction must be added. In equation (9), above, $\bar{y}_k$ was the average from t_k to $t_k + \tau$, but the next average began at $t_{k+1} = t_k + T$. The two-sample variance, $\sigma_y^2(\tau)$, is defined to be restricted to those cases for which $T = \tau$. That is, in words, each sample of $\bar{y}_k$ must be exactly adjacent in time to $\bar{y}_{k+1}$ with no "dead time" between measurements. There are times when this condition is difficult to meet, and corrections must be made. Again, for the model elements of Table 2, the corrections can be found in the literature [14].

A final point on $\sigma_y^2(\tau)$ is to note that there is a bandwidth, f_h, to the measuring system, and this can influence the results. Again, it is a matter of convention to specify f_h with the measurement and to require $2\pi f_h \tau >> 1$ for all measurements.

In practice one can determine a set of $\bar{y}_k$-values, each of which is an average of the fractional frequency fluctuations from nominal. One can then form an estimate, $\hat{\sigma}_y^2(\tau)$, of $\sigma_y^2(\tau)$ with the equation

$$\hat{\sigma}_y^2(\tau) = \frac{1}{m-1} \sum_{k=1}^{m-1} \frac{(\bar{y}_{k+1} - \bar{y}_k)^2}{2} . \qquad (16)$$

For $\tau' = 2\tau$, one can average adjacent values of $\bar{y}_k$ and obtain a new list, $\bar{y}_k'$, where

$$\bar{y}_k' = \frac{\bar{y}_{2k} + \bar{y}_{2k-1}}{2} \qquad (17)$$

and hence determine $\hat{\sigma}_y^2(2\tau)$ based on $m' = \frac{m}{2}$ values.

This process can be repeated for integer multiples of τ out to half of the total data length.

It is then normal to plot $\sqrt{\hat{\sigma}_y^2(\tau)}$ versus τ on log-log graph paper. Typically, regions of the plot can be approximated with straight line segments (see Fig. 2). Since straight lines on log-log paper can be represented by equations of the form $\sigma_y^2(\tau) = A\tau^{\mu}$, one is fitting this form of a model to the $\sigma_y^2(\tau)$ estimates. However, this is essentially equivalent to the models of Table 2 [1]. In fact, Table 3 shows that the models of Table 2 translate into $\sigma_y^2(\tau)$ values varying as τ to some power. With the aid of Table 3, then one can estimate the levels, h_α, of the various noise terms; and, with the aid of the uncertainties listed, obtain confidence intervals for the $\sigma_y(\tau)$ estimates plotted. Thus, the analysis by the use of the two-sample variance allows one to quantify the model parameters (i.e., to measure the stability of the oscillator) and evaluate the adequacy of the model fit.

Although the two-sample variance has required a fair amount of space to explain here, it is really one of the easiest analysis techniques to use. Probably for this reason it is one of the most common techniques used, also. The price one pays for this ease in analysis is some loss in confidence in the results. If one is interested in the spectrum of the frequency fluctuations, then it is probably true that direct spectral estimates are slightly more precise for the measurement process than the two-sample variance, but often this is not a critical issue. Often, "final" results are simply reported as values of the two-sample variance or its square root, since they can be readily translated into the spectral densities via Table 3.

Spectral Density of Phase Fluctuations

One can estimate the power spectral densities of various quantities. Two interesting quantities in addition to y(t) are V(t) and $\phi(t)$ as defined in equations (1) and (2). The rf power spectral density, $S_V(f)$, is just the spectral density one would obtain if he were to analyze directly the output voltage, V(t), of the oscillator. Of course, it is important whenever the rf spectral purity is important (e.g., in communications systems). Included in $S_V(f)$ are effects from the amplitude fluctuations, $\varepsilon(t)$, as well as the phase (frequency) fluctuations, $\phi(t)$. Typically, direct estimates of $S_V(f)$ are limited in resolution to a few Hertz of bandwidth and, thus, are not particularly useful for evaluating long-term performance of the oscillator.

For short-term fluctuations, one often sees estimates of the power spectral density of the phase fluctuations, $S_\phi(f)$. Often it occurs for precision oscillators that $\varepsilon(t)$ is sufficiently small that [1]

$$S_\phi(f) \approx \frac{4}{V_0^2} S_V(\nu_0 \pm f), \tag{18}$$

where $f > 0$. That is, the sideband power at $\pm f$ removed from the carrier frequency, ν_0, is proportional to the phase power spectral density.

There is another useful equation relating $S_\phi(f)$ with $S_y(f)$. Reference [1] shows that

$$S_y(f) = \left(\frac{f}{\nu_0}\right)^2 S_\phi(f). \tag{19}$$

Often one measures the sideband power of the rf spectrum, $S_V(\nu_0 \pm f)$, and uses (18) to estimate $S_\phi(f)$. Occasionally, one devises a circuit to obtain a voltage analog of $\phi(t)$ and spectrum analyzes it directly [13]. Equation (19) provides a link of $S_\phi(f)$ with $S_y(f)$ and hence the rest of the oscillator model.

Systematics

The power spectrum and the two-sample variance are useful tools for the analysis of the random, non-deterministic elements of the model. Their use in the presence of the systematic elements can cause problems [4,15], and for this reason it is often important to remove the systematic elements before the noise analysis. It is important to emphasize that "removal for analysis" does not imply that these elements are discarded--they are recognized, evaluated, and remembered to be reported in the final accounting of the model parameters. Thus, the overall analysis operations are shown in Figure 3.

The methods of separation of the systematics will vary with particular applications. For example, some oscillators display a nearly linear drift in frequency (i.e., y_k) with running time. For analysis by the two-sample variance technique it is often important to "remove" this first. This can be done by means of a linear least squares regression to the y_k, but the drift which is removed should be reported in the final accounting of the model. Again in the presence of periodic terms, the two-sample variance will be very difficult to interpret. Thus, one should either use direct spectrum estimation or devise an appropriate technique to remove periodicities before a two-sample variance is used. Some of the difficulties associated with removing systematics are discussed by Winkler [4].

IV. USES OF THE VARIOUS MEASURES OF FREQUENCY STABILITY

The estimates of the frequency stability of an oscillator or clock have many uses, spanning the range from research to procurement specifications. As mentioned above, the form in which the model is expressed should be influenced by the intended use. For example, if the noise model for a given oscillator included flicker frequency noise as an element, this could be equivalently expressed as a power spectral density varying as $|f|^{-1}$ or a constant $\sigma_y(\tau)$ with varying τ. Which mode of expression one uses is just a matter of convenience.

It is probably true that theoreticians and researchers in the field of frequency instabilities normally use the power spectral densities. It is probably this historical fact which is the source of the mode of expression of the random parts of the model in Tables 1 and 2. That is, the random model elements were expressed in terms of power spectral densities. Also, for diagnostics of oscillators, the use of power spectral densities is very common. A spectrum analyzer can be a very powerful tool in the design, construction, and evaluation of precision signal sources.

Table 4 lists some uses of the measures of frequency instabilities with the commonly used method of analysis. Included in the methods of analysis are references to ARIMA models [5]. ARIMA models provide a powerful and convenient method of analysis, computer simulation, and prediction for random processes and deserve special mention in addition to $S_y(f)$ and $\sigma_y(\tau)$. Although they are not used to any great extent in the measurement of frequency stability, they provide the only practical approach to computer simulation of oscillator performance. It is possible to translate from models based on Table 2 to ARIMA models directly [8,15].

V. SUMMARY

The measurement of frequency stability is the process of evaluating (i.e., quantifying) a set of model parameters. The typical model elements found to be adequate to model the frequency instabilities include (a) random, non-deterministic elements expressed in terms of power-law-types of power spectral densities; (b) systematic terms like linear frequency drift; and (c) environmental sensitivities.

Various mathematical techniques exist to fit these model elements to a particular oscillator, but the most common are power spectrum analysis and the computation of the two-sample variance. Other analysis techniques have been studied and have important advantages for special situations. For example, ARIMA models [5] provide the only practical approach to computer simulation of oscillator performance.

A measurement of the frequency stability of an oscillator includes the values and uncertainties of the model parameters, the range of applicability of the model, and a description of the experiment and analytical techniques.

TABLE 1

Model Elements

1. Random (Noise) Elements

   - Gaussian Noise Elements

     $$S_y(f) = h_{-2}|f|^{-2} + h_{-1}|f|^{-1} + h_0 + h_1|f| + h_2|f|^2$$

   - Sporadic Elements

     Sudden steps in frequency and/or time (phase)

2. Systematic Elements

   - Linear frequency drift
   - Frequency offset
   - Time (phase) offset
   - Periodic terms

3. Environmental Elements

TABLE 2

Names of Noise Types

Name	Frequency Dependence* Power Spectral Density of Frequency	Power Spectral Density of Phase
White Phase Noise	$\|f\|^{2}$	$\|f\|^{0}$
Flicker Phase Noise	$\|f\|^{1}$	$\|f\|^{-1}$
White Frequency Noise	$\|f\|^{0}$	$\|f\|^{-2}$
Flicker Frequency Noise	$\|f\|^{-1}$	$\|f\|^{-3}$
Random Walk Frequency Noise	$\|f\|^{-2}$	$\|f\|^{-4}$

*where

$$\frac{1}{N\tau_0} \leq |f| \leq \frac{1}{2\tau_0}; \ \tau_0 = \text{sampling}$$

interval, and N = total number of data points.

TABLE 3. Translation between $S_y(f)$ and $\sigma_y^2(\tau)$.

Noise Type (Model)	$S_y(f)$	$\sigma_y^2(\tau)$ [1]	Uncertainty [2] in $\sqrt{\sigma_y^2(\tau)}$ for m values of y where m > 5
White Phase Noise*	$h_2 \lvert f \rvert^2$	$h_2 \frac{3f_h}{(2\pi)^2 \tau^2}$	$\pm \frac{0.99}{\sqrt{m}} \cdot \sigma_y(\tau)$
Flicker Phase Noise*	$h_1 \lvert f \rvert$	$h_1 \frac{1}{(2\pi\tau)^2} \quad 3 \quad \gamma + \ln(2\pi f_h \tau) - \ln 2$ $\gamma = 0.577$	$\pm \frac{0.99}{\sqrt{m}} \cdot \sigma_y(\tau)$
White Frequency Noise	h_o	$\frac{h_o}{2\tau}$	$\pm \frac{0.87}{\sqrt{m}} \cdot \sigma_y(\tau)$
Flicker Frequency Noise	$h_{-1} \lvert f \rvert^{-1}$	$h_{-1}\ 2 \ln 2$	$\pm \frac{0.77}{\sqrt{m}} \cdot \sigma_y(\tau)$
Random Walk Frequency Noise	$h_{-2} \lvert f \rvert^{-2}$	$h_{-2} \frac{(2\pi)^2 \tau}{6}$	$\pm \frac{0.75}{\sqrt{m}} \cdot \sigma_y(\tau)$

*The quality f_h is the high-frequency cutoff of the noise; $2\pi\tau f_h >> 1$.

TABLE 4

Uses of Measures of Frequency Stability

Use	Generally Preferred Method of Analysis
Theory/Research	Power Spectral Densities, $S_y(f)$ & $S_\phi(f)$
Diagnostics	Power Spectral Densities, $S_y(f)$ & $S_\phi(f)$
Overall Performance Prediction	Power Spectral Densities, $S_y(f)$ & $S_\phi(f)$
Simple Comparisons	Two-Sample Variances
Simple Estimate of PSD	Two-Sample Variances
Procurement Specs.	Two-Sample Variances
Computer Simulation	ARIMA Models [5]
Prediction	ARIMA Models [5]
Environmental Correlations (Diagnostics)	ARIMA Models [5]

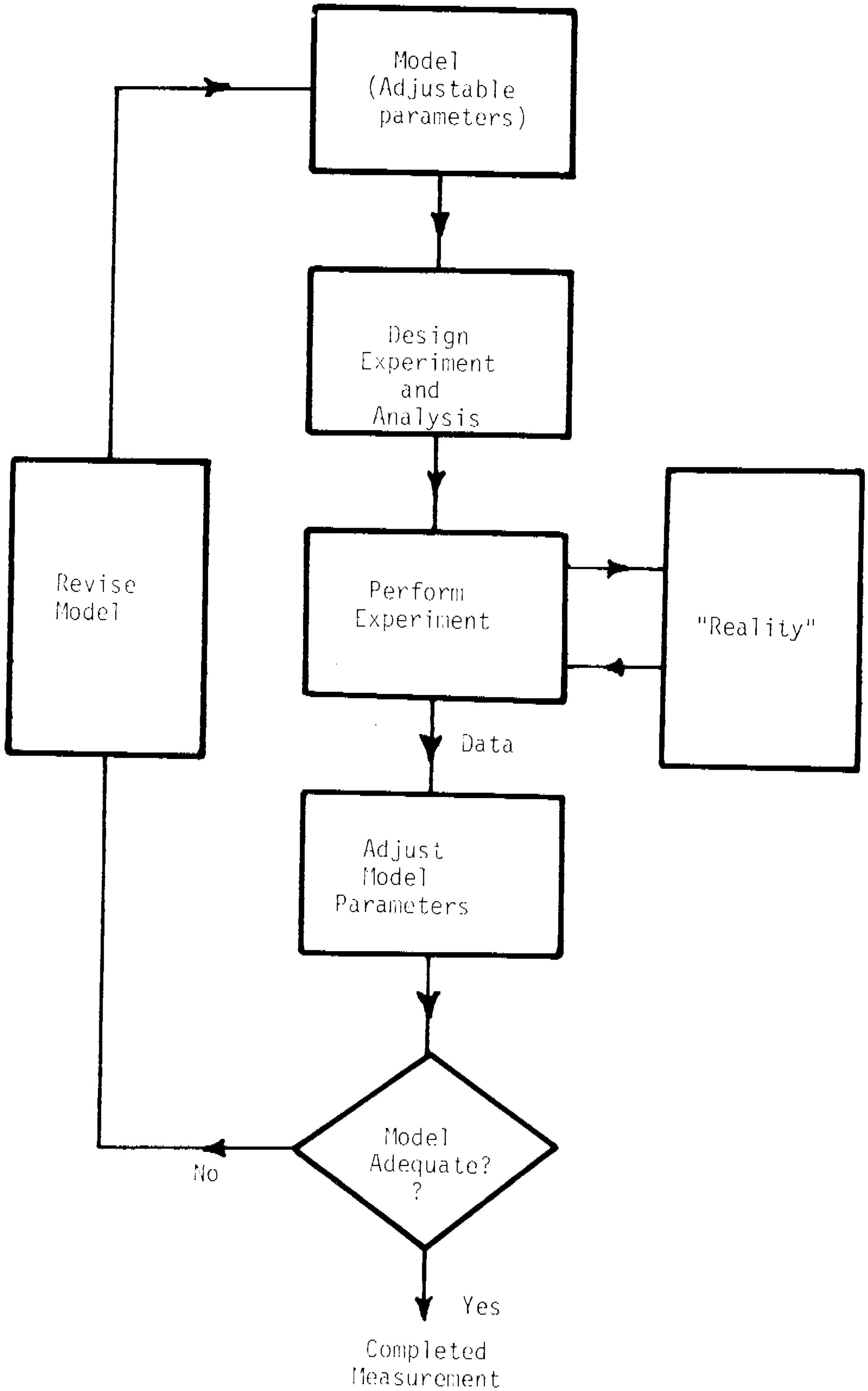

Figure 1. A Model of the Measurement Process.

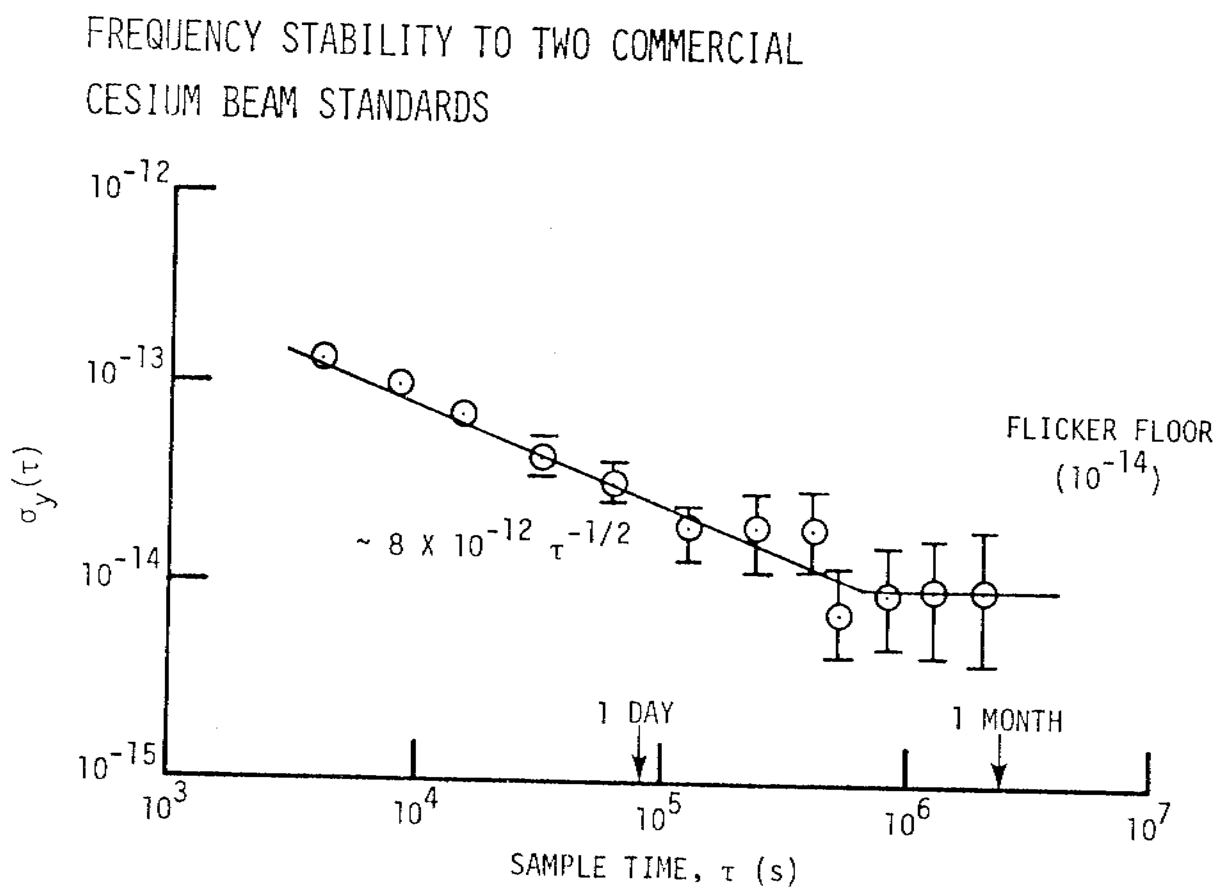

Figure 2. Frequency stability as measured between two state-of-the art commercial cesium frequency standards.

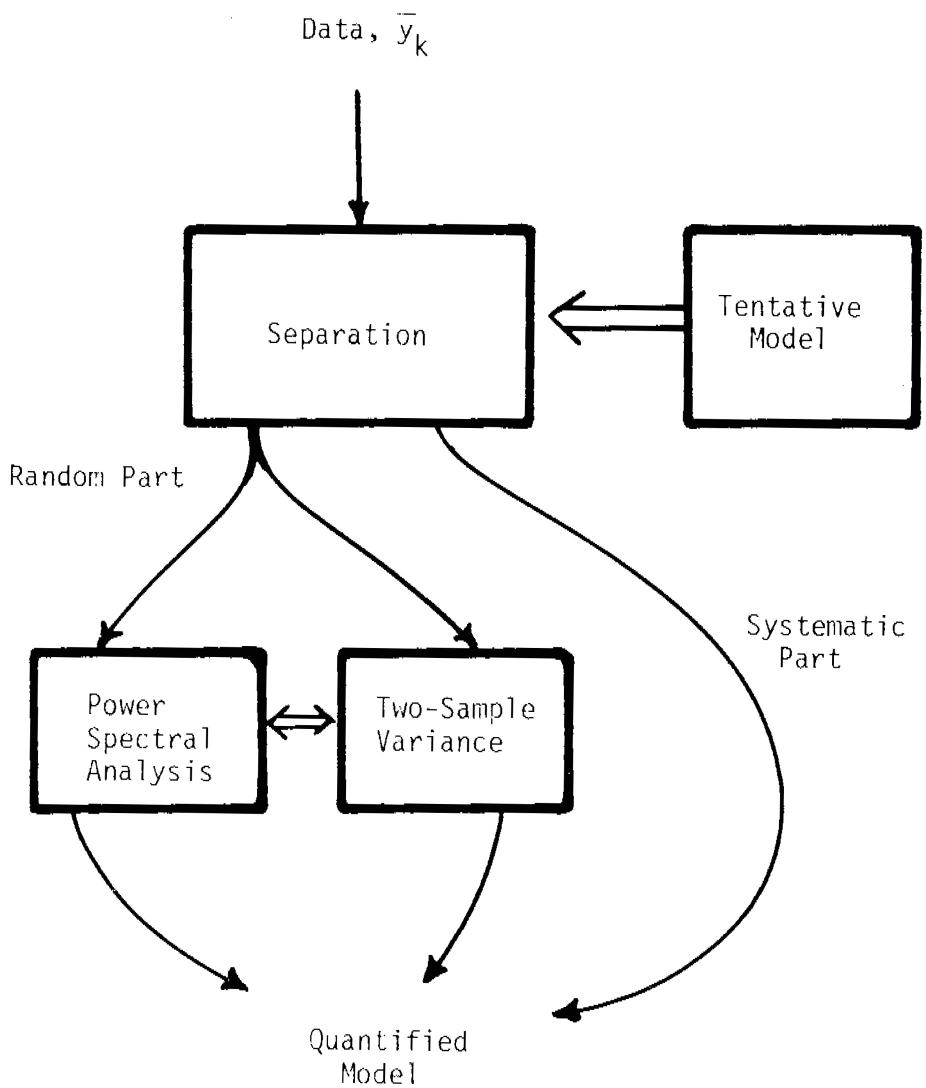

Figure 3. Analysis Techniques.

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