INTERPRETATION AND APPLICATION OF OSCILLATOR INSTABILITY MEASURES USING STRUCTURE FUNCTIONS

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ABSTRACT

This paper is written to cast further light on issues associated with contemporary frequency and phase stability measures of an oscillator. This is accomplished by generalizing known and accepted τ-domain measures of stability through the use of Kolgomorov structure functions. Two sets of stability functions (τ-domain stability measures) are presented and it is shown how they are related to the rms fractional frequency deviation and the two-sample Allan variance. It is further shown that these τ-domain measures of oscillator instability are uniquely related to the f-domain measure $S_y(f)$ by means of the Mellin transform.

Applications of these stability functions to specifying and predicting performance of coherent communication systems, one-way and two-way Doppler measuring, and ranging systems is used in order to emphasize the utility of the theory.

I. INTRODUCTION

A problem of current interest to statistical communication theorists, communication and radar system design engineers and other working groups, see Fig. 1,

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is that of specifying and selecting accurate and stable frequency generators for use in the implementation of radar, communication, navigation, and time service systems. Certain users of frequency generators have had to face, for the most part, the deleterious effects which oscillator instability produces upon system performance by "seat-of-the-pants" engineering. At the same time the users of frequency generators frequently ask the question: What are the stability requirements of the frequency generator(s) needed in a particular application?

In fact, it appears at present that those involved with the process of manufacturing atomic time standards, and electronic time bases have been largely concerned with the characterization of oscillator instability from the viewpoint of developing accurate and stable clocks [1], [2], [3]. This working group has measured the stability of oscillators and compared the measurements with frequency stability measures [1], [2] and suggested mathematical models [1], [2], [4] of the oscillator. On the other hand, statistical communication and radar theorists [5], [6], for the most part, have been developing mathematical models for communication, radar and tracking systems by assuming the availability of ideal (perfect sine wave) frequency generators. The net result of these disjoint ventures, see Fig. 1, is that not enough emphasis has been placed upon determining how oscillator instability degrades system performance measures or how the frequency stability measures proposed in [1], [2] can be used to assess performance. Presently, there is a real need for this in the design of current systems and planning of advanced systems.

This state of affairs, see Fig. 1, is not uncommon to find in many scientific fields where theory, manufacturing and practice are developed and used by widely dispersed workers. At the present the authors feel that there is a lack of understanding among the users pertaining to the differences between oscillator phase instability and frequency instability. There is also a lack of understanding of how to use accepted frequency instability measures. To some engineers, phase and frequency instability imply the same concept; consequently, one of the main purposes of this paper is to offer a unified mathematically characterization of phase instability (time-base jitter) and frequency instability (frequency-base jitter) of an oscillator and demonstrate the degree of sameness of the two concepts, their interconnections with frequency stability measures [1], [2], and present mathematical formulations such that certain users can select the appropriate instability measure for their application. It is hoped that this paper will also spur and motivate a closer relationship between the working groups illustrated in Fig. 1 by providing a systematic forum for discussion.

1.1 Organization of the Paper

Structure functions* (SFs) [7] are introduced so that characterization of oscillator instability is established on a sound mathematical basis regardless of the user application. SFs provide a common mechanism by which all working groups can communicate about the instability of frequency generators. Furthermore, they represent stability functions by which system performance can be predicted or a frequency generator selected. SFs are also introduced so that users of frequency generators can see how current frequency standards enter into the performance expressions of modern communication, Doppler and range measuring systems. There are several other reasons why SFs are introduced [8], [9]. The unifying role they play with respect to frequency stability standards is summarized and their use in various applications as a means of defining system performance is given. The concepts of phase instability (time-base jitter) and frequency instability (frequency-base jitter) are introduced via phase and frequency SFs. In addition, τ-domain to f-domain and f-domain to τ-domain transformations are summarized in terms of Mellin transforms and SFs. Secondly, with the aid of these SFs, alternate interpretations of accepted frequency instability measures [1], [2] and their interconnections are given in hopes that the users will be able to obtain a better understanding of how "oscillator vendor data" can or cannot be used in their particular application. For the sake of brevity, emphasis on the applications have been in the communication and tracking system area.

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*Because of their applications, they are called stability functions in the abstract.

Next, we show the important role which the power spectral density (PSD) of the oscillator frequency process plays with respect to several user applications and emphasize (via theory) that it is the key element which is needed to access the deleterious effects which oscillator instability has on the performance of modern communications and tracking systems. In other words, we show that the PSD of the short term frequency instability is needed in order to define appropriate τ-domain system performance measures. For the engineer involved in the design of communication and tracking systems, it appears that the most important τ-domain performance measure is not always the rms fractional frequency deviation or the Allan variance but is application and performance measure (bit error probability, range or range-rate accuracy, etc.) dependent. Various system performance measures for users are presented in order to support this statement.

On the more controversial side, the authors provide discussions which lead to certain questions regarding the interpretation of the so-called rms fractional frequency deviation as a measure of frequency instability; rather the authors feel that it is a measure of phase instability (time-base jitter) of an oscillator and give an interpretation to support this. On the other hand, the two-sample Allan variance has a direct interpretation in terms of frequency instability (frequency-base jitter) and, as such, provides a measure of the frequency instability of an oscillator. Use of these two measures then motivate a mathematical manipulation that shows that frequency and time are not reciprocally related for real world oscillators. User applications of the two measures are given in the system context in order to offer support for the proposed interpretation and their use [10], [11]. This is accomplished by presenting performance measures for ranging, Doppler and coherent communication systems in terms of structure functions of oscillator instability. Finally, we characterize oscillator instability in terms of SFs of the RF oscillation and show the problems which arise when one attempts to use the RF domain for direct measurement of oscillator instability.

II. STRUCTURE FUNCTIONS (SF) IN OSCILLATOR INSTABILITY THEORY

2.1 SF's of the N^{th} Increment of the Phase and Frequency Noise Process

Let us consider the radian frequency process $\dot{\Phi}(t)$ of an oscillator which we assume takes the form

$$\dot{\Phi}(t) = \underbrace{\omega_0}_{\substack{\text{Mean}\\ \text{Frequency}}} + \underbrace{\sum_{k=1}^{N-1} \frac{\Omega_k}{k!} t^k}_{\substack{\text{Long Term}\\ \text{Frequency}\\ \text{Drift}}} + \underbrace{\dot{\psi}(t)}_{\substack{\text{Short Term}\\ \text{Frequency}\\ \text{Instability}}} \tag{2.1}$$

where $\omega_0 = 2\pi f_0$ is assumed to be the constant mean frequency, the Ω_k's constitute a set of <u>random</u> k^{th}-order frequency drift rate and $\dot{\psi}(t)$ is a stationary, zero mean random process used to characterize the short term oscillator instability. Integrating (2.1) from 0 to t, we have the oscillator <u>phase noise process</u>

$$\Phi(t) = \omega_0 t + \underbrace{\sum_{k=2}^{N} \frac{\Omega_{k-1}}{k!} t^k}_{\substack{\text{Long Term}\\ \text{Phase Drift}}} + \underbrace{[\psi(t)-\psi(0)]}_{\substack{\text{Short Term}\\ \text{Phase Instability}}} + \Phi(0) \tag{2.2}$$

Consider now the N^{th} increment of $\psi(t)$ defined recursively by

$$\Delta^N \psi(t;\underline{\tau}^N) \triangleq \Delta^{N-1}\psi(t+\tau_n;\underline{\tau}^{N-1}) - \Delta^{N-1}\psi(t;\underline{\tau}^{N-1}) \tag{2.3}$$

where $\underline{\tau}^k \triangleq (\tau_1, \tau_2, \ldots, \tau_k)$ is a k-dimensional vector parameter and $\Delta\psi(t;\tau_1) \triangleq \psi(t+\tau_1)-\psi(t)$, which is stationary. For the purpose of our discussion, we define the N^{th} SF of <u>phase instability</u> to be [10]

$$D_\psi^{(N)}(\underline{\tau}^N) = E\{[\Delta_N \psi(t;\underline{\tau}^N)]^2\} \tag{2.4}$$

where $E\{\cdot\}$ is the expectation operator in probability theory. When $\tau_k=\tau$, k=1, ..., N, we shall denote $D_\psi^{(N)}(\underline{\tau}^N)$ by $D_\psi^{(N)}(\tau)$. If $S_{\dot{\psi}}(\omega)$ is the twosided PSD of the process $\dot{\psi}(t)$, then [10]

$$D_{\Phi}^{(N)}(\underline{\tau}^N) = \prod_{k=1}^{N} \tau_k^2 \cdot E\{\Omega_{N-1}^2\} + D_{\psi}^{(N)}(\underline{\tau}^N) \tag{2.5}$$

where

$$D_{\psi}^{(n)}(\underline{\tau}^n) = \frac{2^{2n}}{2\pi} \int_{-\infty}^{\infty} \prod_{k=1}^{n} \sin^2(\omega\tau_k/2) \frac{S_{\dot{\psi}}(\omega)}{\omega^2} d\omega \tag{2.6}$$

for $n \geq 1$. For $M > N$, (2.5) reduces to

$$D_{\Phi}^{(M)}(\underline{\tau}^M) = D_{\psi}^{(M)}(\underline{\tau}^M) \tag{2.7}$$

Analogously, the $(N-1)^{st}$ SF of <u>frequency instability</u> satisfies

$$D_{\dot{\Phi}}^{(N-1)}(\underline{\tau}) = \prod_{k=1}^{N-1} \tau_k^2 E\{\Omega_{N-1}^2\} + D_{\dot{\psi}}^{(N-1)}(\underline{\tau}^{N-1}) \tag{2.8}$$

where

$$D_{\dot{\psi}}^{(n)}(\underline{\tau}^n) = \frac{2^{2n}}{2\pi} \int_{-\infty}^{\infty} \prod_{k=1}^{n} \sin^2(\omega\tau_k/2) S_{\dot{\psi}}(\omega) d\omega \tag{2.9}$$

for $n \geq 1$. For $M \geq N$, then

$$D_{\dot{\Phi}}^{(M)}(\underline{\tau}^M) = D_{\dot{\psi}}^{(M)}(\underline{\tau}^M) \tag{2.10}$$

Equations (2.5)-(2.10) are important in several respects. First of all, if we take high enough increments, the corresponding SF is independent of the drift effect. This is evident from (2.7) and (2.10). Secondly, we note that the usual convergence problem associated with "flicker"-type PSD can be avoided in the τ-domain. For example, suppose $S_{\dot{\psi}}(\omega)$ behaves like $|\omega|^{-\nu}$ with $\nu \geq 1$ as $\omega \to 0$. For a fixed vector $\underline{\tau}^n$, the quantity $\prod_{k=1}^{n} \sin^2(\omega\tau_k/2)$ is proportional to ω^{2n} for $|\omega|$ small. So, as long as $\nu < (2n-1)$ the expression for $D_{\psi}^{(n)}(\underline{\tau}^n)$ is finite. Similarly, $D_{\dot{\psi}}^{n}(\underline{\tau}^n)$ is finite if $\nu < (2n+1)$. Hence, besides combatting the problems associated with the long term frequency drift, the SF approach is useful in treating "flicker"-type noise. An interesting example demonstrating these points is given in [9]. Thirdly, notice that all SFs are characterized in the τ-domain via the PSD $S_{\dot{\psi}}(\omega)$.

Equations (2.6) and (2.9) can be inverted for $S_{\dot\Phi}(\omega)$ using Mellin transforms [10]. In particular, the PSD $S_{\dot\Phi}(\omega)$ can be evaluated via

$$S_{\dot\Phi}(\omega) = \mathcal{M}^{-1}[\mathcal{M}_f(s)] \tag{2.11}$$

where the Mellin transform of the function f(•), $\mathcal{M}_f$ is given via

$$\mathcal{M}_f\left(1 - \sum_{k=1}^{n} s_k\right) = \frac{\mathcal{M}_g(\underline{s}^n)}{\prod_{k=1}^{n} \mathcal{M}_{K}(s_k)} , \tag{2.12}$$

$\mathcal{M}^{-1}[\cdot]$ is the one-dimensional inverse Mellin transform (with inverse transform variable ω), $\mathcal{M}_{K_k}(s_k)$ is the one-dimensional Mellin transform of the functions

$$K_k(\tau_k) = \sin^2\left(\frac{\tau_k}{2}\right) \quad k = 1, \ldots, n$$

and $\mathcal{M}_g(\underline{s}^n)$ is the n-dimensional Mellin transform of the function

$$g(\underline{\tau}^n) = \frac{\pi}{4^n} D_{\dot\Phi}^{(n)}(\underline{\tau}^n) .$$

Alternatively,

$$S_{\dot\Phi}(\omega) = \omega^2 \mathcal{M}^{-1}[\mathcal{M}_f(s)] \tag{2.13}$$

where now

$$g(\underline{\tau}^n) = \frac{\pi}{4^n} D_{\Phi}^{(n)}(\underline{\tau}^n)$$

Unfortunately, taking the inverse Mellin transform in (2.11) and (2.13) is difficult in general and numerical techniques may be needed. For small n and $\tau_1 = \ldots = \tau_n$, a different method for inverting (2.5) and (2.8) was presented in [9]. Figure 2 summarizes τ-domain to f-domain and f-domain to τ-domain transformations which are possible by either of the two approaches.

SFs of the frequency process $\dot\Phi$ and phase process Φ of an oscillator are fundamental to definitions or frequency standards involving oscillator instability. In the following, we give new interpretations to the recommended definitions [1], [2] of instability in terms of the theory based upon structure functions. In particular, we discuss the conditions to be imposed on the phase or frequency process in order to attach meaning to the rms fractional frequency deviation and the two-sample and L-sample Allan variance.

2.2 RMS Fractional Frequency Deviation

For the rms fractional frequency deviation to make sense the radian frequency process $\dot{\Phi}(t)$ has to be modeled as a constant ω_0 plus a stationary frequency component $\dot{\psi}(t)$ whose PSD is well behaved near $\omega = 0$, i.e., $S_{\dot{\psi}}(\omega) \sim |\omega|^{-\nu}$, $\nu < 1$. Then,

$$D_{\psi}^{(1)}(\tau) \triangleq E\{[\psi(t+\tau) - \psi(t)]^2\} < \infty \tag{2.14}$$

is the expected value of the square of the phase accumulated in τ seconds. The <u>true rms fractional frequency deviation</u> defined by Cutler-Searle can be expressed in terms of the first phase structure function as [9]*

$$\frac{\Delta f(\tau)}{f_0} = \sqrt{\frac{D_{\psi}^{(1)}(\tau)}{(\omega_0 \tau)^2}} \quad . \tag{2.15}$$

Furthermore, the statistical average of the <u>measured rms fractional frequency deviation,</u> say $\Delta\hat{f}(\tau)/f_0$, as found using frequency counted data, is easily shown to be an asympotically unbiased estimator of $\Delta f(\tau)/f_0$. The measurement bias is expressed in terms of the first phase SF via

$$\left[\frac{\Delta f(L\tau)}{f_0}\right]^2 = \frac{D_{\psi}^{(1)}(L\tau)}{L^2(\omega_0 \tau)^2} \quad . \tag{2.16}$$

2.3 The Allan Variance

For the two-sample Allan variance to yield a precise meaning, the frequency process $\dot{\Phi}(t)$ must be modeled as a linear frequency drift term and if "flicker"-type noise is present $S_{\dot{\psi}}(\omega)$ must behave like $|\omega|^{-\nu}$ for $\nu < 3$ near $\omega = 0$. Then

*In the original definitions for rms fractional frequency deviation and Allan variances [1], [2], [3], the time average, instead of the present ensemble average, was employed. However, in order to fully exploit various results on stochastic processes in the literature, the ensemble average is used in what follows. Notice that both averages are equivalent if ergodicity holds.

$$\frac{D_{\Phi}^{(2)}(\tau)}{(\omega_0\tau)^2} = E\left\{\left[\frac{\Phi(t+2\tau)-\Phi(t+\tau)}{\omega_0\tau} - \frac{\Phi(t+\tau)-\Phi(t)}{\omega_0\tau}\right]^2\right\} \tag{2.17}$$

is the expected squared fractional deviation of the average frequency. The two-sample Allan variance $E\{\sigma_A^2(2,\tau,\tau)\}$ with zero dead time between measurements can be written as [9]

$$E\{\sigma_A^2(2,\tau,\tau)\} = \frac{D_{\Phi}^{(2)}(\tau)}{2(\omega_0\tau)^2} \quad . \tag{2.18}$$

Except for a factor of $\frac{1}{2}$ and the normalization constant $(\omega_0\tau)^2$, the two-sample Allan variance is exactly the second phase structure function. We also note for small τ that the average in (2.17) is related to the instability of the frequency process, i.e., derivative of the phase process.

If the frequency process $\dot{\Phi}(t)$ does not have any drift and $S_{\dot{\psi}}(\omega)$ behaves like $|\omega|^{-\nu}$ for $\nu < 3$, then the L-sample Allan variance [2], [3, eq. 4] is

$$E\{\sigma_A^2(L,\tau,\tau)\} = \frac{L}{L-1}\frac{1}{(\omega_0\tau)^2}\left[D_{\psi}^{(1)}(\tau) - \frac{1}{L^2}D_{\psi}^{(1)}(L\tau)\right] \tag{2.19}$$

The L-sample Allan variance is not related to the two-sample Allan variance in a simple way, but rather it is an asymptotically unbiased estimator of the rms fractional frequency deviation squared provided the latter is well defined. The estimator bias is expressed in terms of the first phase SF via

$$\frac{1}{(\omega_0\tau)^2}\left[\frac{1}{L-1}D_{\psi}^{(1)}(\tau) - \frac{1}{L(L-1)}D_{\psi}^{(1)}(L\tau)\right] . \tag{2.20}$$

2.4 Relationship Between RMS Fractional Frequency Deviation and Allan Variance

If the assumptions regarding the frequency process $\dot{\Phi}(t)$ in Section 2.2 hold, then the two-sample Allan variance and the rms fractional frequency deviation form a one-to-one correspondence via

$$E\{\sigma_A^2(2,\tau,\tau)\} = 2\left\{\left[\frac{\Delta f(\tau)}{f_0}\right]^2 - \left[\frac{\Delta f(2\tau)}{f_0}\right]^2\right\} \qquad (2.21)$$

The equation can be inverted to yield

$$\left[\frac{\Delta f(\tau)}{f_0}\right]^2 = \tfrac{1}{2}E\{\sigma_A^2(2,\tau,\tau)\} + \tfrac{1}{2}\sum_{k=1}^{\infty} \frac{D_\psi^{(2)}(2^k\tau)}{2(\omega_0 2^k\tau)^2} \quad . \qquad (2.22)$$

In addition, the L-sample Allan variance is related to the rms fractional frequency deviation via

$$E\{\sigma_A^2(L,\tau,\tau)\} = \frac{L}{L-1}\left(\frac{\Delta f(\tau)}{f_0}\right)^2 - \frac{1}{L(L-1)}\frac{D_\psi^{(1)}(L\tau)}{(\omega_0\tau)^2} \qquad (2.23)$$

Provided $D_\psi^{(1)}(\tau) < \infty$ for all τ, then for large L we see that the L-sample Allan variance converges to the mean squared value of the fractional frequency deviation. The authors believe that these facts have not been recognized in previous studies.

In addition, these relationships allow us to identify the bias [12, eq. 15] function $\chi(L)$ as

$$\chi(L) \triangleq \frac{E[\sigma_A^2(L,\tau,\tau)]}{E[\sigma_A^2(2,\tau,\tau)]} = \left(\frac{2L}{L-1}\right)\left[\frac{D_\psi^{(1)}(\tau) - D_\psi^{(1)}(L\tau)/L^2}{D_\psi^{(2)}(\tau)}\right] \qquad (2.24)$$

and if $D_\psi^{(1)}(\tau) < \infty$ for all τ, then for large L ,

$$\chi(L) \cong 2D_\psi^{(1)}(\tau)/D_\psi^{(2)}(\tau)$$

$$= \tfrac{1}{2} + \tfrac{1}{2}\sum_{k=1}^{\infty} 4^{-k}\frac{D_\psi^{(2)}(2^k\tau)}{D_\psi^{(2)}(\tau)} \qquad (2.25)$$

III. INTERPRETATION OF OSCILLATOR INSTABILITY MEASURES

3.1 Phase (Time) and Frequency Instability

In characterizing the performance of an oscillator, it is of fundamental interest to distinguish between the notion of "instability" and "precision" of its frequency as well as its phase. While it is relatively easy (and unambiguous) for one to agree upon a definition of the "precision" of a random quantity in terms of, perhaps, the deviation from its mean value (the relative error), there is no general agreement of what "instability" means. In what follows, we shall take "instability" to mean instability with respect to the passage of time, i.e., how a stochastic quantity behaves (in a probability sense) relative to its past τ seconds earlier. For example, an oscillator which emits a frequency that does not change with respect to time is a "stable" oscillator; the frequency it gives at any particular instant is "precise". Using these notions, we shall attempt to interpret the meanings of phase instability. Conventional measures, in particular, the rms fractional frequency deviation and Allan variance, will serve as the basis for our interpretations.

To formalize the following discussion, we shall assume the instantaneous phase and frequency of the oscillator satisfy

$$\Phi(t) = \omega_0 t + \psi(t) + [\Phi(0) - \psi(0)] \tag{3.1}$$

$$\dot{\Phi}(t) = \omega_0 + \dot{\psi}(t) \tag{3.2}$$

where $\dot{\psi}(t)$ is stationary with finite variance and $E\{\psi(t)\} = E\{\dot{\psi}(t)\} = 0$. Under these assumptions, the instability measures to be discussed are all well-defined quantities.

The instantaneous phase (time) instability of the phase noise process $\psi(t)$ is measured by the increment $\Delta\psi(t;\tau) = \psi(t+\tau) - \psi(t)$ while the rms phase instability is measured by $\sqrt{D_\psi^{(1)}(\tau)}$. Thus the rms fractional frequency deviation is related to the phase instability via (2.15). On the other hand, the instantaneous frequency instability of the frequency noise process $\dot{\psi}(t)$ is measured by the increment $\Delta\dot{\psi}(t;\tau) = \dot{\psi}(t+\tau) - \dot{\psi}(t)$ while the rms frequency instability is measured by $\sqrt{D_{\dot{\psi}}^{(1)}(\tau)}$. The two-sample Allan variance approximates $D_{\dot{\psi}}^{(1)}(\tau)/2\omega_0^2$ with an error

$$\frac{D_{\dot{\psi}}^{(1)}(\tau)}{2\omega_0^2} - E[\sigma_A^2(2,\tau,\tau)] = \frac{2}{\pi\omega_0^2}\int_0^\infty \sin^2\frac{\omega\tau}{2}\left[1-\sin^2\frac{\omega\tau}{2}\right] S_{\dot{\psi}}(\omega)d\omega \geq 0 \tag{3.3}$$

For a PSD $S_{\dot{\psi}}(\omega)$ with power concentrated in the frequency region $\omega\tau \ll 1$ then

$$\frac{D_{\dot{\psi}}^{(1)}(\tau)}{2\omega_0^2} \simeq E[\sigma_A^2(2,\tau,\tau)] \tag{3.4}$$

Thus the two-sample Allan variance is a measure of frequency instability of the oscillator. Table I serves to summarize the results.

From (2.14), (2.15) and (3.1), it is obvious that the rms fractional frequency deviation $\Delta f(\tau)/f_0$ is an instability measure of the phase process Φ or the accuracy (relative error) of the phase increment $\Delta\Phi = \Phi_\tau - \Phi$. Now the frequency accuracy is related to $E\{\dot{\psi}^2\}/\omega_0^2$. If (2.15) is used as a measure of frequency precision, the error introduced in interpreting (2.15) as a measure of frequency precision is

$$\frac{E\{\dot{\psi}^2\}}{\omega_0^2} - \left[\frac{\Delta f(\tau)}{f_0}\right]^2 = \frac{1}{\pi\omega_0^2}\int_0^\infty \left[1 - \mathrm{sinc}^2\left(\frac{\omega\tau}{2}\right)\right] S_{\dot{\psi}}(\omega)d\omega \geq 0 \tag{3.5}$$

where $\mathrm{sinc}(x) \triangleq \sin x/x$. The severity of the error in this interpretation depends on the shape of $S_{\dot{\psi}}(\omega)$ weighted by the weighting function $1-\mathrm{sinc}^2(\omega\tau/2)$.

3.2 Normalized Time Instability Times the Frequency Instability as a Measure of Oscillator Instability

Depending on specific applications, oscillators are used as references for making time (phase) measurements as well as in frequency measurements. Since frequency instability is fundamentally different from phase instability, it seems only fair to specify the performance of an oscillator in terms of both its frequency and phase instability in the measurement (τ) domain. For this purpose, we can define frequency instability $\delta f(\tau)$ to be the rms change in frequency over the observation time τ, through

$$\delta f(\tau) \triangleq \frac{\sqrt{D_{\dot{\psi}}^{(1)}(\tau)}}{2\pi} \cong \sqrt{2}\, f_0\sqrt{E[\sigma_A^2(2,\tau,\tau)]} \tag{3.6}$$

On the other hand, the time-instability $\delta T(\tau)$ can be defined as the rms change in the "time" (see (3.11)) from its mean τ, via

$$\delta T(\tau) \triangleq \sqrt{\frac{D_{\psi}^{(1)}(\tau)}{\omega_0^2}} = \left(\frac{\Delta f(\tau)}{f_0}\right)\cdot\tau \tag{3.7}$$

Phase (Time) Process	Instantaneous Phase Accuracy Measure	Instantaneous Phase Instability Measure	RMS Phase Instability Measure	Accepted Frequency Standard
$\psi(t)$	$\psi - E[\psi]$	$\Delta\psi = \psi_\tau - \psi$	$\sqrt{D_\psi^{(1)}(\tau)/(\omega_0\tau)^2}$	$\Delta f(\tau)/f_0$

Frequency Process	Instantaneous Frequency Accuracy Measure	Instantaneous Frequency Instability Measure	RMS Frequency Instability Measure	Accepted Frequency Standard
$\dot{\psi}(t)$	$\dot{\psi} - E[\dot{\psi}]$	$\Delta\dot{\psi} = \dot{\psi}_\tau - \dot{\psi}$	$\sqrt{D_{\dot{\psi}}^{(1)}(\tau)/2(\omega_0\tau)^2}$ $\cong \sqrt{E[\sigma_A^2(2,\tau,\tau)]}$	$E[\sigma_A^2(2,\tau,\tau)]$

Phase Increment Process	Phase Increment Accuracy Measure	Phase Increment Instability Measure	RMS Phase Increment Instability Measure	Accepted Frequency Standard
$\Delta\psi(t)$	$\Delta\psi - E[\Delta\psi]$	$\Delta(\Delta\psi) = \Delta^2\psi$	$\sqrt{D_\psi^{(2)}(\tau)/2(\omega_0\tau)^2}$ $= \sqrt{E[\sigma_A^2(2,\tau,\tau)]}$	$E[\sigma_A^2(2,\tau,\tau)]$

Table I. Interpretation of Phase and Frequency Instability and the Relationship to the RMS Fractional Frequency Deviation and Two-Sample Allan Variance.

The locus of the point $(\delta f(\tau), \delta T(\tau))$, as a function of the observation time τ, could serve as the overall stability performance guide of an oscillator and is depicted in Fig. 3.

The product: frequency instability times time instability

$$\delta f(\tau)\delta T(\tau) \triangleq \frac{1}{f_0}\sqrt{\frac{D_{\dot{\psi}}^{(1)}(\tau)}{(2\pi)^2} \cdot \frac{D_{\psi}^{(1)}(\tau)}{(2\pi)^2}} \qquad (3.8)$$

of an oscillator serves as a parameter by which various oscillators can be compared for a given frequency of oscillation and a given observation time τ. In Fig. 3, this parameter at $\tau = \tau_1$ is equal to the area of the shaded area. For an ideal oscillator then

$$\delta f(\tau) \cdot \delta T(\tau) = 0 \qquad (3.9)$$

for all τ. Thus any oscillator, which is to be used as a standard in instability measurements, should appear to the oscillator under test to satisfy (3.9) for all τ of interest. Moreover, if we use (3.4) to approximate $D_{\dot{\psi}}^{(1)}(\tau)$ by the Allan variance then we can write

$$\underbrace{\frac{\delta f(\tau)}{f_0}}_{\substack{\text{rms Frequency-}\\ \text{Base Instability}}} \cdot \underbrace{\frac{\delta T(\tau)}{\tau}}_{\substack{\text{rms Time-Base}\\ \text{Instability}}} \cong \sqrt{2}\sqrt{\underbrace{E[\sigma_A^2(2, \tau, \tau)]}_{\substack{\text{Allan}\\ \text{Variance}}}}\ \underbrace{\left(\frac{\Delta f(\tau)}{f_0}\right)}_{\substack{\text{rms Fractional}\\ \text{Frequency Deviation}}} \qquad (3.10)$$

This equation supports our earlier claim that the Allan variance is a measure of oscillator frequency instability while the rms frequency deviation is a measure of oscillator phase (time) instability. The authors believe that the above construction has provided insight into the meaning of the Allan variance and the rms fraction frequency deviation.

To gain further insight into these performance measures, let us assume that the random process $f(\tau) \triangleq \Delta\dot{\psi}(\tau)/2\pi$ and $T(\tau) \triangleq \Delta\psi(\tau)/\omega_0$ are jointly Gaussian. If the process $\dot{\psi}(t)$ is stationary, it can be shown that $\Delta\psi(\tau)$ and $\Delta\dot{\psi}(\tau)$ are uncorrelated. Hence $f(\tau)$ and $T(\tau)$ are uncorrelated Gaussian processes with zero mean and standard deviations $\delta f(\tau)$ and $\delta T(\tau)$. If we plot the contour of constant probability density as a function of τ as in Fig. 4, we observe the effect of the degradation of oscillator stability as τ increases.

3.3 Is the Nominal Frequency of a Real World Oscillator Reciprocally Related to the Nominal Period?

If the oscillator phase process $\Phi(t)$ in (3.1) is used for a clock (period $T_0 \triangleq 2\pi/\omega_0$), the time $T_c(t)$ registered by the clock is given by

$$T_c(t) = [\Phi(t)-\Phi(0)]/\omega_0 \tag{3.11}$$

During the interval $(t, t+T_0)$, the clock time accumulated is

$$\Delta T_c(T_0) = T_0 + \frac{\Delta\psi(T_0)}{\omega_0} \tag{3.12}$$

The time average frequency during this time interval is

$$\overline{f}(T_0) = \frac{1}{2\pi T_0} \int_t^{t+T_0} \dot{\Phi}(\xi)d\xi = f_0 + \frac{\Delta\psi(T_0)}{2\pi T_0} \tag{3.13}$$

Hence the product $\Delta T_c(T_0) \times \overline{f}(T_0)$ is

$$\Delta T_c(T_0) \times \overline{f}(T_0) = 1 + \frac{\Delta\psi(T_0)}{\pi} + \frac{(\Delta\psi(T_0))^2}{(2\pi)^2} \tag{3.14}$$

which is random with mean

$$E\{\Delta T_c(T_0)\times\overline{f}(T_0)\} = 1 + \left(\frac{\Delta f(T_0)}{f_0}\right)^2 \tag{3.15}$$

$$= 1 + D_\psi^{(1)}(T_0)/(2\pi)^2$$

From (3.15), it is clear that the rms fractional frequency deviation (or first phase SF) characterizes the uncertainty in the product $\Delta T_c(T_0) \times \overline{f}(T_0)$. Since $(\Delta f(T_0)/f_0)^2$ is not zero for real world oscillators, the period and (time average) frequency are not inversely related for any physical clock. Instead the mean of the product differs from unity by the square of the rms fractional frequency deviation.

IV. EFFECTS OF OSCILLATOR INSTABILITY IN APPLICATIONS IN TERMS OF SF'S

The characterization and specification of oscillator instability must ultimately be made in the light of its effects on the performance of systems that rely on the oscillation. The utility of SF's will now be demonstrated in terms of a number of user applications. It shall become evident that, in every application, the PSD $S_{\dot{\psi}}(\omega)$ of the stationary frequency noise is the key to predicting performance due to oscillator instability. Only a summary account is given herein; detailed analytical developments are anticipated in future articles.

4.1 Effect of Oscillator Instability on Phase-Locked Loop (PLL) Tracking

One interesting application of SF's is found in studying the tracking behavior of a PLL. In addition to a first-order statistical characterization, i.e., the phase-error variance, the first SF of the phase error is also an important performance measure of a tracking loop. It is important in assessing the effect of oscillator instabilities on tracking performance and it is related to the average hold-in time of the PLL.

As an example, it was shown in [10] (assuming the loop bandwidth is small compared to τ, i.e., $W_L\tau \ll 1$) that the first phase-error SF of a first-order loop due to oscillator instability is given by

$$D_{\varphi}^{(1)}(\tau) = D_{\psi_1}^{(1)}(\tau) + D_{\psi_2}^{(1)}(\tau) \qquad (4.1)$$

where φ is the loop phase error, ψ_1 represents the phase noise on the transmitted oscillation and ψ_2 represents the phase noise process produced by the frequency generation in the receiver. Notice that in this case, the phase instability measure (3.7) of the oscillators are the quantities of interest in predicting the ability of the loop to track the input process in the presence of oscillator phase noise.

4.2 Effect of Oscillator Instability on One-Way and Two-Way Doppler Measurements

In a practical Doppler measurement system, the Doppler information is usually extracted from the local phase estimate increment $\Delta\Theta(\tau)$ generated by the receiver PLL. For one-way Doppler measurements, the error contribution due to oscillator instabilities can be shown to be contained in the equation

$$D_{\Theta_e}^{(1)}(\tau) = \mathbb{D}_{\psi_1}^{(1)}(\tau) + \mathbb{D}_{\psi_2}^{(1)}(\tau) + D_{\psi_3}^{(1)}(\tau) \qquad (4.2)$$

where Θ_e denote the error in the local phase estimate, ψ_1 represents the transmitter oscillator instabilities, ψ_2 represents the receiver VCO instability and ψ_3 represents the receiver reference instability. The quantities $D_{\dot\psi_1}^{(1)}(\tau)$ and $D_{\dot\psi_2}^{(1)}(\tau)$ denote the first SF of the loop filtered ψ_1 and ψ_2 process which are directly related to the PSD $S_{\dot\psi}(\omega)$. Thus the key role which $S_{\dot\psi}(\omega)$ plays is again manifested.

For two-way Doppler measurements [6], the situation is more complicated. The first SF of the error in the local phase estimate due to oscillator instabilities is given by

$$D_{\Theta_e}^{(1)}(\tau) = D_{\psi_0}^{(1)}(\tau) + D_{\psi_1}^{(1)}(\tau) + D_{\psi_2}^{(1)}(\tau) + D_{\psi_3}^{(1)}(\tau) \qquad (4.3)$$

where ψ_0 represents the transmitter oscillator instability, ψ_1 represents the vehicle receiver VCO instability and ψ_2 represents the ground receiver, VCO instability, and ψ_3 represents the ground receiver reference instability [5].

In the case that: (1) the transmitter and receiver reference signals are derived from the same timing source and (2) the static phase gain and the receiver filtering can be neglected, then the error contribution due to the reference oscillator instability $\psi_0=\psi_3=\psi$ alone is $D_{\psi}^{(2)}(\tau, T)$ which is explicitly (see (2.6))

$$D_{\psi}^{(2)}(\tau, T) = \frac{16}{\pi} \int_0^{\infty} \sin^2\left(\frac{\omega\tau}{2}\right) \sin^2\left(\frac{\omega T}{2}\right) \frac{S_{\dot\psi}(\omega)}{\omega^2}\, d\omega \qquad (4.4)$$

where T is the round-trip delay time of the signal. Hence we see that the second SF of the phase noise is the important quantity to specify the performance of the reference oscillation. It can be shown that $D_{\psi}^{(2)}(\tau, T)$ is related to the two-sample Allan variance with non-zero dead time $T-\tau$ between measurements $E\{\sigma_A^2(2, T, \tau)\}$ via

$$E\{\sigma_A^2(2, T, \tau)\} = \frac{1}{2(\omega_0\tau)^2} D_{\psi}^{(2)}(\tau, T) \qquad (4.5)$$

Under the same assumptions, if the Doppler frequency shift is derived from the loop frequency estimate, the error contribution from the oscillator instability $\dot\psi$ is $D_{\dot\psi}^{(1)}(\tau)$. In this case, the first SF of the frequency noise process of the reference is the instability measure of interest.

4.3 Effect of Oscillator Instability on Range Measurements

In one-way ranging, the vehicle transmits a ranging signal to the tracking station. Range from the tracking station to the vehicle can then be determined from a measurement of the local phase estimate of the receiver PLL relative to the phase of a reference. The uncertainty introduced in the range estimate due to oscillator instabilities alone is related to the variance in the phase estimate error

$$\sigma^2_{\varphi_e} = \tilde{\sigma}^2_{\psi_1} + \tilde{\sigma}^2_{\psi_2} + \sigma^2_{\psi_3} \tag{4.6}$$

where

$$\tilde{\sigma}^2_{\psi_1} = \frac{1}{2\pi}\int_{-\infty}^{\infty} |1 - H_{\varphi}(j\omega)|^2 \frac{S_{\dot{\psi}}(\omega)}{\omega^2}\, d\omega$$

$$\tilde{\sigma}^2_{\psi_2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} |1 - H_{\varphi}(j\omega)|^2 \frac{S_{\dot{\psi}_2}(\omega)}{\omega^2}\, d\omega$$

$$\sigma^2_{\psi_3} = E\{\psi_3^2\}$$

and ψ_1 represents the transmitted reference instability, ψ_2 represents the receiver VCO instability, ψ_3 represents the receiver reference instability and $H_{\varphi}(j\omega)$ is the receiver PLL closed-loop transfer function.

In two-way measurements, a ranging signal is transmitted to the vehicle to be tracked and is returned by it to the tracking station. Range is then determined from a measurement of the phase of the returned tone relative to a reference (possibly the same one as the original transmitted tone). The uncertainty introduced in the error in the phase estimate due to oscillator instabilities alone is related to the variance of the phase estimate error

$$\sigma^2_{\varphi_e} = \tilde{\sigma}^2_{\psi_0} + \tilde{\sigma}^2_{\psi_1} + \tilde{\sigma}^2_{\psi_2} + \sigma^2_{\psi_3} \tag{4.7}$$

where the σ's are defined by expressions similar to (4.6). Here ψ_0 represents the instability of the transmitted signal, ψ_1 represents the vehicle transponder VCO instability, ψ_2 represents the receiver VCO instability, ψ_3 represents the instability in the reference used for phase comparison.

If the effect of the PLL's are neglected and the transmitted tone and the reference tone are derived from the same oscillator, then the error contribution to the range measurement due to the oscillator phase noise ψ is related to $D_{\psi}^{(1)}(T)$ where T is the round-trip path delay. Notice that $D_{\psi}^{(1)}(T)$ is proportionate to the rms fractional frequency deviation.

If the phase error of the PLL in the tracking station is used to measure range-rate via $\Delta\varphi(\tau)/\tau$ or $\dot{\varphi}$, expressions similar to (4.7) and (4.8) can be obtained in terms of the appropriate quantities.

4.4 Effect of Oscillator Instability on Coherent Communication System Performance

In digital communication systems, the bit error probability [6] is an important performance measure and is related to the statistics of a random variable which depends upon [11] the phase error φ and its increment $\Delta\varphi \triangleq \Delta\varphi(T)$. Then the probability of error will depend upon the instability and the accuracy of φ. The first SF $D_{\psi}^{(1)}(T)$ given by (4.1) becomes the important parameter in determining the bit error probability. The effect of the oscillator instabilities then enters into the bit error probability evaluation through, for example, (4.1) and therefore can be used to select frequency generators for system implementation.

4.5 Application of SFs in Studying Timing Standards [9]

In timing standards, it is important to specify the instability $\dot{\psi}(t)$ as well as the drift term as in (2.1). The use of SF's provides a convenient means for such measurements. The scheme is first to work with successively higher increments until the drift problem and the singularity problem associated with "flicker"-type PSD are alleviated. The "measured" SF's will then be inverted via (2.11) or (2.13) to get an estimate for $S_{\dot{\psi}}(\omega)$. After obtaining this estimate, one can design an optimal estimation procedure to evaluate the drift. An example of such an approach was discussed in [9].

4.6 Miscellaneous Applications of Structure Functions

All signalling schemes in telecommunication and radio engineering depend on a stable frequency reference. In a time-division multiple access (TDMA) system, the accumulated phase noise between "marker" intervals of duration $T-\tau$ is an important factor in evaluating synchronization performance. The rms phase noise accumulated between marker interval is characterized by $\sqrt{D_\psi^{(2)}(\tau, T)}$ which is related to the two-sample Allan variance with dead zone (see (4.5)).

In communication systems employing differential phase-shift keying (DPSK), it turns out that the second SF $D_\psi^{(2)}(\tau, \tau)$ of the transmitter oscillator phase noise ψ is an important parameter in specifying achievable system data rates.

Because of the path delays involved in network synchronization, the SF approach is also important in studying network synchronization and specifying the requirements on oscillator stability.

V. SF'S AND THEIR RELATIONSHIP TO THE RF OSCILLATION

Frequently the PSD of the RF oscillation is used as a means of obtaining the "PSD" of the phase noise process. Here we derive an expression for the N^{th} moment function of the RF oscillation and show how the SF of the phase process enters into the measurement as well as the problems associated with this approach. In order to proceed with this approach, several restrictive assumptions are required if this approach is to be tractable.

For the sake of simplicity in what follows, let us work with the complex oscillation

$$s(t, \Phi(t)) = \sqrt{2P} \exp[j\Phi(t)] \tag{5.1}$$

where P represents the mean square power of the oscillation and $\Phi(t)$ is given by (2.2). For $k = 0, \ldots, N$, let us define the random variables

$$z_k = \begin{cases} \binom{N}{N-k} s\left(t + \sum_{m=1}^{k} \tau_m\right) & k = 0, 2, \ldots \\ \binom{N}{N-k} s^*\left(t + \sum_{m=1}^{k} \tau_m\right) & k = 1, 3, \ldots \end{cases} \tag{5.2}$$

where ξ^* denotes the complex conjugate of ξ. Then the $(N+1)^{st}$ moment function of the random variables $\{z_0, \ldots, z_N\}$ is related to the N^{th} SF of the phase process $\Phi(t)$ via

$$E[z_N \cdots z_0] = (\sqrt{2P})^{N+1} E\{\exp[j\Delta^N \Phi(t;\underline{\tau}^N)]\} \tag{5.3}$$

Notice that if the N^{th} increment of the phase process $\Phi(t)$ is stationary, then the quantity $E\{\exp[j\Delta^N\Phi(t;\underline{\tau}^N)]\}$ is independent of t and equal to the characteristic function of the N^{th} increment, say, $\Delta^N\Phi(t=0;\underline{\tau}^N)$, evaluated at unity. Furthermore, if $\Delta^N\Phi(t;\underline{\tau}^N)$ is Gaussian, then

$$E[z_N \cdots z_0] = (\sqrt{2P})^{N+1} \exp[-D_\Phi^{(N)}(\underline{\tau}^N)/2] \tag{5.4}$$

which generalizes earlier work when $N > 1$. As an example, if the complex oscillation satisfies

$$s(t,\Phi(t)) = \sqrt{2P}\exp\{j[\omega_0 t + \psi(t) - \psi(0) + \Phi(0)]\} \tag{5.5}$$

then

$$E[s_\tau s^*] = 2P\exp[-D_\psi^{(1)}(\tau)/2]\exp(-j\omega_0\tau) \tag{5.6}$$

The real part of (5.6) satisfies

$$\mathrm{Re}\{E[s_\tau s^*]\} = 2P\cos\omega\tau\exp[-\tfrac{1}{2}D_\psi^{(1)}(\tau)] \tag{5.7}$$

It can be shown that if $\psi(t)$ is stationary, Gaussian, then the time average correlation $\overline{E[r(t+\tau)r(t)]}$ of the process

$$r(t) \triangleq \mathrm{Re}\{s(t)\} = \sqrt{2P}\cos(\omega_0 t + \psi(t) - \psi(0) + \Phi(0)) \tag{5.8}$$

where $\mathrm{Re}\{\xi\}$ denotes the real part of ξ, is given by [13, pp. 108-119]

$$\begin{aligned} \overline{E[r_\tau r]} &\triangleq \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} E[r(t+\tau)r(t)]dt \\ &= \tfrac{1}{2}\mathrm{Re}\{E[s_\tau s^*]\} \\ &= P\cos\omega_0\tau\exp[-D_\psi^{(1)}(\tau)/2] \end{aligned} \tag{5.9}$$

Notice that in this case, the first phase SF $D_\psi^{(1)}(\tau)$, which is related to phase instability, is the quantity of interest in characterizing the instability of an RF oscillation. If we now assume that $\psi(t)$ is stationary then the PSD of $\sin[\omega_0 t+\psi(t)]$ is given by [14]

$$S_r(\omega) = P[S_{r_0}(\omega-\omega_0) + S_{r_0}(\omega+\omega_0)] \tag{5.10}$$

where

$$S_{r_0}(\omega) = \exp(-\sigma_\psi^2)\left[\delta(\omega) + \frac{S_{\dot\psi}(\omega)}{\omega^2} + \frac{1}{2!}\left(\frac{S_{\dot\psi}(\omega)}{\omega^2} * \frac{S_{\dot\psi}(\omega)}{\omega^2}\right) + \frac{1}{3!}\left(\frac{S_{\dot\psi}(\omega)}{\omega^2} * \frac{S_{\dot\psi}(\omega)}{\omega^2} * \frac{S_{\dot\psi}(\omega)}{\omega^2}\right) + \ldots\right] \tag{5.11}$$

and the symbol * denotes convolution. Obviously, the first term corresponds to unmodulated carrier and the remaining terms are due to oscillator instability. Thus, it is clear that the "tails" of $S_r(\omega)$ do not correspond to $S_{\dot\psi}(\omega)$!

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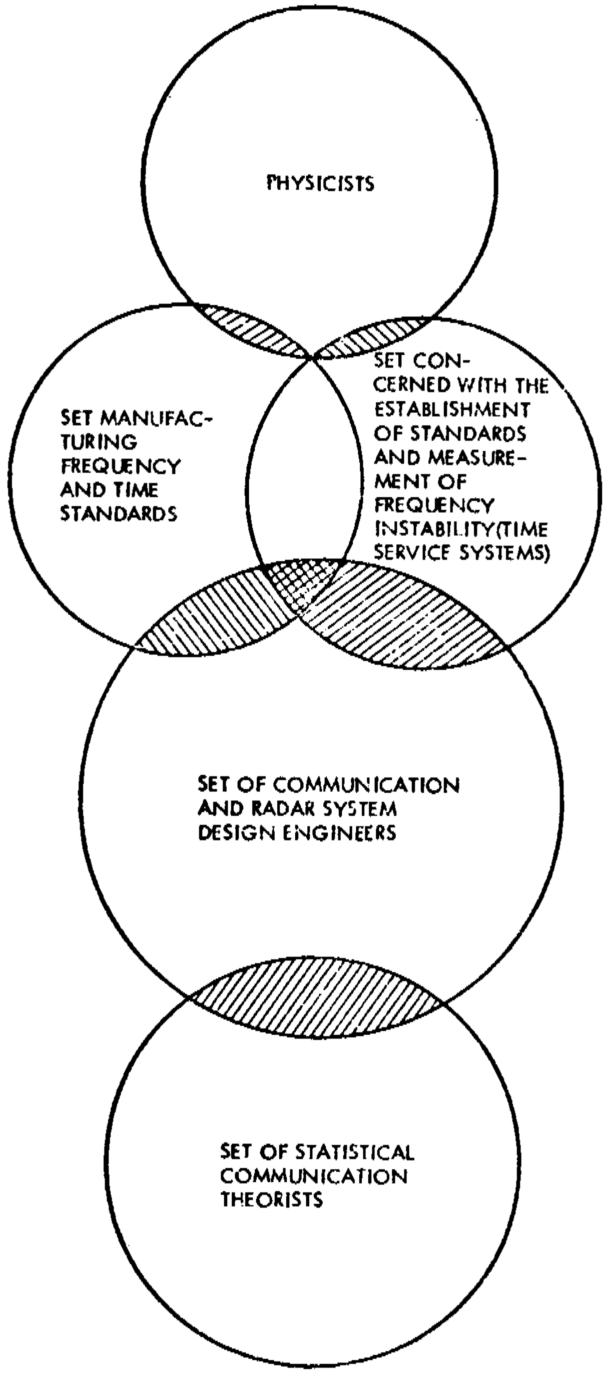


Figure 1. Characterization of the State-of-Affairs Among Various Working Groups and Users of Frequency Generators.

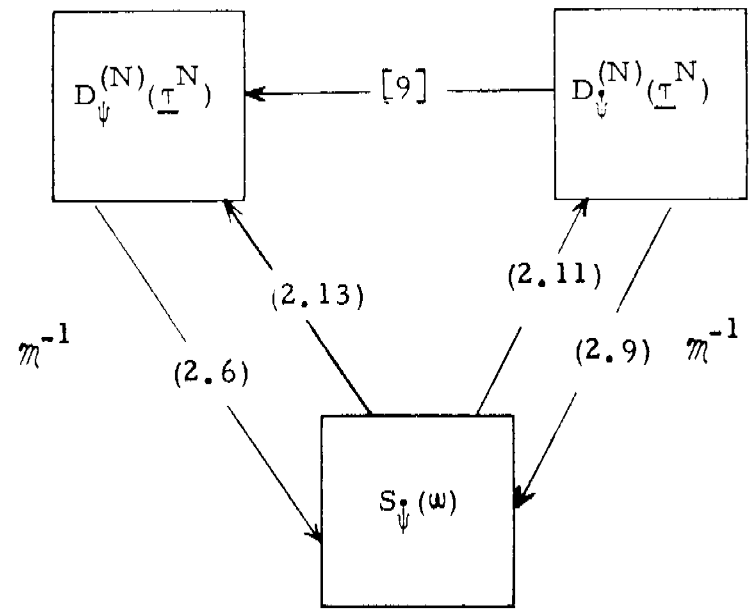


Figure 2. τ-Domain to f-Domain and f-Domain to τ-Domain Transformations.

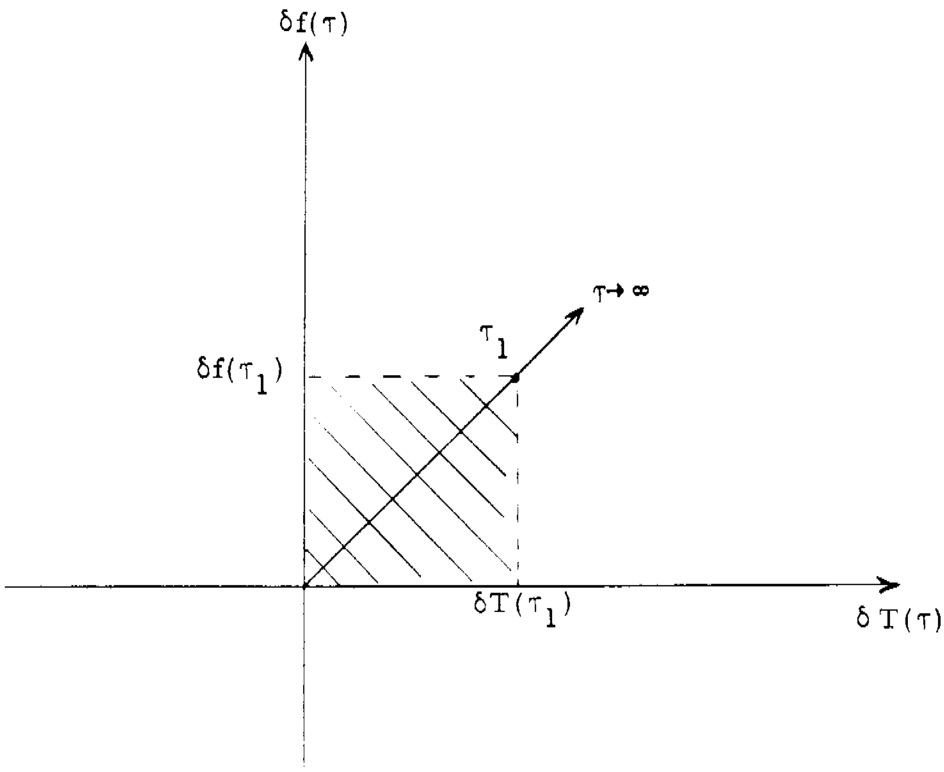
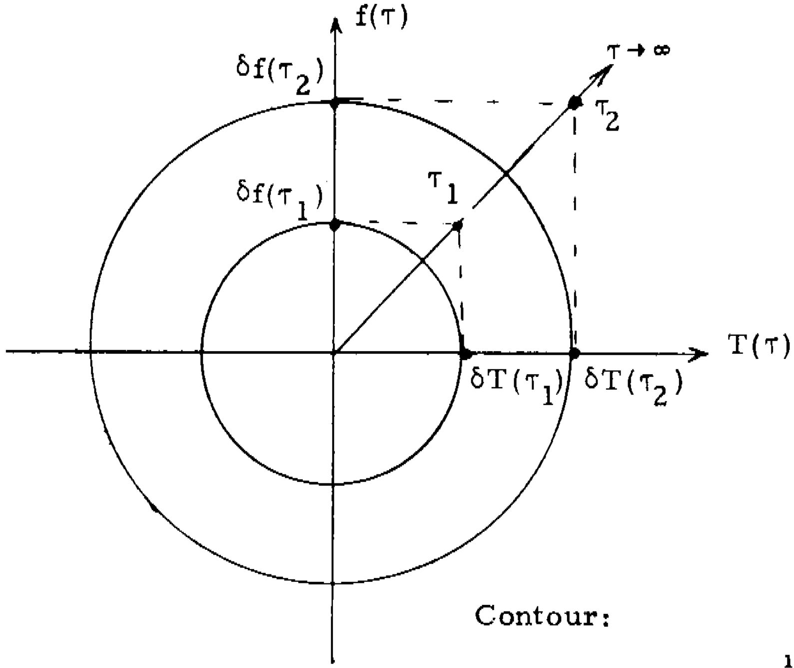


Figure 3. An Overall Frequency/Time Instability Measure.



$$p(f(\tau), T(\tau)) = \frac{e^{-\frac{1}{2}}}{2\pi \delta f(\tau) \delta T_c(\tau)}$$

Figure 4. Contour of Constant Probability Density as a Function of τ.