NBS TECHNICAL NOTE **683**

Models for the Interpretation of Frequency Stability Measurements

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ERRATA

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Page 3, Fig. 2.1, Model Translations:

Eq. (4.2) and Eq. (4.1) should be changed to Eq (1.2) and (1.1), respectively.

Page 9

Equation (3.7) should read: $y_i - \phi y_{i-1} = a_i$

Page 16, Table 4.1

Flicker phase information should be corrected to reposition { in last column, as follows:

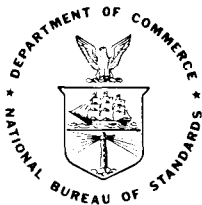$$h_1 \cdot \frac{1}{(2\pi\tau)^2} \{3[\gamma + \ln(2\pi f_h \tau)] - \ln 2\}$$

Page 24, line 7

Change "as in Ref. [14]." to read, "as in Ref [38]."

Models for the Interpretation of Frequency Stability Measurements

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The results of measurements of frequency stability are normally
interpreted in the terms of models. Typically, these models are
expressed as a power-law for the power spectral density (e.g.,
$S_y(f) = h_{-1}f^{-1}$). The experiments which provide the basis for
these models are always limited in their range and include neither
zero nor infinite Fourier frequency. Some authors have neglected
the limits to the actual physical devices and the underlying data
and have encountered mathematical difficulties with the models at
either zero or infinite frequency or both. These problems are
associated with the model and not the actual oscillators being
modeled. By carefully taking into account actual limits of the
device and constraining discussions to be concerned only with
quantities which can be observed in a practical sense, one can
avoid the problems of non-convergence and non-stationarity.

Even for stationary and convergent models, however, there is grow-
ing evidence that the typical Gaussian models may be inadequate.
There appear to be occasional (sporadic) steps in the frequency
of oscillators of a magnitude which is difficult to explain with
conventional models.

Key Words: Flicker noise; frequency; frequency stability; oscil-
lators; phase noise; stationary models.

1. Introduction

This paper reviews models used in interpreting the measurements of frequency sta-
bility. The purpose of this review is to identify models which are useful for the mea-
surements. The reader is specifically referred to Slepian's paper [1], "On Bandwidth,"
and the discussion of "reality" (his "facet A") and models for reality (his "facet B").

In the literature one can find papers which propose many models for frequency
instabilities. Some of these models are simply expressed in mathematical terms but have
very questionable applicability to real devices. One often finds that these models (not
the devices which they were intended to simulate) have severe pathologies such as non-
existent instantaneous frequencies. This paper explores some of these model pathologies
and attempts to put them into perspective with the important goal: to adequately describe
real oscillators.

The paper also details the elements of conventional models currently in use, discusses
some of the experimental evidence supporting these models, and presents some results
suggesting an inadequacy of these models.

2. Present Models

2.1 The Measurement of Frequency Stability

The measurement of frequency stability has received a great deal of attention for
well over ten years. Some of the key events which have occurred recently are as follows:

 a. In November 1964 NASA and IEEE sponsored a Symposium on Short-Term Frequency
  Stability [2].

 b. A special issue of the Proceedings of the IEEE was published [3] in February 1966
  which was devoted to frequency stability.

c. Following this publication, a subcommittee of the IEEE was formed to define precisely the term "frequency stability." To date, the only publication of this subcommittee [4] appeared in October 1970 as an NBS Tech Note and was later published in May 1971. This document proposed the power spectral density of the fractional frequency fluctuations and the "Two-Sample Variance" as the recommended measures of frequency stability.

d. Additional papers have appeared [5,6,7,8,9,10], giving confidence intervals on the "Two-Sample Variance" and suggesting other variations of measurement methods which lead to convergent results even in the presence of flicker FM. (While there are numerous references to frequency stability, this list provides access to a really vast field.)

Insofar as oscillator models are concerned, nearly all current publications acknowledge the presence of fluctuations of frequency whose power spectral density varies as the reciprocal Fourier frequency--i.e., flicker FM--over substantial spectral regions.

2.2 Models and noise simulation

Given a data set there are various means of analyzing the data and model building.

Once a model has been devised it is possible to use the model to simulate performance or predict future values of the noise in an optimal way. In use today there are three means of data analysis: (1) Power spectral density estimation [11]; (2) Fitting of Auto-Regressive, Integrated, Moving Average (ARIMA) models [12]; and (3) the use of the two-sample variances [4], $\sigma_y^2(\tau)$. Methods (1) and (3) have been recommended [4] as preferred measures of frequency stability.

It seems that the power spectral density is almost always the choice of the theoretician for use in models and design applications. For short-term stability measurements (f > 1 Hz) an experimenter will often use an analog spectrum analyzer and obtain spectra "on line." For longer term data (f < 1 Hz) one could make use of digital computers and obtain spectrum estimates following any of a number of texts [11]. Unfortunately, the process of spectrum estimation tends to be a rather specialized field, and many experimenters do not avail themselves of these techniques.

Another, more recent technique is the fitting of particular models (ARIMA models) to the data [12]. This has the very practical advantage that when this process is completed, one has immediately a means of simulating the data on a computer, optimally predicting future values, and estimating the power spectral density. Like direct spectrum estimation, however, the analysis techniques tend to be specialized, and not many experimenters make use of this technique. Of course, this may change in the future, but so far it is little used for frequency-stability measurements.

Perhaps the most common means of analyzing frequency instabilities of oscillators is the use of the two-sample variance, $\sigma_y^2(\tau)$, and the use of tables as in Ref. [4] to infer the associated spectral density. This common use of $\sigma_y^2(\tau)$ probably arises from (1) the relative simplicity of its estimation from experimental data and (2) the common occurrence of power-law-type spectral densities for the observed instabilities. In a certain practical

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sense $\sigma_y^2(\tau)$ allows a very rapid estimation of the spectral density $S_y(f)$. In the sense of obtaining the best confidence in the result, the spectrum estimated in this way probably does not represent the most efficient use of the data, but it is a quick-and-easy method.

Since there exist three essentially equivalent methods of describing a given oscillator's instabilities, it is of value to be able to translate theoretically between these different cases. Table 4.1 provides a translation between $\sigma_y^2(\tau)$ and $S_y(f)$ for those few special cases given (i.e., power-law spectral densities). While it is true that the cases given in Table 4.1 are very commonly encountered in practice, they do not suffice for the general problem. Cutler [4] has provided an equation for translating from the (one-sided) power spectral density to the two-sample variance (also see Ref. [13]):

$$\sigma_y^2(\tau) = 2 \int_0^\infty df \; S_y(f) \cdot \frac{\sin^4 \pi f \tau}{(\pi f \tau)^2} \tag{1.1}$$

At present no simple, closed form of the inverse of this relation is known to the author; however, Lindsey and Chie [6] have shown that $S_y(f)$ can be given in terms of the Mellin transforms of the Kolmogorov structure functions [6,14], which in turn are closely related to the two-sample variance.

In contrast, if one knows the parameters $\left\{\phi_i, \; \theta_i\right\}$, σ_a^2 for an ARIMA model [12], then one can immediately estimate the spectrum by the relation [12]:

$$S_y(f) = 2\sigma_a^2 \; \frac{\left| 1 - \theta_1 e^{-i2\pi f} - \theta_2 e^{-i4\pi f} - \ldots - \theta_q e^{-i2\pi q f} \right|^2}{\left| 1 - \phi_1 e^{-i2\pi f} - \phi_2 e^{-i4\pi f} - \ldots - \phi_p e^{-i2\pi p f} \right|^2} \tag{1.2}$$

Now, however, the inference of the ϕ's and Θ's from $S_y(f)$ is not so simple. Appendix B provides a means of estimating the ARIMA parameters from $S_y(f)$ by a graphical technique.

Figure 2.1 provides a map for the translations discussed above.

There have been other suggested measures of frequency instability also, but these are not in common use. They include the Hadamard Variance [15], the Kolmogorov structure functions [6], and others [5,7,16].
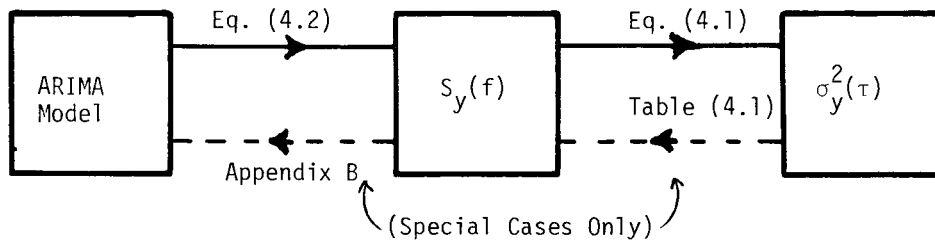

Fig. 2.1 Model Translations

2.3 Model elements

In general, the elements of a model for the instabilities of an oscillator can be divided into those caused by variations in the environment (e.g., temperature, acceleration,

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magnetic field, etc.) and those intrinsic to the oscillator.   Although the environmentally dependent elements are very important (in some cases the most important elements), this paper will treat only the intrinsic elements--the elements present for oscillators in nearly ideal environments.

2.3a Deterministic elements

Deterministic elements are elements of the model which, in principal, can be known with perfect confidence. For example, quartz crystal oscillators often display a very stable, continuous, linear drift of frequency. The deterministic elements for most oscillators include:

> Periodic terms,
> Phase (time) offset,
> Frequency offset, and
> Linear frequency drift.

While other deterministic elements are possible, these are the more common ones. There are problems concerning how one estimates these quantities [4].

2.3b Gaussian noise elements

The typical models [4] used for the noise observed on oscillators can be expressed as a specific functional form for the power spectral density of the fractional frequency fluctuations, $y(t)$. An often used functional form for the power spectral density is a power-law model:

$$S_y(f) = h_{-2}|f|^{-2} + h_{-1}|f|^{-1} + h_0 + h_1|f|^1 + h_2|f|^2$$

for   (2.1)

$$0 < f_\ell \leq |f| \leq f_h < \infty \, .$$

While other functional forms could certainly be treated within the model framework, (2.1) is the most common form. To date, almost all models have been Gaussian in nature, even if one does not assume a power-law dependence as in (2.1).

Typically, the behavior outside of the interval (f_ℓ, f_h) is unspecified because it hasn't been measured. One should note this explicit statement on the limits of the Fourier frequencies, which ought to conform to the region of direct observation. Beyond this range one is free to choose suitably convergent forms for the spectrum. While it is true that one must assume some form for the spectrum outside of this range, one might just as well choose a convenient form which does not complicate life too much.

2.3c Sporadic elements

There have been suggestions that, on occasion, the frequency of an oscillator suddenly takes on a new and permanent average value [17]. These changes have limited documentation, and it is not clear whether these frequency steps are distinct and additional to the other parts of the model or just an unsuspected visual aspect of the data. A third possibility is that the "flicker part" of the model manifests itself as sudden changes in frequency [18]. Still, its spectrum could be proportional to $|f|^{-1}$. One possible model is sporadic flicker noise as postulated by Mandelbrot [19]. Table 2.1 summarizes the model elements.

In terms of these model elements, the analysis scheme in general should follow that of Fig. 2.2.

Fig. 2.2 Methods of Data Analysis

Gaussian noise elements

Usual model

$$S_y(f) = h_{-2}|f|^{-2} + h_{-1}|f|^{-1} + h_o + h_1|f| + h_2|f|^2$$

for $\dfrac{1}{N\tau} \leq |f| \leq \dfrac{1}{2\tau}$

Other $S_y(f)$ also used.

Systematic elements

Linear frequency drift

Frequency offset

Time (phase) offset

Periodic terms

Sporadic elements

Sudden steps in frequency

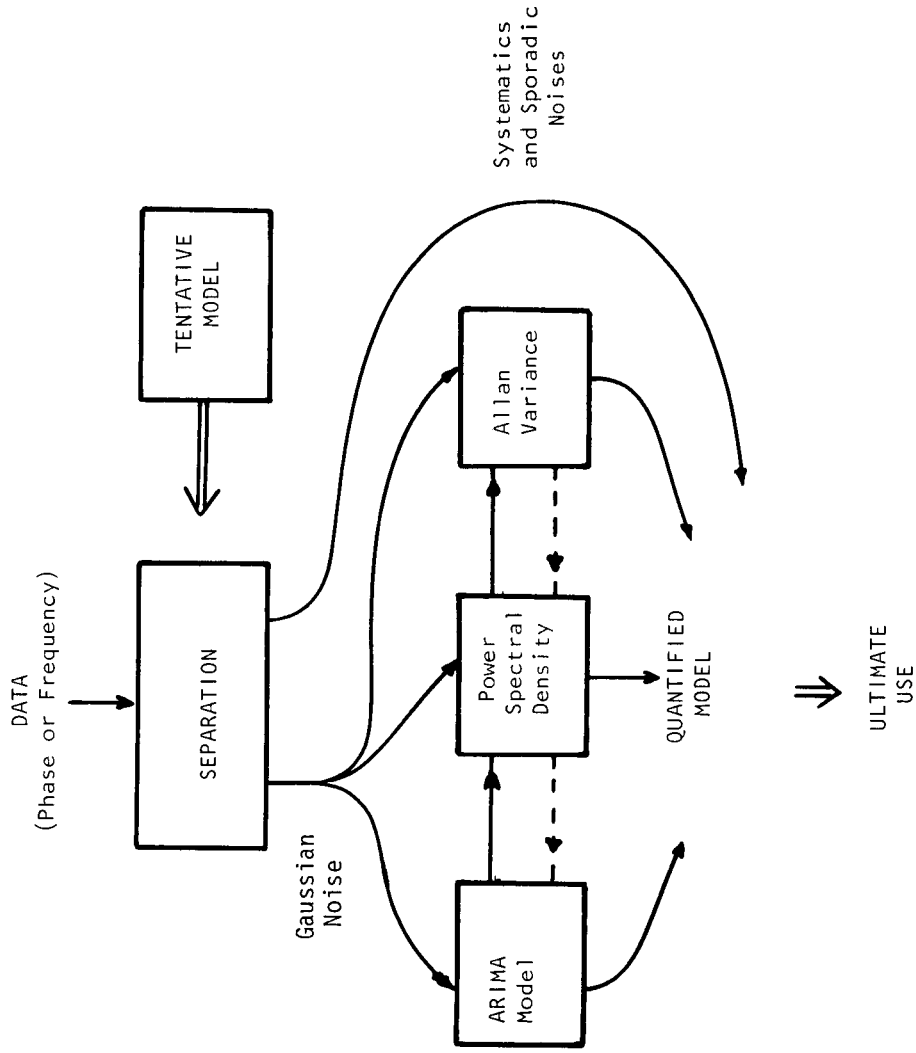Sudden steps in time (phase)

Environmental elements

Table 2.1 Model Elements

2.4 Flicker noise models

During the same period of time considered above, various models of flicker noise have
also been developed.

a.   A transfer function model was proposed [20], which generated flicker noise, but
its use was very inefficient. (The discrete model was simply in error when the impulse
response was taken as proportional to $\tau^{-2/3}$.)

b.   Efficient analog and numerical approximations to flicker noise were presented in
1971 [21]. Although done independently of Ref. [12], these models are clearly ARIMA
models and directly amenable to the treatments of Ref. [12]. There is an important
difference between Refs. [12] and [21] in the sense that [21] is synthetic in its
treatment where [12] is _analytic_. That is, Ref. [21] considers the problem of synthe-
sizing a (ARIMA) model which approximates a given power spectral density. In con-
trast, Ref. [12] _analyzes_ a specific noise sample by fitting ARIMA models to it. One
can then estimate the power spectral density of the sample noise by means of the
ARIMA parameters.

The noise model of Ref. [21] is the typical model used in current simulation
studies [22,23].

c.   Independently and nearly concurrently to Ref. [21], Mandelbrot [24] developed
an efficient Gaussian noise generator which produces numbers which approximate a
flicker process. This model is obtained by superposing several bandlimited, indepen-
dent white noises in just such a way as to obtain an arbitrarily good approximation to
a flicker noise over a finite spectral region. Although there are many similarities
to Ref. [21], the model of Ref. [24] is not as useful in that it does not have a simple
inverse and the methods of Ref. [12] are not particularly useful for this model. This
is true even though the band-limiting filters used are of the Auto Regressive (AR) form.

d.   It has been pointed out that diffusion processes [25] can give rise to flicker
noise. This is based on the recognition that _Heaviside's_ [26] semi-infinite diffuser
is the physical analog of Abel's [27] half-order integral.

e.   Halford [28] showed that there is a broad class of Poisson processes which
give rise to flicker noise. From the computer simulation point of view, these Poisson
noise samples are not nearly as easy to generate as the models given in Refs. [21]
and [24].

f.   The models presented above are all similar in the sense that they usually generate
Gaussian noises. Unique among all of the models presently known to simulate flicker
noise is the sporadic model presented by Mandelbrot [19], which is _non_ Gaussian, at
least in orders higher than the first. A similar model has been presented by Rutman
and Ubersfeld [18]. The possible applicability of this sporadic model to real oscil-
lators appears very interesting in view of recent results in several laboratories.
An example of a sporadic flicker model is presented in Appendix C.

3. Model Pathologies

Over the past many years people have become progressively more interested in the measurement of frequency stability. Today, oscillators exist whose frequencies might change only a few parts in 10^{15} from one measurement to the next. The documentation of such outstanding performance in exact quantitative terms is indeed of interest.

The process of this documentation always involves the acquisition of data. It is significant to note that these data always have an upper frequency limit and a finite duration. If one were to consider the Fourier transform of these data, it is clear that only a finite range of Fourier frequencies could be resolved, which does not include zero Fourier frequency. If the measurement data spanned a duration T in time and had an upper frequency limit f_h, then it is clear that the Fourier transform is meaningful for Fourier frequencies in the range from $1/T$ to f_h.

It is perhaps surprising (but true) that most model pathologies arise out of the simple error of extrapolating experimental results beyond their range of applicability and direct observation. By "model pathology" I specifically mean that the model (facet B) is pathologic, not the device or data (facet A) which were intended to be modeled.

These model pathologies fall naturally into two categories: the first pathology considered arises from extrapolating behavior too high in frequency. This pathology is called the "ultraviolet catastrophe." The other pathology arises from extrapolating behavior too low in frequency and is called the "infrared catastrophe." In model pathology, a concomitant of the infrared catastrophe is the "non-stationarity bugaboo." I don't mean to imply that all oscillators must of necessity be "stationary" in their performance; I only mean that the blind extrapolation of results to zero Fourier frequency should not be used as an indictment against the usefulness of stationary models.

3.1 The ultraviolet catastrophe

Many models of oscillators recognize the existence of "white noise" perturbing the instantaneous frequency of the oscillator over some significant range of Fourier frequencies [29]. In long term this gives rise to a "Brownian Motion" of the phase fluctuations.

Now if one forgets about all of the band-limiting effects in the real oscillator, amplifiers, etc., one can become concerned with the mathematically ideal Brownian Motion. This Brownian Motion is characterized as having independent increments; that is, if $\phi(t)$ is the Brownian Motion of phase, then

$$\Delta\phi(t) = \phi(t + \Delta\tau) - \phi(t) \tag{3.1}$$

is independent (i.e., random, uncorrelated) of $\Delta\phi(t')$ provided the two intervals do not overlap. The result is that the limit as $\Delta\tau \to 0$ of the quantity $\frac{\Delta\phi(t)}{\Delta\tau}$ does not exist [30]. That is, one cannot even speak of an instantaneous frequency defined by the relation [4].

$$\nu(t) = \frac{1}{2\pi} \frac{d\phi}{dt} . \tag{3.2}$$

Of course, the problem arises from the assumption that the differences, $\Delta\phi(t)$, are independent for all non-overlapping $\Delta\tau > 0$, even, say, $\Delta\tau = 10^{-20}$ sec. If one looks at

real electronics, one knows that transistors and even the wires and connectors have very substantial frequency-limiting effects. The net result is that in real electronic circuits all orders of differentiation can be assumed to exist with complete safety [1]. One just does not have to be concerned with the idea that coherent x-rays or γ-rays might be coming out of the output connector! (One should note that if the spectrum of the signal is band limited,* then the spectrum of the frequency fluctuations is similarly band limited.) Thus, this is a pathology of the model and not of the device.

3.2 The infrared catastrophe and stationarity

Many experiments on oscillators reveal estimates of the power spectral density of the frequency fluctuations varying as $|f|^\alpha$ for $\alpha \approx -1$ over a significant but limited range of the Fourier frequencies [29,31,32,33,34,35]. (This is normally called "Flicker Noise" frequency modulation.) If one now considers the "ideal" model of flicker noise with an assumed spectral density varying as $|f|^{-1}$ for all f, then some significant problems arise.

First, it is obvious that the integral

$$\int_{f_1 > 0}^{\infty} |f^{-1}|\ df = \infty \ ,$$

(3.3)

so one has problems with the "ultraviolet catastrophe." I assume that this aspect of model pathology has been adequately handled above. Also, however, one has

$$\int_0^{f_1 > 0} |f|^{-1}\ df = \infty,$$

(3.4)

which imples an infinite amount of power in the very low frequencies--i.e., an "infrared catastrophe." While some authors are not too bothered with either the infrared or the ultraviolet catastrophe itself, they point out that Flicker noise implies a non-stationary model [5,36], and hence one is hard put to even speak of a power spectral density at all. Thus, the assumption of a power spectral density varying as $|f|^\alpha$ for $\alpha \leq -1$ and for all f is not even self-consistent.

This problem is slightly more subtle than the ultraviolet catastrophe and perhaps explains why many people become hung up on this issue. In order to put this problem into perspective, it is of interest to consider a specific thought experiment.

Let us suppose that we have a rather large set of experimental data in the form of a single time series, x_i for $1 \leq i \leq N$. I assume that the x_i are the result of measurements at regular intervals τ and that we have already carried out many experiments on this data set. In particular, I assume that I have looked at the time series

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* See Ref [1] for a discussion of band limits.

$$\Delta x_i = x_{i+1} - x_i \text{ for } 1 \leq i \leq N-1 \tag{3.5}$$

and discovered that the Δx_i are <u>consistent with</u> a model of random, uncorrelated (i.e. "white") numbers with, say, zero mean, a Gaussian distribution, and finite variance, σ^2, just to be specific.

In a loose use of the words, one might say that an acceptable model for the x_i is a Brownian Motion, but one shouldn't get trapped into <u>either</u> the ultraviolet or infrared catastrophe too easily. One might, however, get concerned about the stationarity of the x_i, since a Brownian Motion is nonstationary [5,36].

At this point it is of value to introduce another model which can also fit the observations. I will introduce this one model in two different but equivalent forms. First, consider the apparatus of Fig. 3.1.
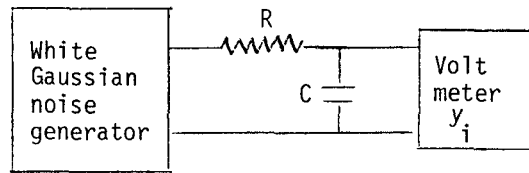

Fig. 3.1. Possible Model

If one reads the meter at intervals τ, one can obtain a time series y_i for $1 \leq i \leq N$. Now, if one selects the filter time constant, RC, such that

$$RC >> N\tau, \tag{3.6}$$

the y_i would have essentially the same statistical properties as the x_i. Indeed, y_i could also be modeled well by a Brownian Motion.

An equivalent mathematical model would be to let [12]

$$y_i - \phi y_{i-1} = a_1 \tag{3.7}$$

where the a_i are random, normal deviates with variance σ^2, and ϕ is a constant chosen such that

$$\frac{1}{1-\phi} >> N,$$

$$\phi < 1. \tag{3.8}$$

The point of significance is that the y-model <u>is</u> stationary [12]. This is easy to see from Fig. 1, since the y_i are the output of a linear, realizable filter processing a stationary noise. Thus, the y-process has a spectral density, and it varies as

$$S_y(f) \sim |f|^{-2}$$

$$\text{for } \frac{1}{N\tau} \leq |f| \leq \frac{1}{2\tau} . \tag{3.9}$$

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Now the question of significance is which model--the y-model or the Brownian Motion model--is the best model for the χ_i? I think that it is clear that the data themselves cannot yield an answer to this question! One might have aesthetic preferences, but one model is every bit as good as the other as far as objective tests are concerned. I should mention that the same statements could have been made for flicker noises rather than Brownian Motions, but the details are just a bit more involved.

There is another question which is not answerable from the data: Is the underlying χ-process stationary? The corollary to this is to note that even if the χ_i were non-stationary (i.e., a Brownian Motion), the χ_i could be well modeled by a stationary process (y_i) over a finite range (1,N). (Since the average frequency of an oscillator is normally set by the manufacturer and even calibrated later, average values are of little consequence.) The trivial nature of this stationarity issue is further emphasized by the example in Appendix B.

On the basis of aesthetics (or "parsimony" [12]) one can object to the introduction of parameters like RC or ϕ, since, by construction, they are not observable. Indeed, if one predicts that a specific measurement depends upon one of these parameters in an important way, then he would conclude that the measurement requires more data than is available [1]. Thus, one is led to building "cutoff independent" [37] measures of frequency stability.

On the other hand, one can make a plausibility argument for the existence of an actual cutoff frequency, even though to measure it requires more data than is currently available. One notes that the frequency of most quartz crystal oscillators is perturbed by a flicker noise ($|f|^{-1}$), at least over a finite range of Fourier frequencies, not including zero frequency. If it did, in fact, include zero Fourier frequency, then its output frequency now would almost surely be infinite (at least if one thinks of the crystal as infinitely old). Actually, one would just be unable to bring its frequency within specifications manually. But, since I would conclude that such an oscillator were malfunctioning (output frequency outside of specifications), I would not take data on it in the first place. One could go further to the ridiculous and ask if the finite age of the universe doesn't itself preclude the existence of low-frequency divergent Fourier frequency components down to zero frequency.

The point is that data like the χ_i series (facet A) can be well modeled (facet B) by either stationary, convergent models or non-stationary, divergent ones. In terms of observation, neither model is more correct nor incorrect than the other. Which one a person chooses to use is a matter of convenience and simple prejudice. Most papers on frequency stability prefer the use of stationary statistics and the intuitively familiar concepts like power spectral densities. Those who do not prefer this approach may not be wrong, but they are dealing with a less familiar branch of statistics; and, based on experience, they seem to be prone to extrapolating results beyond the range of direct observation with its attendant problems.

In the interests of completeness, I should note that there are other types of non-stationarity besides low-frequency divergences. The conclusions presented here

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are explicitly restricted to apparent low-frequency divergences only and not to other types of non-stationarity.

In short, the infrared catastrophe is unobservable because of finite time limits, and the ultraviolet catastrophe is not real because of finite pass bands of all real electronics. The challenge is to develop useful, consistent models which reflect these points rather than to be caught up into the mathematical complexities of unreal models which ignore the real constraints of the equipment and the data-acquisition process which were intended to be modeled.

In what follows, it will often be convenient to use an expression of the form, "the noise has a $|f|^{-2}$ spectral density," for example. This does not mean to imply that this spectral form continues to either zero or infinite frequency. What it does mean is that there are many stationary and convergent models which are consistent with the available data and which exhibit a predominantly $|f|^{-2}$ spectral character over the pertinent range of Fourier frequencies.

It is true that many calculated quantities might depend on the detailed assumptions of the spectral density of the model outside of the range of meaningful Fourier frequencies—that is, outside of the range of direct observation. This, however, is not a weakness in the model but rather serves to remind the theoretician that he is extrapolating beyond the range of reliable data.

A simple example is worthwhile. Consider a function $\chi(t)$ which experimentally appears to "have a $|f|^{-2}$ power spectral density." What one concludes from this is that the conventionally defined variance of χ will depend critically on the low frequency limit of the $|f|^{-2}$ behavior. Indeed, if experiment has not revealed this low-frequency cutoff, then any estimate of this variance is questionable at best. (This is in contrast to Ref. [19].) Clearly, the assumption that there is no lower cutoff frequency (i.e, zero Fourier frequency) is as unfounded as any other assumption. Also, even if it were known (by virtue of other information) that χ was "really" a Brownian Motion, it would still be possible to model χ arbitrarily well over any finite spectral range (not including zero Fourier frequency) with a stationary process which "has a $|f|^{-2}$ spectral density."

As noted above, the goal is to develop good and useful models for the interpretation of stability measurements of real oscillators. To the extent that the development of non-stationary and/or divergent models aids the realization of that goal, one should support the development. To the extent that it is merely a mathematical exercise, it can be appreciated for that, also. Otherwise, much of the discussions of non-stationarity and divergences are counterproductive in that they raise false issues (e.g., the detailed behavior at unobservably small Fourier frequencies). Ref. [1] is quite germane to the discussions above.

It is probably true that several authors have erred in not being adequately explicit and precise in their treatment of the infrared and ultraviolet catastrophes, although their results seem to work. On the other hand, there have been authors who have carefully and rigorously pursued these catastrophes to their illogical conclusions. The purpose of

this paper is to be a bit more careful and explicit regarding model assumptions and to come to the accepted conclusions with (hopefully) some improvement in believability. The results themselves are not new.

4. Some Stationary and Convergent Models

It is worthwhile to provide a specific example of a stationary, convergent model with all orders of derivatives existing. The model presented also has application to some real oscillators. The objective is to develop a mathematical model for an oscillator whose frequency is thought to be perturbed by a (band limited) white noise.

Specifically, a model for the _phase_ fluctuations will be developed which is stationary and convergent. In particular, we will require the model to be a reasonable fit to the data between the Fourier frequencies of, say, one cycle per one hundred years ($\sim 3 \times 10^{-10}$ Hz) and a few kilohertz--a range of about 14 decades. The upper frequency cuoff, of course, has an obvious physical basis in bandwidths of narrow-band amplifiers and various processing equipment. The low-frequency limit involves times which are reasonably longer in duration than any manufacturer might be expected to investigate.

The model power spectral density of the phase is taken to be

$$S_\phi(f) = h_0 \nu_0^2 \; \frac{e^{-\alpha^2 f^2}}{f_\ell^2 + f^2} \; , \tag{4.1}$$

where h_0 is a constant, ν_0 is the nominal frequency of the oscillator, f_ℓ is the lower cutoff frequency ($f_\ell \sim 3 \times 10^{-10}$ Hz), α is a time constant associated with the upper cutoff frequency ($\alpha \sim 10^{-5}$ sec.), and f is the Fourier frequency. Making use of Equation (8) of Ref. [4], the power spectral density of the fractional frequency fluctuations,

$$y(t) \equiv \frac{1}{2\pi\nu_0} \; \frac{d\phi}{dt} \; , \tag{4.2}$$

for the model is just

$$S_y(f) = h_0 \; \frac{f^2 \; e^{-\alpha^2 f^2}}{f_\ell^2 + f^2} \tag{4.3}$$

$$\approx h_0 \; \text{for } f_\ell < f < \frac{1}{\alpha} \; .$$

A plot of (4.3) is shown in Fig. 4.1, which indeed shows the "white" character of the frequency fluctuations over the requisite band of frequencies. It is probably worth noting again that the detailed behavior of the actual oscillator may differ substantially from the model outside of the specified range. However, meaningful measurements cannot reasonably depend on non-observables. Thus, our interest is to develop measures of frequency stability which do not depend critically on these non-observables. Indeed, we will see that one can define a measure of frequency stability which is quite insensitive to f_ℓ.
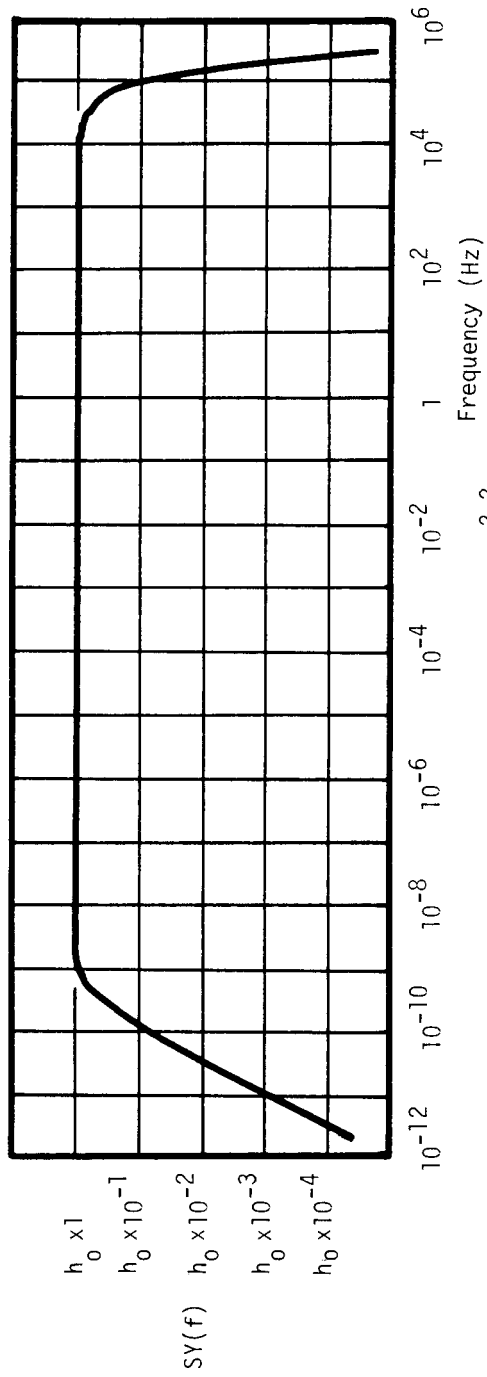
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Fig. 4.1 Model Spectrum, $S_y(f) = h_o \dfrac{f^2 e^{-\alpha^2 f^2}}{f_\ell^2 + f^2}$

$(f_\ell = 3 \times 10^{-10}; \ \alpha = 10^{-5})$

Since, for the model, $\phi(t)$ is stationary and has a power spectral density given by (4.1), we can obtain the autocorrelation function of the phase, $R_\phi(\tau)$. The details of this calculation can be found in Appendix A. For the case where

$$f_\ell \alpha \equiv 3 \times 10^{-15} << 1$$

one obtains

$$R_\phi(0) \approx \frac{\pi \nu_0^2 h_0}{2f_\ell} \tag{4.4}$$

and if $\tau >> \frac{\alpha}{\pi} \sim 3 \times 10^{-6}$ sec., one obtains

$$R_\phi(\tau) \approx \frac{\pi \nu_0^2 h_0}{2f_\ell} \left[1 - 2\pi f_\ell \tau + \frac{(2\pi f_\ell \tau)^2}{2!} - \cdots \right] . \tag{4.5}$$

We are now in a position to calculate the mean square of the change in average frequency over an interval τ for the model. Specifically, one makes use of (4.2) to note that

$$\bar{y}_k \equiv \frac{1}{\tau} \int_{t_k}^{t_k + \tau} y(t)dt = \frac{\phi(t_k + \tau) - \phi(t_k)}{2\pi \nu_0 \tau} , \tag{4.6}$$

where $t_{k+1} = t_k + \tau$.

Thus, the "two-sample variance," defined by the relation

$$\sigma_y^2(\tau) \equiv \left\langle \frac{(\bar{y}_{k+1} - \bar{y}_k)^2}{2} \right\rangle , \tag{4.7}$$

becomes

$$\sigma_y^2(\tau) = \frac{1}{8\pi^2 \nu_0^2 \tau^2} \left\langle \left[\phi(t_k + 2\tau) \right. \right.$$
$$\left. \left. - 2\phi(t_k + \tau) + \phi(t_k) \right]^2 \right\rangle \tag{4.8}$$

where the brackets denote infinite time average. Equation (4.8) can be expressed in terms of the autocovariance functions, $R_\phi(\tau)$, noting that

$$R_\phi(\tau) = \left\langle [\phi(t_n + \tau) \cdot \phi(t_n)] \right\rangle . \tag{4.9}$$

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The result is

$$\sigma^2(\tau) = \frac{1}{8\pi^2 \nu_0^2 \tau^2} \left[6R_\phi(0) - 8R_\phi(\tau) + 2R_\phi(2\tau) \right].$$

(4.10)

Substitution of (4.4) and (4.5) into (4.10) yields

$$\sigma_y^2(\tau) \approx \frac{h_0}{2\tau} \left[1 - \mathcal{O}(\pi f_\ell \tau) \right].$$

(4.11)

However, by assumption

$$\pi f_\ell \tau \ll 1,$$

that is, all measurements involve averages of duration τ, which are quite small compared to one hundred years, $\frac{1}{f_\ell}$. Thus,

$$\sigma_y^2(\tau) \approx \frac{h_0}{2\tau} ,$$

(4.12)

in complete agreement with other results [4,29,31,32].

This same treatment can be extended to other power-law types of power spectral densities. In particular, one can assume a stationary model for the phase such that

$$S_\phi(f) = h_\beta \nu_0^2 \frac{e^{-\alpha^2 f^2}}{(f_\ell^2 + f^2)^{\frac{\beta+2}{2}}}$$

(4.13)

For this development one makes use of pair 569 in Campbell and Foster [38] instead of 444. The mathematics is a bit more involved than for white noise, but essentially the same outline can be followed. The detailed calculations will not be performed here, but the results are provided in Table 4.1

Noise Type	Spectral Density* of fractional frequency fluctuation $S_y(f)$	Spectral Density* of phase fluctuations $S_\phi(f)$	Two-Sample Variance $\sigma_y^2(\tau)$
White phase noise	$h_2|f|^2$	$h_2\nu_0^2$	$h_2 \cdot \dfrac{3f_h}{(2\pi\tau)^2}$
Flicker phase	$h_1|f|$	$\dfrac{h_1\nu_0^2}{|f|}$	$h_1 \cdot \dfrac{1}{(2\pi\tau)^2} \, 3\{[\gamma+\ln(2\pi f_h\tau)]-\ln 2\}$ $\gamma = 0.577 \ldots$
White frequency	h_0	$\dfrac{h_0\nu_0^2}{|f|^2}$	$\dfrac{h_0}{2|\tau|}$
Flicker frequency	$h_{-1}|f|^{-1}$	$\dfrac{h_{-1}\nu_0^2}{|f|^3}$	$h_{-1} \cdot 2\ln 2$
Random walk frequency	$h_{-2}|f|^{-2}$	$\dfrac{h_{-2}\nu_0^2}{|f|^4}$	$h_{-2} \cdot \dfrac{(2\pi)^2|\tau|}{6}$

Table 4.1

It is perhaps worth emphasizing that for white frequency noise there is no problem with stationarity of the frequency fluctuations. For the phase, however, a "convergence factor," $\dfrac{1}{f_\ell^2+f^2}$, was used to obtain a stationary model for both phase and frequency. By explicitly displaying a high-frequency cutoff, $e^{-\alpha^2 f^2}$, we avoided all problems of infinite variances and the non-existence of certain derivatives (e.g., frequency). Of course, the high-frequency cutoff of the model has its physically observable counterpart in reality in the form of numerous band-limiting devices.

It is also worthwhile making one other observation: The technique followed above is very close to the development of generalized functions [39] or the theory of distributions [40]. Thus, it is not surprising that calculations based on the use of generalized functions have yielded the same results [31,32].

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*Assumes that spectral density is valid between limits $0 < f_\ell \leq f \leq f_h < \infty$, that $f_\ell\tau \ll 1$, and that $f_h\tau \gg 1$. The high-frequency cutoff, f_h, is important for some noises and should be measured.

5. The Adequacy of Present Models

5.1 Frequency steps and sporadic noise

If a high-quality oscillator were disturbed, it would not be surprising to note a sudden change in its output frequency. However, people have suggested [17], from time to time, that even the best clocks and oscillators display these sudden changes in frequency even though they are in a nearly ideal environment. Although there is limited objective documentation of these frequency steps ("F-steps") in the open literature [41,42], people in the field often discuss their occurrences.

At the present time there appear to be three possible positions one might assume relative to these F-steps: (1) It is a characteristic of the Gaussian noise processes (i.e., Flicker FM), which cause the fluctuations to <u>appear</u> to undergo sudden steps in frequency, but the Gaussian noise models are adequate to explain the appearance; (2) the Gaussian noise models used to explain frequency and phase fluctuations need to be augmented to include F-steps; and (3) there are apparent (non-Gaussian) F-steps, but they are a consequence of sporadic noise [19], and this sporadic noise simultaneously explains both F-steps and flicker FM--these are two facets of the same process.

These three possibilities are considered separately below:

(1) <u>No F-steps</u>. Figure 5.1 is the plot of the simulated phase of an oscillator which is perturbed by a pure (Gaussian) flicker noise. This plot was obtained by the use of an ARIMA model as discussed in Ref. [21]. Since this is a computer-simulated noise, it is known that no special, <u>ad hoc</u>, additions to the model are needed to explain the changes in slope (e.g., at point A). Indeed, the appearance of this plot should be typical of the phase plots of any oscillator perturbed by ficker FM. The implication is that F-steps may just be a consequence of the visual appearance of the data.

This hypothesis, however, can be tested by objective means and has been tested to some extent [41,42]. Basically, one can define the problem to be one of detecting a "signal" (an F-step) in the presence of "noise" (all of the Gaussian noises perturbing the oscillator). The optimum detector (in a signal-to-noise ratio sense) is just a low-pass filter processing the first derivative of the frequency. Detection occurs if the output of the filter exceeds some preset condition of acceptability. Figure 5.2 shows the result of the use of such a detector on actual clock data.

Since one has assumed that the "noise" part is Gaussian, it is a simple matter to calculate the rate of <u>false</u> detection and compare this to the observed rate of detection. In the test runs made to date on real oscillators (totaling ten runs), one should have detected two false steps. In actuality, 12 to 15 F-steps were detected, which raises severe questions concerning the "no F-step" option; that is, the pure Gaussian model (without steps) seems inadequate to explain the observed results. This is consistent with other observations.
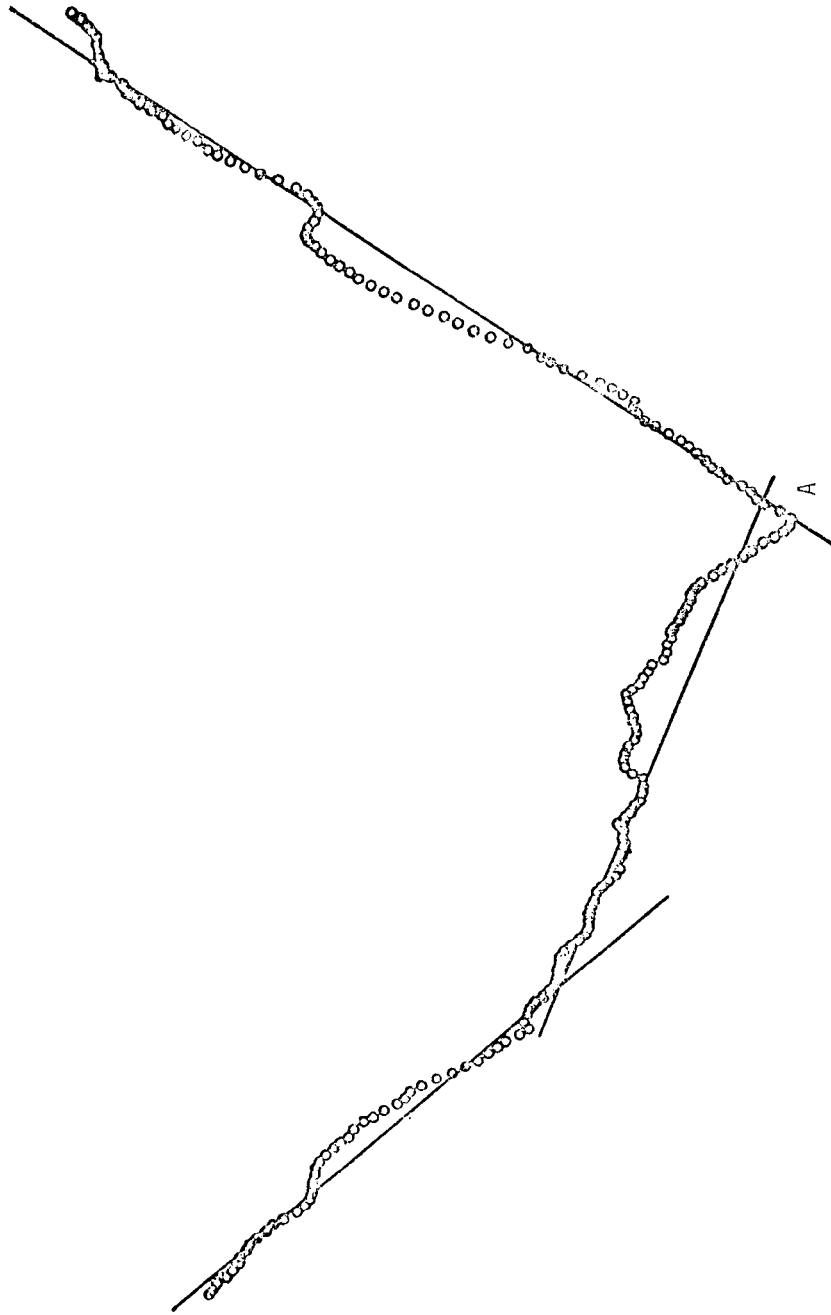
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Fig. 5.1 Simulated sample of phase noise with $S_y(f) \sim |f|^{-1}$
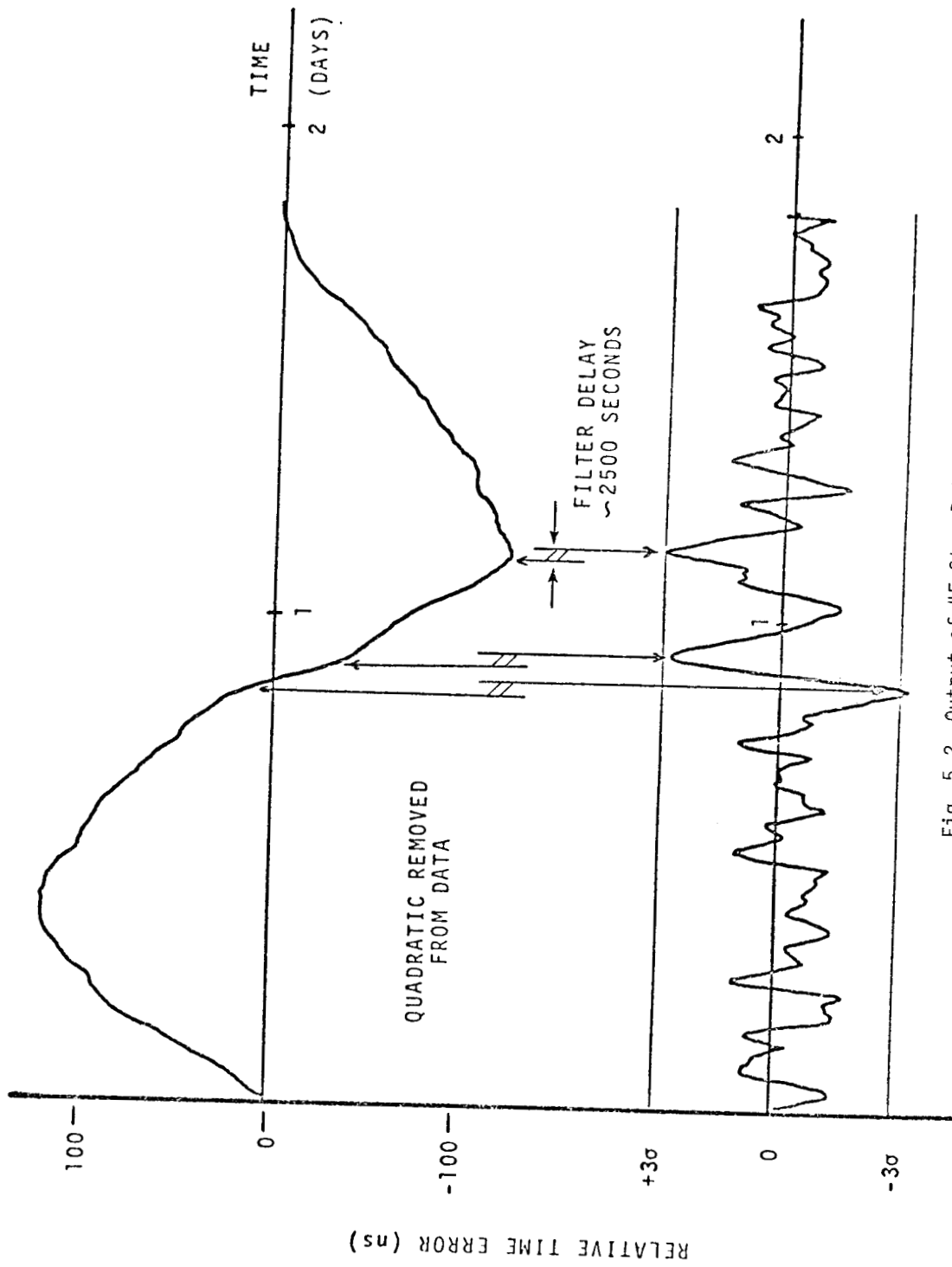
Fig. 5.2 Output of "F-Step Detector"

(2) Gaussian noise and F-steps. The F-step detector discussed above was designed according to the assumptions that this model was correct; and in fact, F-steps were detected. The observations are consistent with this model.

The ad hoc addition of F-steps to the Gaussian noise models appears adequate, but still left unresolved are the very difficult questions of rates of occurrences of F-steps, amplitude distributions, and correlations. This model has numerous parameters, only a few of which have been resolved by experiment. In the sense of Ref. [12], this does not seem to be a "parsimonious" model--it may, however, be a good model.

(3) Sporadic models for flicker FM. Ref. [19] presents a class of non-Gaussian models which are capable of generating flicker noise. That is, when noise samples are subjected to typical tests (e.g., the determination of two-sample variances), the results are totally consistent with the flicker assumption. Not only do samples of this sporadic noise satisfy the flicker diagnostics, but the probability of detecting "F-steps" with the step detector described above can be made similar to the actual oscillator results.

Thus, at least superficially, one has two viable models for the description of oscillator performance. There remain two significant questions: (a) is there any real difference in the two models; and (b), if there is, which model is the best model? The first of these questions is discussed in the next section.

5.2 Model implications

In the previous section, three models of oscillator noise were proposed:

1. No F-steps, just Gaussian noise,

2. Gaussian noise and F-steps,

3. Gaussian noise (short-term) and sporadic noise.

For the sake of argument, let's suppose that model 3 above represents "reality," and that, incorrectly, one has chosen model 2 for use in designing a clock system.

It is important to note that the apparent F-steps in model 3 are eventually self-correcting; no adjustments should be made in long term. There would be other corrections which might be used to advantage if model 3 were accepted.

Since F-steps (model 2) are detectable with reasonable probability, one can design a system to detect and "remove" these steps. However, errors of detection and errors in quantifying the sizes of the steps will naturally occur. Since these errors will tend to be random and uncorrelated, the resulting clock system will become perturbed by a random walk of frequency.

While it is possible that model 3 ("reality") was not as divergent in long term as a random walk of frequency, the act of "correcting" the F-steps guarantees that the resulting system will have random walk of frequency perturbations. Thus, the use of model 2 could lead to an overall deterioration of the long-term stability of such a clock system relative to a much simpler use of the clock data based on model 3. Indeed, one can ask if this state of affairs is not a reality in some major time scales currently in widespread use.

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While there may well be other implications to the use of models 2 and 3, the above example at least shows that the differences are not trivial. Also, model 2 has been applied to the "correction" of F-steps in actual clock systems [43] so it is a very real problem.

6. Conclusion

In all theoretical developments of oscillator statistics, models are used. Typical oscillator models assume a power-law dependence on the Fourier frequency for the power spectral density (e.g., $S_y(f) = h_{-1}f^{-1}$) over a range of Fourier frequencies. Although the experimental basis for these models involves only a finite range for the Fourier frequency components not including zero or infinity, some authors extrapolate these models beyond the range of observation. The error committed by extrapolating a non-integrable spectrum toward the high-frequency end is called the "ultraviolet catastrophe" and toward the low end is called the "infrared catastrophe." A concomitant of the "infrared catastrophe" is the "non-stationarity bugaboo." These are problems of oscillator models, not of actual oscillators [1].

One can show that a properly constructed, stationary model with a (very, very) low-frequency cutoff cannot practically be distinguished from a model with an "infrared catastrophe." This is true provided (1) that average values are not important, and (2) that one is not willing to perform experiments which require hundreds or thousands or even more years for the answer. Both of these conditions are true for oscillator models, since (1) the absolute phase is only observable modulo 2π for phase noise models, and the frequency of the oscillator is conditioned by the manufacturer to be within specifications and hence average frequency is not important for frequency models; and (2) practical uses of the stability measurements preclude extended experiments.

Some authors have objected to the use of models with low-frequency cutoffs because important quantities (like some variances) depend critically on the exact cutoff frequency. However, the important point to note is that if a quantity depends critically on the power in spectral components which have not been measured, then one is on shaky ground at best to estimate such a quantity at all. For example, if it happens to be true that an experiment suggests that a substantial fraction of the total noise power might lie in Fourier components below the reciprocal data length, then the experimenter can make no believable estimate of quantities which depend critically on the total noise power. This is not a fault of models; it is a real problem caused by limited data. Thus, one can object to stationary models with low-frequency cutoffs only on aesthetic grounds or their failure to model other aspects (not low-frequency dependent) of oscillator behavior.

As for the "ultraviolet catastrophe," all real electronic devices have very finite frequency limits. In fact, one can safely assume that all orders of differentiation of the phase exist as well. Thus, the "ultraviolet catastrophe" has its resolution in the very real, measurable band limits of the oscillator circuitry.

While one can construct stationary, convergent Gaussian models for the noise observed on real oscillators, there is growing evidence that these models might be

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inadequate. It has been suggested that one might have to add to the typical noise models the possibility of sudden steps in the instantaneous frequency of the oscillator. Rather than this ad hoc approach, it might be possible to model both the flicker noise and the steps with the sporadic noise models of Mandelbrot. It is not yet known which, if any, of these models suggested above is best.

Appendix A. A Stationary, Convergent Model

Consider the two functions

$$A(f) = \frac{1}{f_\ell^2 + f^2} \tag{A-1}$$

and

$$B(f) = e^{-\alpha^2 f^2} . \tag{A-2}$$

The Fourier transforms are

$$a(t) = \frac{\pi}{f_\ell}\ e^{-2\pi f_\ell |t|} \tag{A-3}$$

and

$$b(t) = \frac{\pi}{\alpha}\ e^{-\left(\frac{\pi t}{\alpha}\right)^2} \tag{A-4}$$

respectively, where use has been made of transform pairs 444 and 710, respectively, from Campbell and Foster [38].

If we now assume that the (one-sided) power spectral density of the phase is

$$S_\phi(f) = \nu_0^2\ h_o\ A(f)B(f), \tag{A-5}$$

it is possible to calculate the autocovariance function, $R_\phi(t)$, as the Fourier transform of (A-5). Or, more simply, $R_\phi(t)$ can be calculated from the convolution of $a(t)$ with $b(t)$. That is,

$$R_\phi(t) = \frac{\nu_0^2 h_o}{2}\ \int_{-\infty}^{\infty} a(t-u)b(u)du, \tag{A-6}$$

or, explicitly,

$$R_\phi(t) = \frac{\pi\sqrt{\pi}\ \nu_0^2 h_o}{\alpha f_\ell 2}\ \left\{ e^{-2\pi f_\ell t} \int_{-\infty}^{t} e^{-\left(\frac{\pi u}{\alpha}\right)^2 + 2\pi f_\ell u}\ du \right.$$

$$\left. + e^{2\pi f_\ell t} \int_{t}^{\infty} e^{-\left(\frac{\pi u}{\alpha}\right)^2 - 2\pi f_\ell u}\ du \right\} . \tag{A-7}$$

(The factor of 1/2 arises from the assumption that $S_\phi(f)$ is a one-sided density). By changing variables in the integrals in (A-7), $R_\phi(t)$ can be written in the form

$$R_\phi(t) = \frac{\pi \nu_o^2 h_o}{4 f_\ell} e^{\left(f_\ell \alpha\right)^2} \left\{ e^{-2\pi f_\ell t} \left[1 + \operatorname{erf}\left(\frac{\pi t}{\alpha} - f_\ell \alpha\right)\right] \right.$$

$$\left. + e^{2\pi f_\ell t} \left[1 - \operatorname{erf}\left(\frac{\pi t}{\alpha} + f_\ell \alpha\right)\right] \right\} \tag{A-8}$$

where

$$\operatorname{erf}(\chi) \equiv \frac{2}{\sqrt{\pi}} \int_0^\chi e^{-\chi^2} d\chi \tag{A-9}$$

as in Ref. [14].

For $f_\ell \sim 3 \times 10^{-10}$ Hz and $\alpha \sim 10^{-5}$ sec., then $f_\ell \alpha \sim 3 \times 10^{-15} \ll 1$. Thus,

$$R_\phi(0) \approx \frac{\pi \nu_o^2 h_o}{2 f_\ell} , \tag{A-10}$$

and

$$R_\phi(\tau) \approx \frac{\pi \nu_o^2 h_o}{2 f_\ell} \left[1 - 2\pi f_\ell \tau + \frac{(2\pi f_\ell \tau)^2}{2!} - \cdots \right] , \tag{A-11}$$

for $\frac{\alpha}{\pi} \ll \tau \ll 1/f_\ell$. As noted in the text, $R_\phi(t)$ diverges as f_ℓ approaches zero; however, for positive, non-zero values of f_ℓ one can have a stationary and convergent model with all derivatives defined.

Appendix B. Building an ARIMA Model to Fit a Given Spectrum

In general, ARIMA models can approximate a wide range of spectral shapes. This appendix, however, will consider only a rather restricted set of these models. Also, the objective of this appendix is to provide a means of building an ARIMA model to fit a prescribed spectrum. Other techniques [12] emphasize the building of ARIMA models to fit a particular data set regardless of its spectrum.

One method of simulating a flicker noise with ARIMA models has already been published [21]. The treatment here will be a graphical approach using Bode plots.

Consider a random, uncorrelated, Gaussianly distributed, discrete time series, a_t, with mean zero and variance σ_a^2. Also consider a time series ω_t deduced from the a_t by the equation

$$\omega_t - \phi_1 \omega_{t-1} - \phi_2 \omega_{t-2} - \cdots - \phi_p \omega_{t-p} = a_t - \theta_1 a_{t-1}$$
$$- \theta_2 a_{t-2} - \cdots - \theta_q a_{t-q},$$
(B-1)

where the ϕ_i's and θ_i's are constants. We can define the index-lowering operator, B, by the relation

$$B\chi_t \equiv \chi_{t-1}.$$
(B-2)

This allows (B-1) to be rewritten in the form

$$(1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p) \omega_t = (1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_q B^q) a_t$$
(B-3)

The expression in parentheses on the left side of (B-3) is called the Auto Regressive (AR) operator of order p. That on the right is called the Moving Average (MA) operator of order q.

We can further define another time series, Z_t by the relation

$$\Delta^d Z_t = \omega_t,$$
(B-4)

where $\Delta Z_t \equiv (1-B)Z_t \equiv Z_t - Z_{t-1}$. That is, Z_t is the d-fold (finite) integral of ω_t. Thus, Z_t is an Auto Regressive, Integrated, Moving Average process defined by the ϕ_i's, θ_i's, σ_a^2, and the d differences of (B-4). In particular it is an ARIMA (p,d,q) process.

In order for ω_t to be stationary and invertible, there are restrictions on the ϕ's and θ's [12]. However, for the present discussions it will suffice to consider only the following two processes

$$\omega_t = (1 - \theta B) a_t$$
(B-5)

$$(1 - \phi B)\chi_t = y_t$$
(B-6)

For this case it is sufficient that $-1 < \theta < 1$ and $-1 < \phi < 1$.

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In these equations one sees that if one sets $\phi = \theta$ and $y_t = \omega_t$, then one obtains $\chi_t = a_t$. That is, if one considers (B-5) and (B-6) to define two digital filters, then (B-5) and (B-6) are inverse filters for each other when $\phi = \theta$. Thus, it is a simple matter to extend any understanding one gains about one filter to the other.

It is of value to consider the effect of a single MA filter of the form (B-5) when θ is further restricted to the interval $0 \leq \theta < 1$. Figure (B-1) shows the general behavior of the transfer function (magnitude squared) of such a filter and the straight-line approximations to this transfer function. The "knee" in the approximation occurs at a frequency f_c which is related to θ by the empirical relationship

$$\theta \approx \frac{1-\pi f_c}{1+\pi f_c} , \tag{B-7}$$

which will not be derived here.

Similarly, Fig. (B-2) is the transfer function of the AR filter with $\phi = \theta = 0.728$ as in Fig. (B-1). Hence, also,

$$\phi \approx \frac{1-\pi f_c}{1+\pi f_c} \tag{B-8}$$

We can now construct an approximation to a process which is a mixture of flicker noise and white noise. Figure (B-3) shows the Bode plot of the desired filter transfer function. The frequency axis of Fig. (B-3) is in terms of cycles per data spacing. Thus, the Nyquist frequency is just 1/2. For the sake of the example, we will assume that it is sufficient to approximate the spectrum to a lower frequency of .002 (cycles per data spacing).

Figure (B-4) superimposes the Bode plot of the filter which will be used onto the spectrum of Fig. (B-3). From each of the "knees" in the approximation one can determine the appropriate ϕ or θ for a filter to be cascaded in series.

Thus, from "knees" at the frequencies of .0233 and .0033, one obtains ϕ's for the AR filters (eq. B-6) of .8636 and .9795, respectively. These numbers have been calculated from eq. (B-8). Similarly, from the "knees" at the frequencies of .062 and .0087, one obtains θ values of .6740 and .9468, respectively, for the MA filters. One decides to use an AR filter if the function turns downward in response with increasing frequency as in Fig. (B-2). Correspondingly, one chooses a MA filter when the function turns upward as in Fig. (B-1).

The final filter can be obtained by cascading the output of one filter to the input of the next in the form

$$\omega_t = (1 - .6740B)a_t$$

$$(1 - .8636B)\ \chi_t = \omega_t \tag{B-9}$$

$$y_t = (1 - .9468B)\chi_t$$

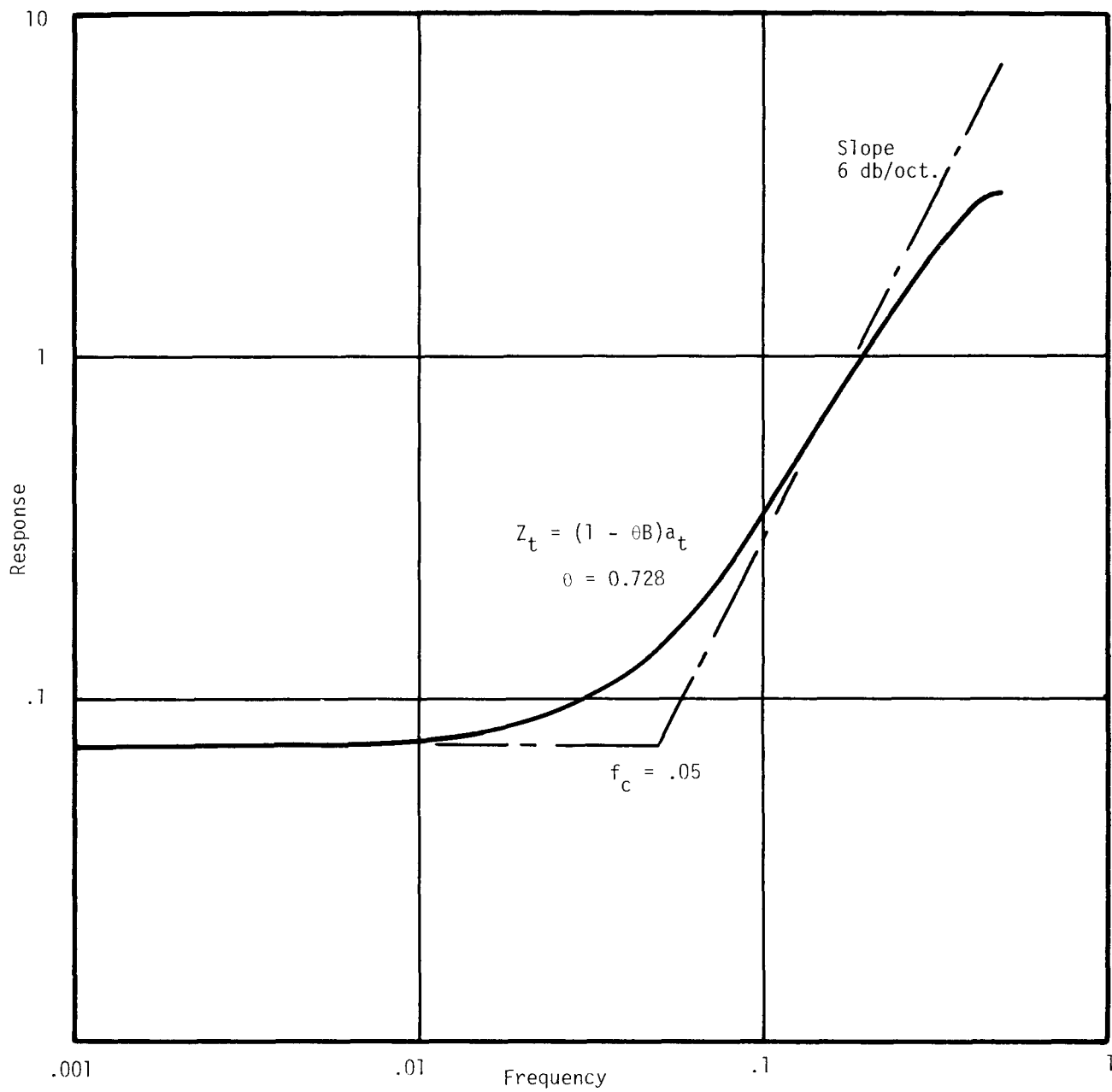$$(1 - .9795B)\ z_t = y_t$$

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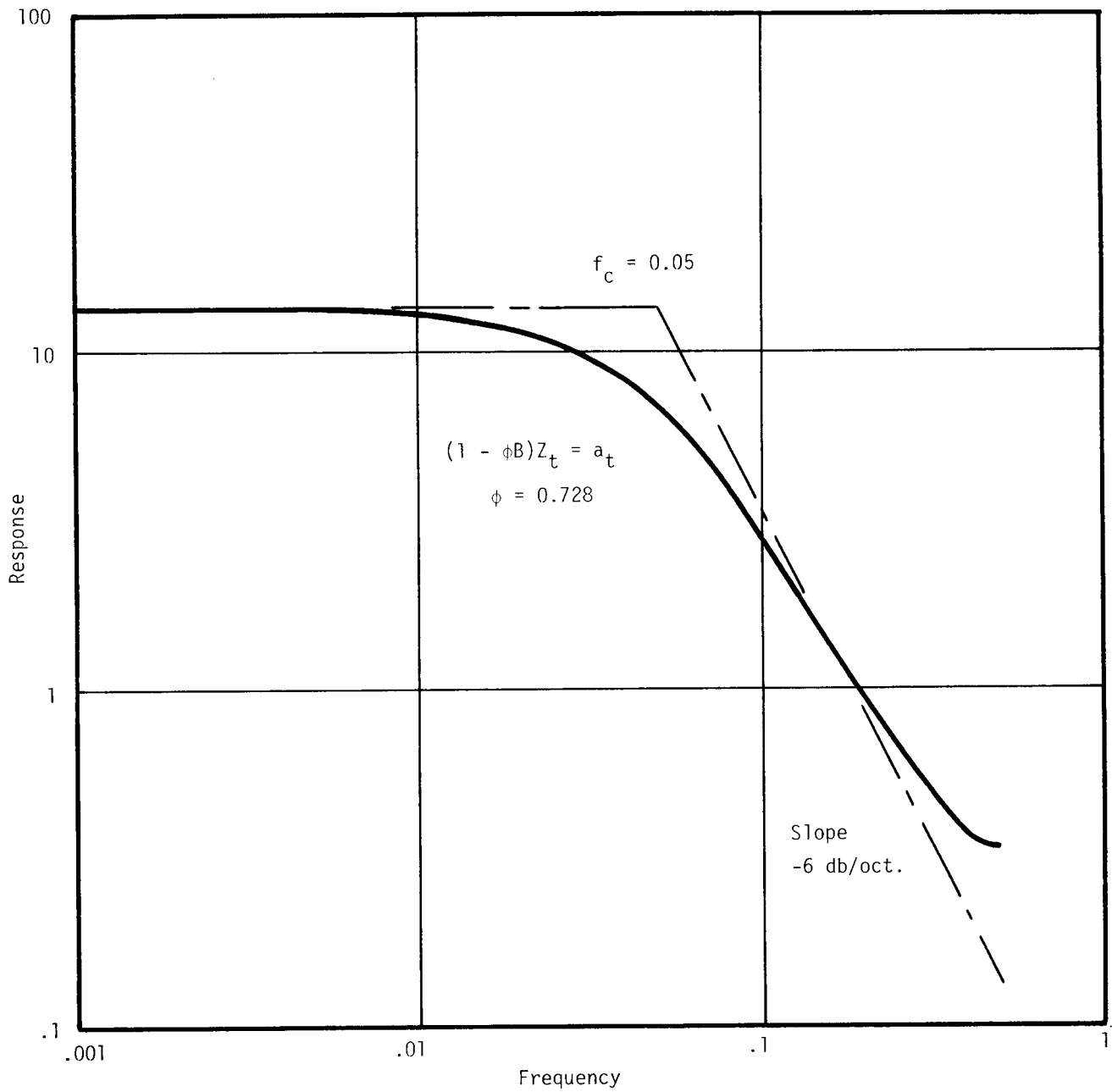Fig. (B-1) Transfer Function of Simple MA-Filter

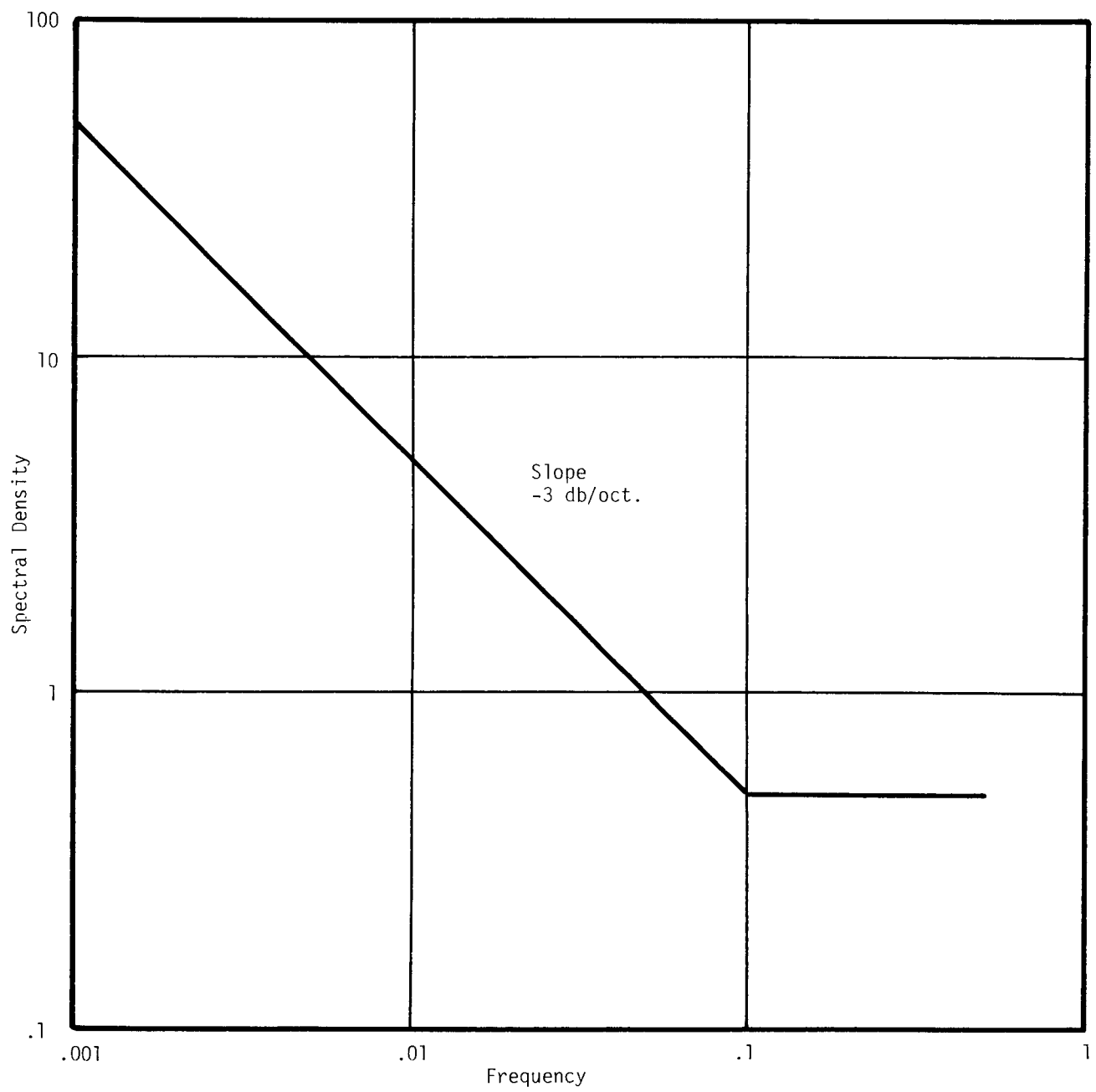Fig. (B-2)   Transfer Function of Simple AR-Filter
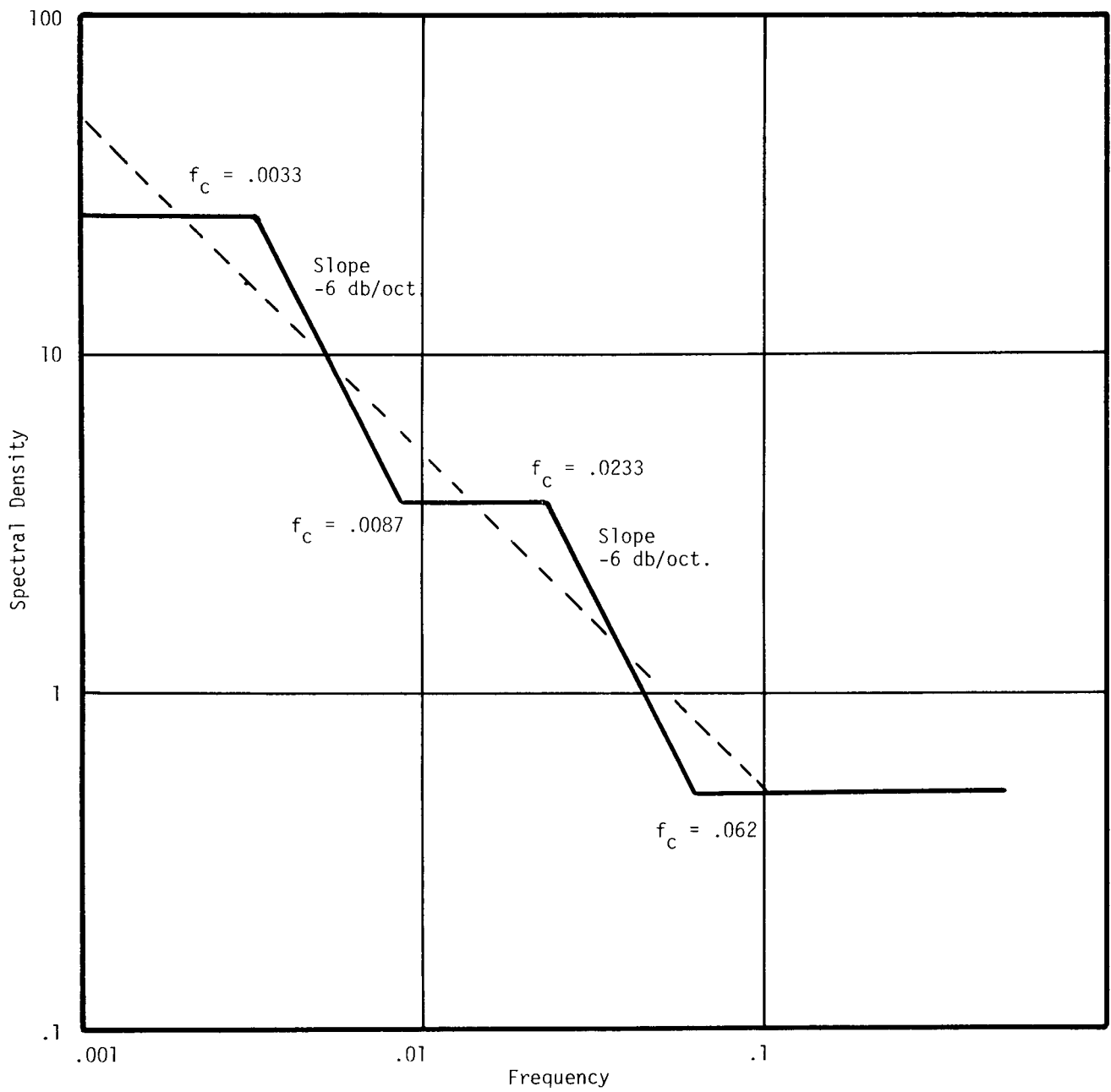
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Fig. (B-3) Bode Plot of Desired Spectrum

Fig. (B-4) Bode Plot of Filter Cascade

where a_t is the input to the cascade and z_t is the output. One can eliminate ω_t, χ_t and y_t and write an equivalent expression in the form

$$(1 - \phi_1 B - \phi_2 B^2)z_t = (1 - \theta_1 B - \theta_2 B^2)a_t, \qquad \text{(B-10)}$$

where

$$\phi_1 = 1.8431, \quad \theta_1 = 1.6208$$
$$\phi_2 = -.8459, \quad \theta_2 = -.6381 . \qquad \text{(B-11)}$$

Making use of equation (4.2) one can obtain the spectrum corresponding to (B-10). Figure (B-5) is a plot of the spectrum superimposed on the Bode plot of Figure (B-3). The parameter $\sigma_a^2 = 0.319$ was selected to make the spectrum fit the Bode plot in amplitude.

In this example, one could let z_t model the frequency of an oscillator, and it is clearly stationary [12]. To model the phase, one could define a parameter ν_t by the relation

$$(1 - \phi'B)\nu_t = z_t \qquad \text{(B-12)}$$

where $\frac{1}{1-\phi'}$ is large compared to any data sample to be tested.

In the current example f_ℓ was taken as .002. Thus, one could select $\phi' \simeq .99999$ to be quite adequate. Note that ν_t is also stationary.

By selecting $\phi' = .99999$ one obtains a stationary model for ν_t if one so desires. Of course, for actually generating a data set one could use $\phi' = 1$, since no observational difference in the simulated data would result--that is why ϕ' was chosen as close to unity as it was. Thus, the entire difference between a stationary and a nonstationary model (ν_t) for the phase is centered in whether one chooses .99999 or unity for ϕ'. Further, this choice is totally unobservable for short data sets ($N \sim 1/f_\ell = 500$).

Clearly, these techniques could be used to develop an ARIMA model to simulate data over a much broader range of frequencies than done here. In any event, equations (B-10) and (B-12) allow one to simulate a stationary noise with the desired spectrum.
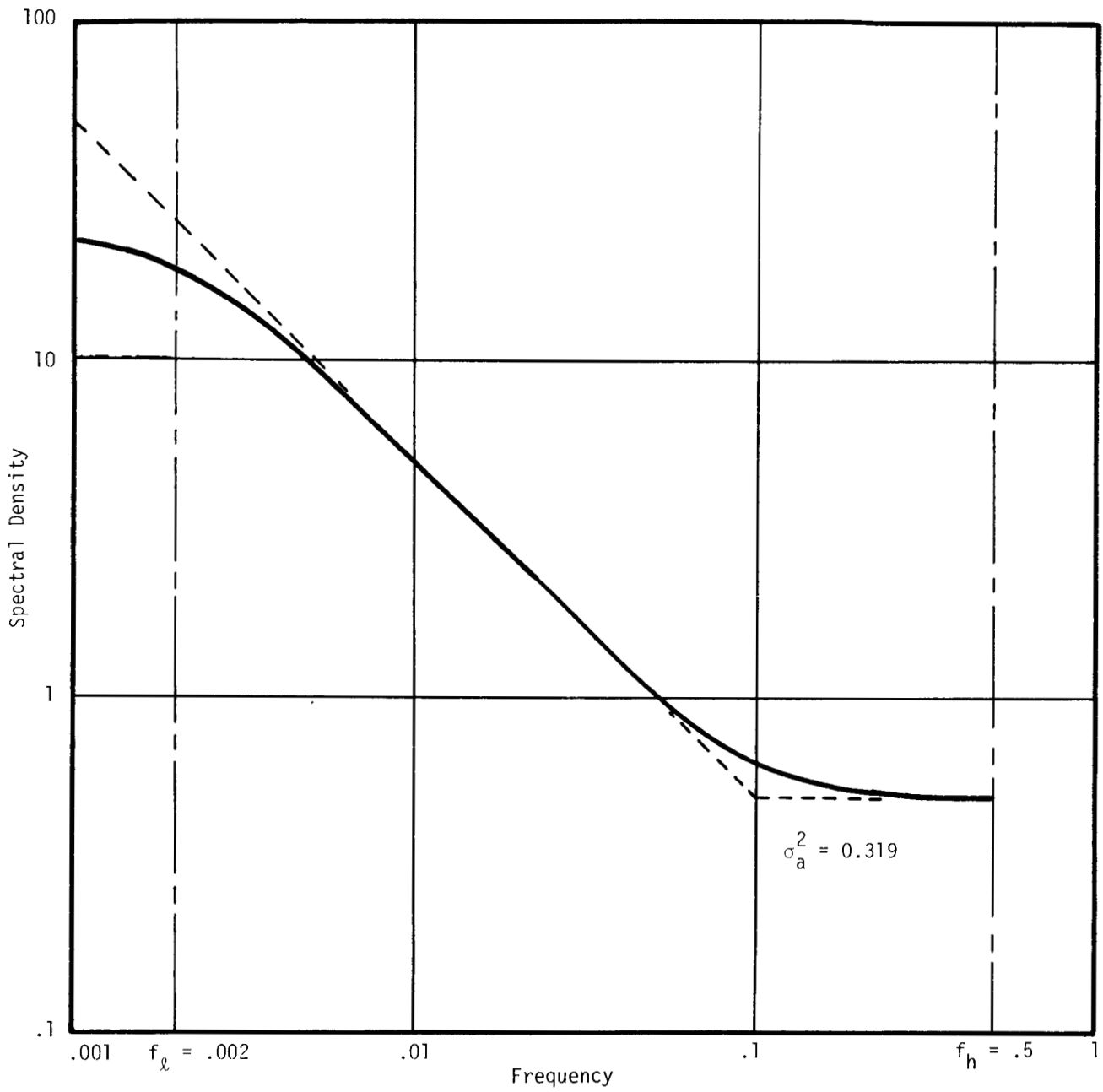
Fig. (B-5) Actual Spectrum of ARIMA (2,0,2) Model

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Appendix C. An Example of a Sporadic Noise with f^{-1} Spectral Density

Consider the identically distributed, independent, random variables ω_k with Gaussian distribution, zero mean, and unit variance. Also consider the independent, random variables r_n, with a rectangular distribution in the range $\varepsilon \leq r_n \leq 1$, where $0 \leq \varepsilon < 1$. That is, the distributions are as indicated in Figure (C-1).

Next define $m_n = \dfrac{1}{r_n}$. Since the r_n are rectangularly distributed,

$$\rho(r) = \begin{cases} \dfrac{1}{1-\varepsilon} \;, & \varepsilon \leq r \leq 1 \\[3mm] 0 & \text{otherwise,} \end{cases} \tag{C-1}$$

then the distribution of m_n is given by

$$\rho'(m)dm = \rho(r)dr = \frac{1}{1-\varepsilon} \left| d\left(\frac{1}{m}\right) \right|. \tag{C-2}$$

Thus, as shown in Fig. (C-2),

$$\rho'(m) = \begin{cases} \dfrac{1}{1-\varepsilon} \left(\dfrac{1}{m^2}\right) \;, & \text{for } 1 \leq m \leq \dfrac{1}{\varepsilon} \\[3mm] 0 \;, & \text{otherwise.} \end{cases} \tag{C-3}$$

One can obtain a sporadic noise, [19], χ_n, whose spectral density is approximately f^{-1} (flicker noise) over the range $\varepsilon < f < 1/2$, by following the procedure below: Define:

$$\chi_n = \begin{cases} \omega_0, & \text{for } 0 \leq n \leq m_1 \\[1mm] \omega_1, & \text{for } m_1 < n \leq m_1 + m_2 \\[1mm] \omega_2, & \text{for } m_1 + m_2 < n \leq m_1 + m_2 + m_3 \\[1mm] \;\vdots \\[1mm] \omega_k, & \text{for } t_k < n \leq t_{k+1} \\[1mm] \;\vdots \end{cases} \tag{C-4}$$

where $t_k = t_{k-1} + m_k$. An example of such a noise is presented in Fig. (C-3).
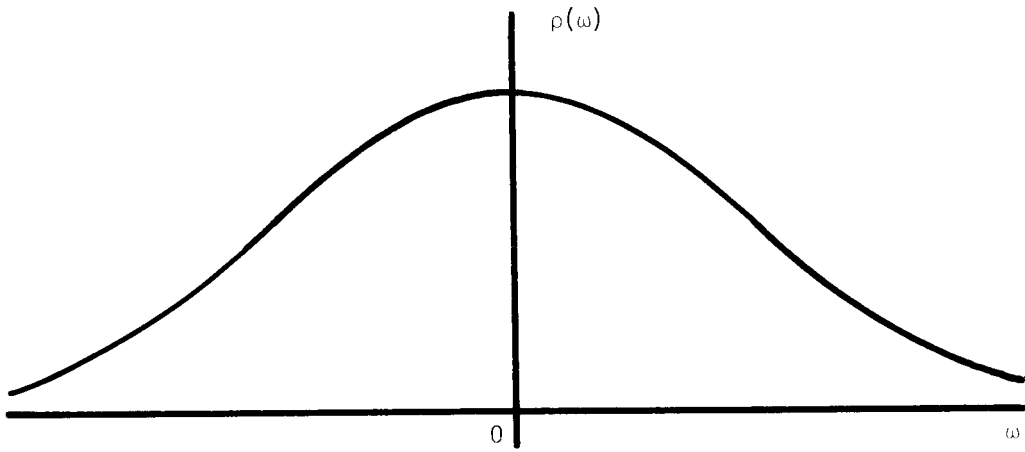
Fig. (C-1)a. Distribution Density of ω
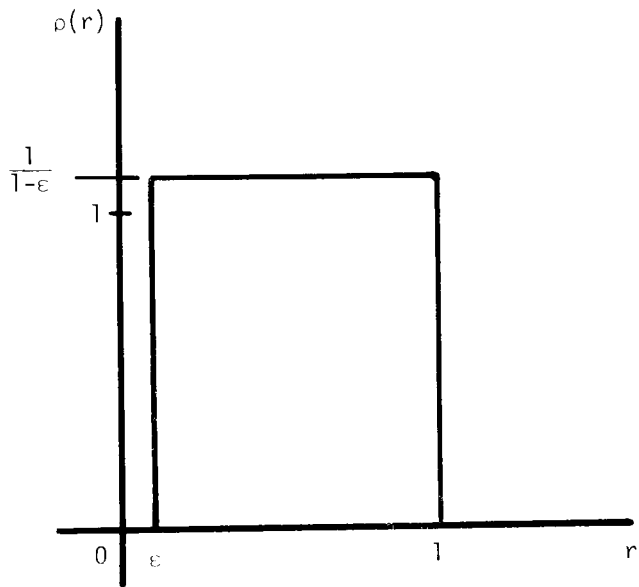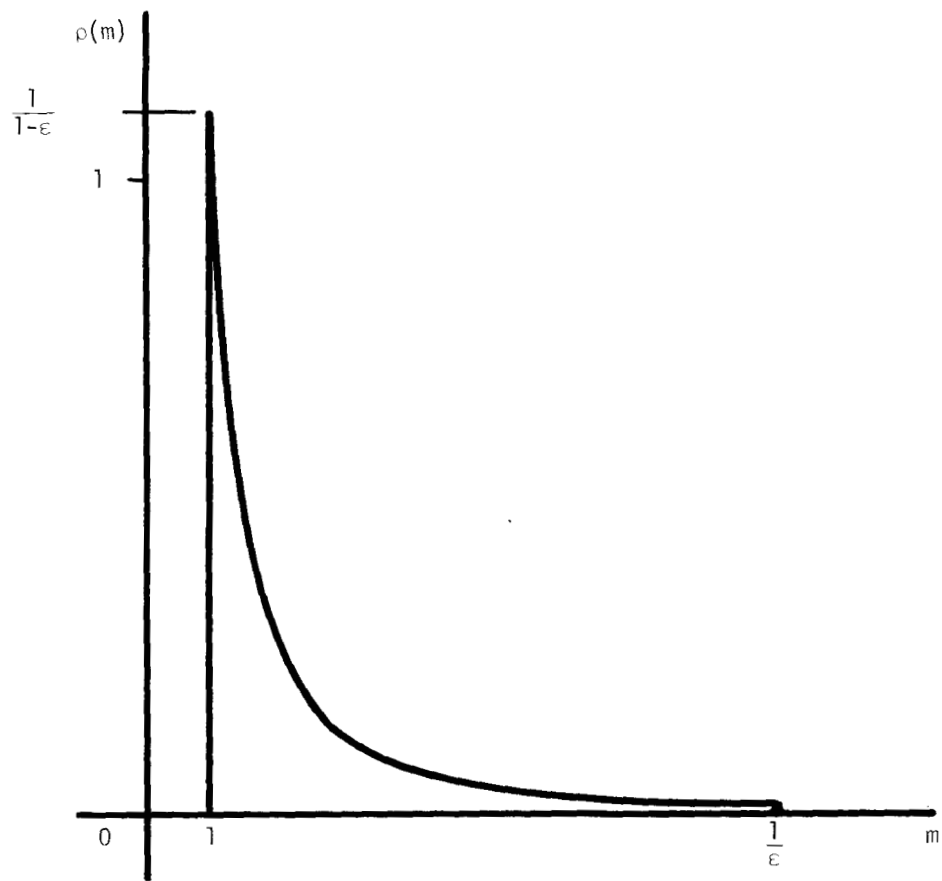

Fig. (C-1)b. Distribution Density of r

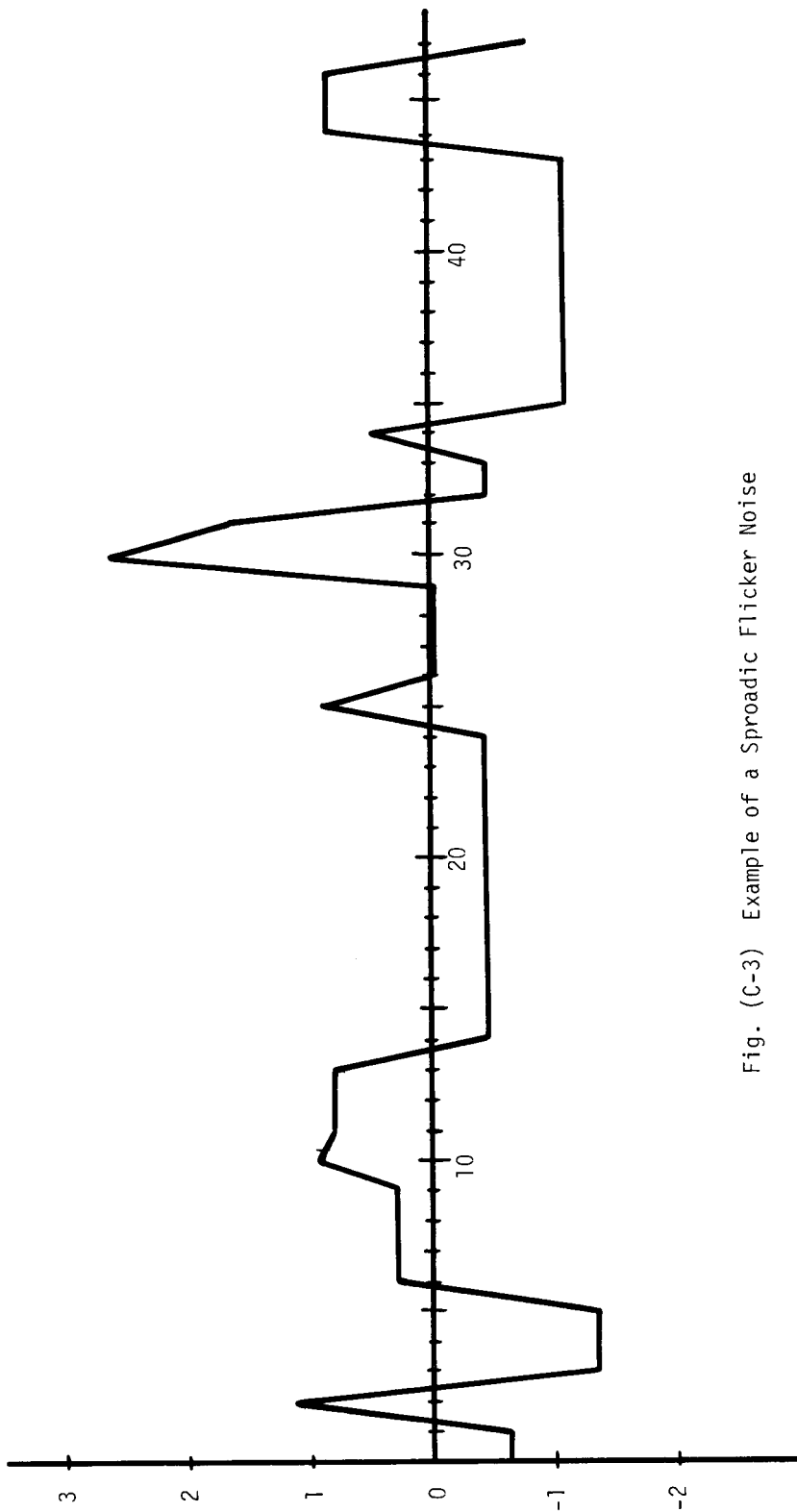Fig. (C-2) Distribution density of $m = \dfrac{1}{r}$

Fig. (C-3) Example of a Sproadic Flicker Noise

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References

[1] Slepian, D., On bandwidth, Proc. IEEE, Vol. 64, No. 3, pp. 292-300, (March 1976).

[2] Proc. IEEE-NASA Symposium on Short-Term Frequency Stability, NASA SP-80, Goddard Space FLight Center, Greenbelt, Md., November 23-24, 1964.

[3] Proc. IEEE, vol. 54, No. 2 (Feb. 1966).

[4] Barnes, J. A., et al., Characterization of frequency stability, IEEE Trans. Instrum. Measure, vol. IM-20, pp. 105-120 (May 1971).

[5] Boileau, E., Picinbono, B., The statistical study of phase fluctuations and oscillator stability, IEEE Trans. on I&M, vol. IM-25, No. 1, pp. 66-75 (March 1976).

[6] Lindsey, W. C., Chie, C. M., Theory of oscillator instability based upon structure functions. To be published.

[7] Rutman, J., Characterization of frequency stability: A transfer function approach and its application to measurements via filtering of phase noise, IEEE Trans. on I&M, IM-23, No. 1, pp. 40-48 (March 1974).

[8] Lesage, P., Audoin, C., Characterization of frequency stability: uncertainty due to the finite number of measurements, IEEE Trans. on I&M, IM-22, No. 2, pp. 157-161 (June 1973).

[9] Lesage, P. Audoin, C., Correction to: characterization of frequency stability: Uncertainty due to finite number of measurements, IEEE Trans. on I&M, IM-23, No. 1, p. 103 (March 1974).

[10] Lesage, P., Audoin, C., Correction to 'characterization of frequency stability: uncertainty due to the finite number of measurements.' To be published.

[11] Jenkins, G. M., Watts, D. G., Spectral analysis and its applications, Holden-Day, San Francisco (1968).

[12] Box, G.E.P., Jenkins, G. M., Time series analysis, Holden-Day, San Francisco, Calif. (1970).

[13] Rutman, J., Comment on 'characterization of frequency stability,' Trans. on I&M, IM-21, No. 1, p. 85 (Feb. 1972).

[14] Kolmogorov, A. N., Dissipation of energy in locally isotropic turbulence, Akad. Nauk. USSR, vol. 32, No. 16 (1941).

[15] Baugh, R., Frequency modulation analysis with the Hadamard variance, Proc. 25th Annual Symp. on Frequency Control, April 26-28, 1971.

[16] Cornelissen, G. (Ph.D. Dissertation), Brussels (1976).

[17] Winkler, G.M.R., private communication.

[18] Rutman, J., Ubersfeld, J., A model for flicker frequency noise of oscillators, Proc. IEEE, vol. 60, No. 2, pp. 233-235 (Feb 1972).

[19] Mandelbrot, B., Some noises with 1/f spectrum, a bridge between direct current and white noise, IEEE Trans. on Info. Theory, vol. IT-13, No. 2, pp. 289-298 (April 1967).

[20] Barnes, J. A., Allan, D. W., A statistical model of flicker noise, Proc. IEEE, vol. 54, No. 2, pp. 176-178 (Feb. 1966).

[21] Barnes, J. A., Jarvis, S., Efficient numerical and analog modeling of flicker noise processes, NBS Tech. Note 604 (June 1971).

[22] Wilson, R. G., Prediction of phase (time) errors in oscillator-driven clocks, Aerospace Report No. TOR, 0075 (5461-04)-1, The Aerospace Corp. (June 1975).

[23] Mealy, G. L., Sensor data bank development: time standard error models and data file, Tech. Report AFAL-TR-75-65, Air Force Avionics Lab. (June 1975).

[24] Mandelbrot, B. B., A fast fractional Gaussian noise generator, Water Resources Research, vol. 7, No. 3 (June 1971).

[25] Barnes, J. A., The generation and recognition of flicker noise, NBS Tech. Note 9284 (June 1967).

[26] Heaviside, O., Electromagnetic theory, pp. 128-129, Dover Publications, New York, N. Y. (1950).

[27] Courant, R., Differential and integral calculus, II, Chapt. IV, Sect. 7, Interscience, New York, N. Y. (1957).

[28] Halford, D., A general mechanical model for $|f|^\alpha$ spectral density noise with special reference to flicker noise $1/|f|$, Proc. IEEE, vol. 56, No. 3, pp. 251-258 (March 1968).

[29] Cutler, L., and Searle, C., Some aspects of the theory and measurement of frequency fluctuations in frequency standards, Proc. IEEE, vol. 54, pp. 136-154 (Feb. 1966).

[30] Doob, J. L., The Brownian movement and stochastic equations, Annals of Mathematics, vol. 43, No. 2 (April 1942).

[31] Barnes, J. A., Atomic timekeeping and the statistics of precision signal generators, Proc. IEEE, vol. 54, pp. 207-220 (Feb. 1966).

[32] Allan, D. W., Statistics of atomic frequency standards, Proc. IEEE, vol. 54, pp. 221-230 (Feb. 1966).

[33] Atkinson, W. R., Fey, L., Newman, J., Spectrum analysis of extremely low frequency variations of quartz oscillators, Proc. IEEE (Corres.), vol. 51, p. 379 (Feb. 1963).

[34] Fey, L., Atkinson, W. R., Newman, J., Obscurities of oscillator noise, Proc. IEEE (Corres.), vol. 52, pp. 104-105 (Jan. 1964).

[35] Troitsky, V., Certain problems of the theory of fluctuations in oscillators. The influence of flicker noise, Isv. Vysshikh Uchebn. Zavedenii, Radiofiz., vol. 1, pp. 20-33 (1958).

[36] DePrins, J., et al., Frequency-domain interpretation of oscillator phase stability, IEEE Trans. on IPM, vol. IM-18, No. 4, pp. 251-261 (Dec. 1969).

[37] Barnes, J. A., Allan, D. W., Effects of long-term stability on the definition and measurement of short-term stability, IEEE Symposium on Short-Term Frequency Stability, Goddard Space Flight Center, Greenbelt, Maryland, pp. 119-123, NASA SP-80.

[38] Campbell, G. A., Foster, R. M., Fourier integrals for practical applications, D. van Nostrand, N. Y., N. Y. (1948).

[39] Lighthill, M., Introduction to Fourier analysis and generalized functions, Cambridge (1962).

[40] Zemanian, A. H., Distribution theory and transform analysis, McGraw-Hill, N. Y. (1965).

[41] Allan, D. W., et al., Performance, modeling, and simulation of some cesium beam clocks, Proc. of the 27th Annual Symposium on Frequency Control, pp. 334-346 (1973).

[42] Barnes, J. A., Unpublished lecture notes (August 1975).

[43] Winkler, G.M.R., Hall, R. G., Percival, D. B., The U. S. Naval Observatory clock time reference and the performance of a sample of atomic clocks, Metrologia, vol. 6, No. 4, pp. 126-134 (Oct. 1970).

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