

VARIANCES BASED ON DATA WITH DEAD TIME BETWEEN THE MEASUREMENTS

James A. Barnes
Austron, Inc.
Boulder, Colorado 80301

and

David W. Allan
Time and Frequency Division
National Institute of Standards and Technology
Boulder, Colorado 80303

The accepted definition of frequency stability in the time domain is the two-sample variance (or Allan variance). It is based on the measurement of average frequencies over adjacent time intervals, with no "dead time" between the intervals. The primary advantages of the Allan variance are that (1) it is convergent for many encountered noise models for which the conventional variance is divergent; (2) it can distinguish between many important and different spectral noise types; (3) the two-sample approach relates to many practical implementations; for example, the rms change of an oscillator's frequency from one period to the next; and (4) Allan variances can be easily estimated at integer multiples of the sample interval.

In 1974 a table of bias functions which related variance estimates with various configurations of number of samples and dead time to the Allan variance was published [1]. The tables were based on noises with pure power-law spectral densities.

Often situations occur that unavoidably have dead time between measurements, but still the conventional variances are not convergent. Some of these applications are outside of the time-and-frequency field. Also, the dead times are often distributed throughout a given average, and this distributed dead time is not treated in the 1974 tables.

This paper reviews the bias functions $B_1(N, r, \mu)$, and $B_2(r, \mu)$ and introduces a new bias function, $B_3(2, M, r, \mu)$, to handle the commonly occurring cases of the effect of distributed dead time on the computed variances. Some convenient and easy-to-interpret asymptotic limits are reported. A set of tables for the bias functions are included at the end of this paper.

Key words: Allan variance; bias functions; data sampling and dead time; dead time between the measurement; definition of frequency stability; distributed dead time; two-sample variance

1. Introduction

The sample mean and variance indicate respectively the approximate magnitude of a quantity and its uncertainty. For many situations a continuous function of time is sampled, or measured, at fairly regular intervals. Sampling is not always instantaneous. It takes a finite time and provides an "average reading." If the underlying process (or noise) is random and uncorrelated in time, then the fluctuations are said to be "white" noise. In this situation, the sample mean and variance calculated by the conventional formulas,

$$m = \frac{1}{N} \sum_{n=1}^N \bar{y}_n, \quad (1)$$
$$s^2 = \frac{1}{N-1} \sum_{n=1}^N (\bar{y}_n - m)^2,$$

provide the needed information. The "bar" over the y in eq (1) above denotes the average over a finite time interval. In time and frequency work, y is defined as the average fractional (or normalized) frequency deviation from nominal over an interval τ and at some specified measurement time. As in science generally, the physical model determines the appropriate mathematical model. For the white noise model, the sample mean and variance are the mainstays of most analyses.

Although white noise is a common model for many physical processes, more general noise models are being identified and used. In precise time and frequency measurement, for example, there are two quantities of great interest: instantaneous frequency and phase. These two quantities by definition are exactly related by a differential. (We are NOT considering Fourier frequencies at this point.) That is, the instantaneous frequency is the time rate of change of phase. Thus, if we were employing a model of white frequency-modulation (white FM) noise, then the phase noise is the integral of the white FM noise, commonly called a Brownian motion or random walk. Therefore, depending on whether we are currently interested in phase or frequency, the sample mean and variance may or may not be appropriate.

By definition, white noise has a power spectral density (PSD) that is constant with Fourier frequency. Since random walk noise is the integral of white noise, the power spectral density of a random walk varies as $1/f^2$ (where f is the Fourier frequency) [2]. We encounter noise models whose power spectral densities are various power laws of their Fourier frequencies. Flicker noise is very common and is defined as a noise whose power spectral density varies as $1/f$ over a relevant spectral range. If an oscillator's instantaneous frequency is well modeled by flicker noise, then its phase would be the integral of the flicker noise. It would have a PSD which varied as $1/f^3$.

Noise models whose PSD's are power laws of the Fourier frequency but not integer exponents are possible as well but not as common. This paper considers power-law PSD's of a quantity $y(t)$; $y(t)$ is a continuous sample function which can be measured at regular intervals. For noises whose PSD's vary as f^α with $\alpha < -1$ at low frequencies, the conventional sample mean and variance given in eq (1) do not converge as N gets large [2, 3]. This lack of convergence renders the sample mean and variance ineffective and often misleading in some situations.

Although the sample mean and variance have limitations, other time-domain statistics can be convergent and quite useful. The quantities that we consider in this paper depend significantly on the details of the sampling procedures. Indeed, each sampling scheme has its own bias, and this is the motivation for the bias functions discussed in this paper.

2. The Allan Variance

Recognizing that for particular types of noise, the conventional sample variance fails to converge as the number of samples, N , grows, Allan suggested that we set $N = 2$ and average many of these two-sample variances to get a convergent and stable measure of the spread of the quantity in question [3]. This is what has come to be called the Allan variance.

More specifically let us consider a sample function of time as indicated in figure 1. A measurement consists of averaging $y(t)$ over the interval τ . The next measurement begins at a time T after the beginning of the previous measurement interval. There is no logical reason why T must be as large as τ or larger--if $T < \tau$, then the second measurement begins before the first is completed, which is unusual but possible. When $T = \tau$, there is no dead time between measurements.

The accepted definition of the Allan variance is the expected value of a two-sample variance with no dead time between successive measurements. In symbols, the Allan variance is given by

$$\sigma_y^2(\tau) = \frac{1}{2}E[(\bar{y}_{n+1} - \bar{y}_n)^2], \quad (2)$$

where there is no dead time between the two sample averages for the Allan variance and the $E[\cdot]$ denotes the expectation operator.

3. The Bias Function $B_1(N, r, \mu)$

Define N to be the number of sample averages of $y(t)$ used in eq (1) to estimate a sample variance ($N = 2$ for an Allan variance). Also define r to be the ratio of T to τ ($r = 1$ when there is no dead time between measurements). The parameter μ is related to the exponent of the power law of the PSD of the process $y(t)$. If α is the exponent in the power-law spectrum for $y(t)$, then the Allan variance varies as τ raised to the μ power, where α and μ are related as shown in figure 2 [2-4]. We can use estimates of μ to infer α , the spectral type. The ambiguity in α for $\mu = -2$ has been resolved by using a modified $\sigma_y^2(\tau)$ [5-7].

Often data cannot be taken without dead time between sample averages, and it is useful to consider other than two-sample variances. We will define the bias function $B_1(N, r, \mu)$ by the ratio,

$$B_1(N, r, \mu) = \frac{\sigma^2(N, T, \tau)}{\sigma^2(2, T, \tau)}, \quad (3)$$

where $\sigma^2(N, T, r)$ is the expected sample variance given in eq (1) and based on N measurements at intervals T and averaged over a time r and $r = T/\tau$. In words, $B_1(N, r, \mu)$ is the ratio of the expected variance for N measurements to the expected variance for two samples (everything else held constant). The variances on the right in eq (3) depend implicitly on the noise type even though μ or α are not shown as independent variables. The noise-type parameter, μ , is shown as an independent variable for all of the bias functions in this paper, because the values of the ratio of these variances explicitly depend on μ as will be derived later in the paper. Allan showed that if N and r are held constant, then the α , μ relationship shown in figure 2 is the same; that is, we can still infer the spectral type from the r dependence using the equation $\alpha = -\mu - 1$, $-2 \leq \mu < 2$ [3].

4. The Bias Function $B_2(r, \mu)$

The bias function $B_2(r, \mu)$ is defined in [1] by the relation,

$$B_2(r, \mu) = \frac{\sigma^2(2, T, r)}{\sigma^2(2, \tau, r)} = \frac{\sigma^2(2, T, r)}{\sigma_y^2(r)}. \quad (4)$$

In words, $B_2(r, \mu)$ is the ratio of the expected two-sample variance with dead time to that without dead time (with N = 2 and r the same for both variances). A plot of the $B_2(r, \mu)$ function is shown in figure 3. The bias functions B_1 and B_2 represent biases relative to N = 2 rather than infinity; that is, the ratio of the N sample variance (with or without dead time) to the Allan variance and the ratio of the two-sample dead-time variance to the Allan variance respectively.

5. The Bias Function $B_3(N, M, r, \mu)$

Consider the case where a great many measurements are available with dead time between each pair of measurements ($T_o > \tau_o$). The measurements are averaged over the time interval τ_o , the spacing between the beginning of one measurement to the next is T_o , and it may not be convenient to retake the data. We might want to estimate the Allan variance at, say, multiples M of

the averaging time τ_0 . If we average groups of the measurements of $y(t)$, then the dead times between the original measurements are distributed periodically throughout the new average measurements (see figure 4). Define

$$\bar{\bar{y}}_i = \frac{1}{M} \sum_{n=i}^{M+i-1} \bar{y}_n, \quad (5)$$

where \bar{y}_i are the raw or original measurements based on dead time $T_0 - \tau_0$.

Also define the two-sample variance with distributed dead time as

$$\sigma^2(2, M, T, \tau) = \frac{1}{2} E[(\bar{\bar{y}}_i - \bar{\bar{y}}_{i+M})^2], \quad (6)$$

with $\tau = M\tau_0$ and $T = MT_0$.

We can now define B_3 as the ratio of the N-sample variance with distributed dead time to the N-sample variance with dead time accumulated at the end as in figure 1:

$$B_3(N, M, r, \mu) = \frac{\sigma^2(N, M, T, \tau)}{\sigma^2(N, T, \tau)}. \quad (7)$$

Although $B_3(N, M, r, \mu)$ is defined for general N , the tables in the Appendix confine treatment to the case where $N = 2$. There is little value in extending the tables to include general N . Though the variances on the right in eq (7) depend explicitly on N , T and τ , the ratio $B_3(N, M, r, \mu)$ depends on the ratio $r = T/\tau$, and on μ as developed later in this paper.

In words, $B_3(2, M, r, \mu)$ is the ratio of the expected two-sample variance with periodically distributed dead time, as shown in figure 4, to the expected two-sample variance with all the dead time grouped together as shown in figure 1. Both the numerator and the denominator have the same total averaging time and dead time, but they are apportioned differently. The product $B_2(r, \mu) \cdot B_3(2, M, r, \mu)$ is the distributed dead-time variance over the Allan variance for a particular T , τ , M and μ .

Some useful asymptotic forms of B_3 can be found. In the case of large M and $M > r$, we may write that

$$B_3 \approx \frac{1 + \mu}{3}, \quad 1 \leq \mu \leq 2, \quad (8)$$

$$B_3 \approx \frac{4 \ln(2)}{2 \ln(r) + 3}, \quad \mu = 0.$$

One simple and important conclusion from these two equations is that for the cases of flicker FM noise and random-walk FM noise, the r^μ dependence for large r is the same whether or not there is periodically distributed dead time. The values of the variances differ only by a constant, and in the latter case the constant is 1. This conclusion is also true for white FM noise, and in this case the constant is also 1.

In the cases $r \gg 1$ and $-2 \leq \mu \leq -1$, we may write for the asymptotic behavior of B_3

$$B_3 \approx M^\alpha, \quad \alpha = -\mu - 1, \quad (9)$$

as was determined empirically. In this region of power-law spectrum the B_3 function has an M^α dependence for an f^α spectrum.

6. The Bias Functions

The bias functions can be written fairly simply by first defining the function,

$$F(A) = 2A^{\mu+2} - (A + 1)^{\mu+2} - |(A - 1)|^{\mu+2}. \quad (10)$$

The bias functions become

$$B_1(N, r, \mu) = \frac{1 + \sum_{n=1}^{N-1} \frac{N-n}{N(N-1)} \cdot F(nr)}{1 + \frac{1}{4} F(r)}, \quad (11)$$

$$B_2(r, \mu) = \frac{1 + \gamma F(r)}{2(1 - 2^\mu)}, \quad (12)$$

as given in [1], and

$$B_3(2, M, r, \mu) = \frac{2M + M \cdot F(Mr) - \sum_{n=1}^{M-1} (M-n)[2F(nr) - F((M+n)r) - F((M-n)r)]}{(M^{\mu+2})[F(r) + 2]}, \quad (13)$$

as indicated in the appendix.

For $\mu = 0$, eqs (11), (12), and (13) are the indeterminate form 0/0 and must be evaluated by l'Hôpital's rule. Special attention must also be given when expressions of the form 0^0 arise. We verified a random sampling of the table entries using noise simulation and Monte Carlo techniques. No errors were detected. The results in this paper differ some from those in [8], which suggests that there may be some mistakes. Tables for the three bias functions are listed at the end of the paper (note that the computer print-out did not have a symbol for Greek mu $\equiv \mu$).

7. Examples of the Use of the Bias Functions

The spectral type, that is, the value of μ , may be inferred by varying τ , the sample time. However, another useful way of determining the value of μ is by using $B_1(N, r, \mu)$ as follows: calculate an estimate of $\sigma_y^2(N, T, \tau)$ and $\sigma_y^2(2, T, \tau)$ and hence $B_1(N, r, \mu)$; then use the tables to infer the value of μ .

Suppose one has an experimental value for $\sigma_y^2(N_1, T_1, \tau_1)$ and its spectral type is known, that is, μ is known. Suppose also that one wishes to know the variance at some other set of measurement parameters, N_2, T_2, τ_2 . An unbiased estimate of $\sigma_y^2(N_2, T_2, \tau_2)$ may be calculated by the equation:

$$\sigma_y^2(N_2, T_2, \tau_2) = \left(\frac{\tau_2}{\tau_1}\right)^\mu \left(\frac{B_1(N_2, r_2, \mu) B_2(r_2, \mu)}{B_1(N_1, r_1, \mu) B_2(r_1, \mu)} \right) \sigma_y^2(N_1, T_1, \tau_1) \quad (14)$$

where $r_1 = T_1/\tau_1$ and $r_2 = T_2/\tau_2$.

Since the time-domain definition for frequency stability is the Allan variance, it behooves us, where possible, to relate other variances to the Allan variance. If we have an N-sample variance on data with dead-time $T-\tau$ and we know the power-law spectral type (the value of μ), then we may write

$$\sigma_y^2(\tau) = \frac{\sigma_y^2(N, T, \tau)}{B_1(N, \tau, \mu) B_2(\tau, \mu)}. \quad (15)$$

If we have an N-sample variance where each data entry is an average of M samples with distributed dead time, then we may write

$$\sigma_y^2(\tau) = \frac{\sigma_y^2(N, M, T, \tau)}{B_1(N, \tau, \mu) B_2(\tau, \mu) B_3(N, M, \tau, \mu)}. \quad (16)$$

8. Conclusion

For some important power-law spectral density models often used in characterizing precision oscillators ($S_y(f) \sim f^\alpha$, $\alpha = -2, -1, 0, +1, +2$), we have studied the effects on variances when there is dead time between the frequency samples, and the frequency samples are averaged to increase the integration time. Since dead time between measurements is a common problem throughout metrology, the analysis here has broader applicability than just to time and frequency. Specifically, this kind of analysis has been used with gage blocks and standard volt cells--showing that the classical variance may be non-convergent in some cases [9].

Heretofore, the Allan variance has been shown to have some convenient theoretical properties in relation to power-law spectra as the integration or sample time is varied (if $\sigma_y^2(\tau) \sim \tau^\mu$, then $\alpha = -\mu - 1$, $-2 < \mu \leq 2$). Since $\sigma_y(\tau)$, by definition, is estimated from data with no dead time, the sample or integration time can be unambiguously changed to investigate the τ dependence. From our analysis, we have concluded that for the asymptotic limit of several samples being averaged with dead time present in the data, the τ dependence of the variances is the same. The $\alpha = -\mu - 1$ relationship still remains valid for white FM noise ($\mu = -1$, $\alpha = 0$), flicker FM noise ($\mu = 0$, $\alpha = -1$), and for

random-walk FM noise ($\mu = +1$, $\alpha = -2$). The asymptotic limit is approached as the product of number of samples averaged and the initial data sample time, τ_0 , becomes larger than the dead time ($M > r$). The variances so obtained differ only by a constant, which can be calculated as given in this paper.

A knowledge of the appropriate power-law spectral model is required to translate a distributed dead-time variance to the corresponding value of the Allan variance. In principle, the power-law spectral model can be estimated from the r^μ dependence, using the variance analysis on the data as outlined above.

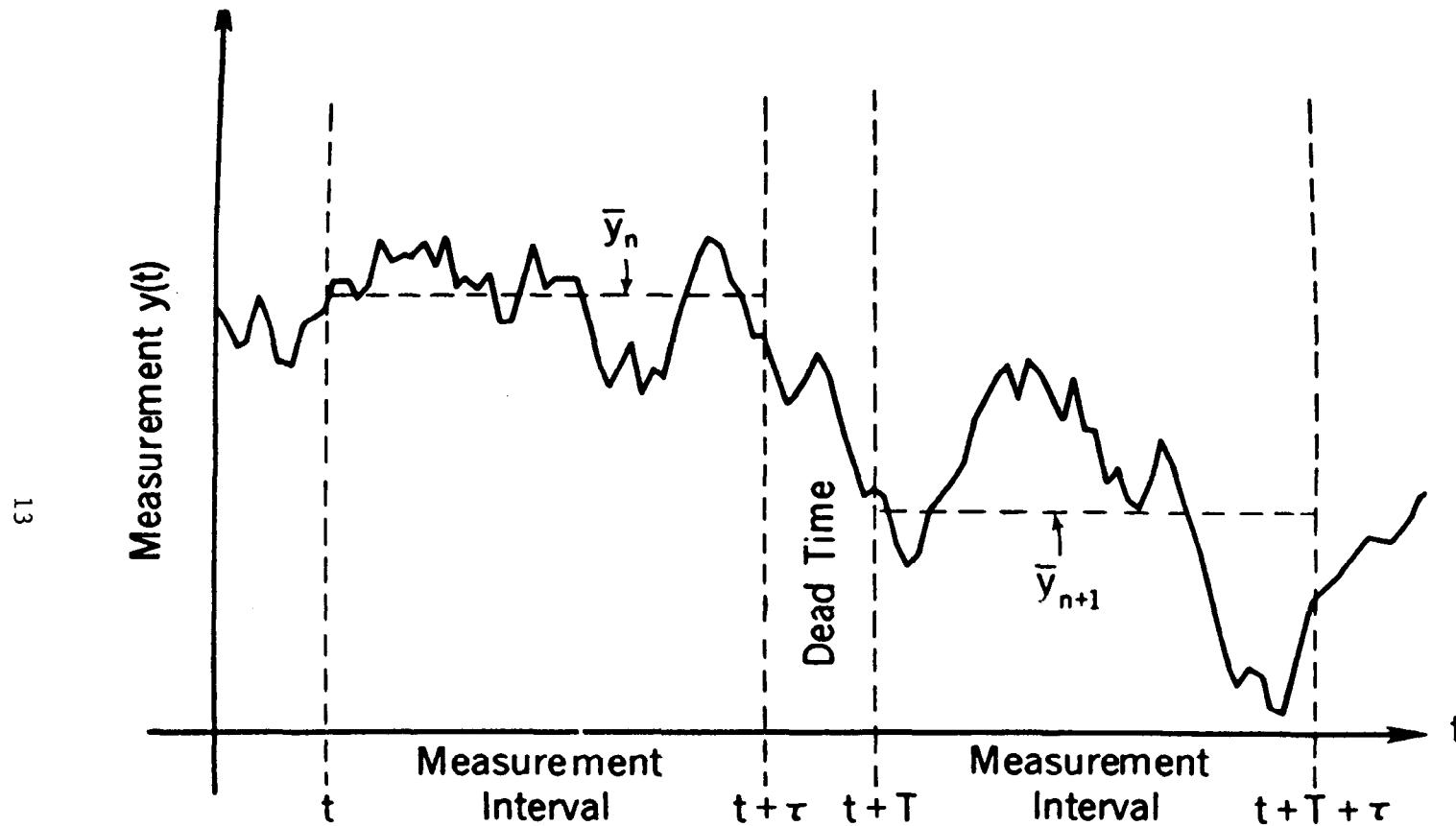
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Table 1. Table of some bias function identities

$B_1(2, r, \mu)$	= 1
$B_1(N, r, 2)$	= $(N(N+1))/6$
$B_1(N, 1, 1)$	= $N/2$
$B_1(N, 1, \mu)$	= $(N(1-N^\mu))/(2(N-1)(1-2^\mu))$ for $\mu \neq 0$ = $N \ln(N)/[2(N-1) \ln(2)]$ for $\mu = 0$
$B_1(N, 1, \mu)$	= 1 for $\mu < 0$ = $[2/(N(N-1))] \sum_{n=1}^{N-1} (N-n) \cdot n^\mu$ for $\mu > 0$
$B_1(N, r, -1)$	= 1 if $r \geq 1$
$B_1(N, r, -2)$	= 1 if $r \neq 1$ or 0
$B_2(0, \mu)$	= 0
$B_2(1, \mu)$	= 1
$B_2(r, 2)$	= r^2
$B_2(r, 1)$	= $(3r - 1)/2$ if $r \geq 1$
$B_2(r, -1)$	= r if $0 \leq r \leq 1$ = 1 if $r \geq 1$
$B_2(r, -2)$	= 0 if $r=0$ = 1 if $r=1$ = $2/3$ otherwise
$B_3(2, M, 1, \mu)$	= 1
$B_3(2, M, r, -2)$	= M
$B_3(2, r, \mu)$	= 1
$B_3(2, M, r, 2)$	= 1
$B_3(2, M, r, -1)$	= 1 for $r \geq 1$



TWO MEASUREMENTS OF A SET WITH DEAD TIME

Figure 1. Illustration of two fractional frequency samples with dead time, $T-\tau$, between the samples. This is two of a set of adjacent frequency measurements, each averaged over an interval τ , needed to calculate a two-sample variance from a data set.

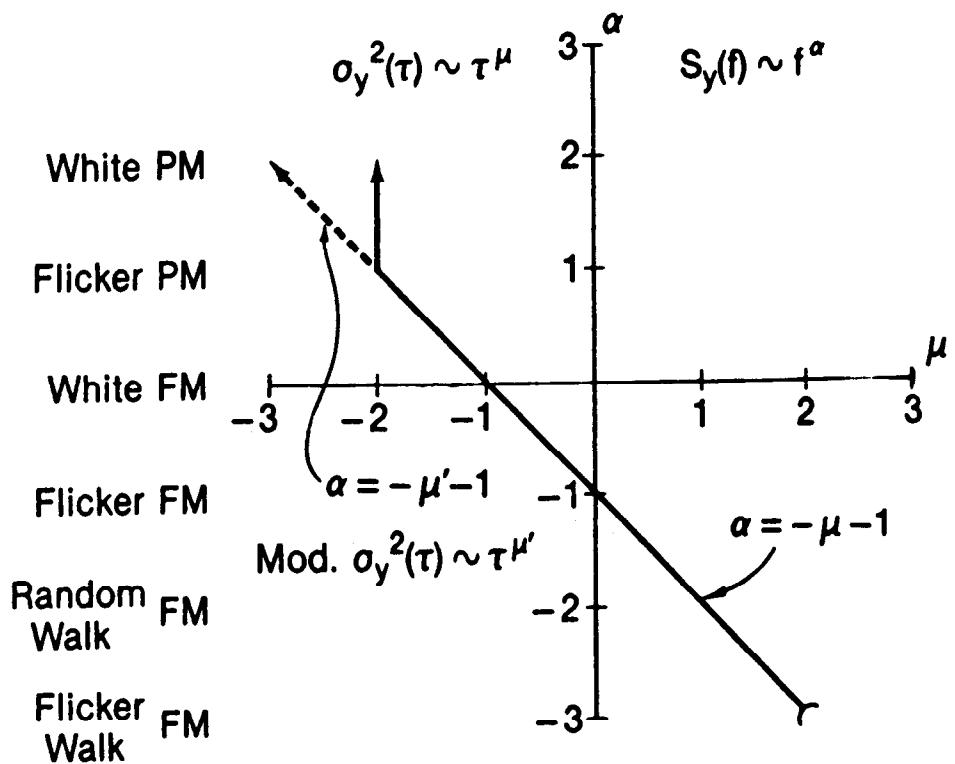
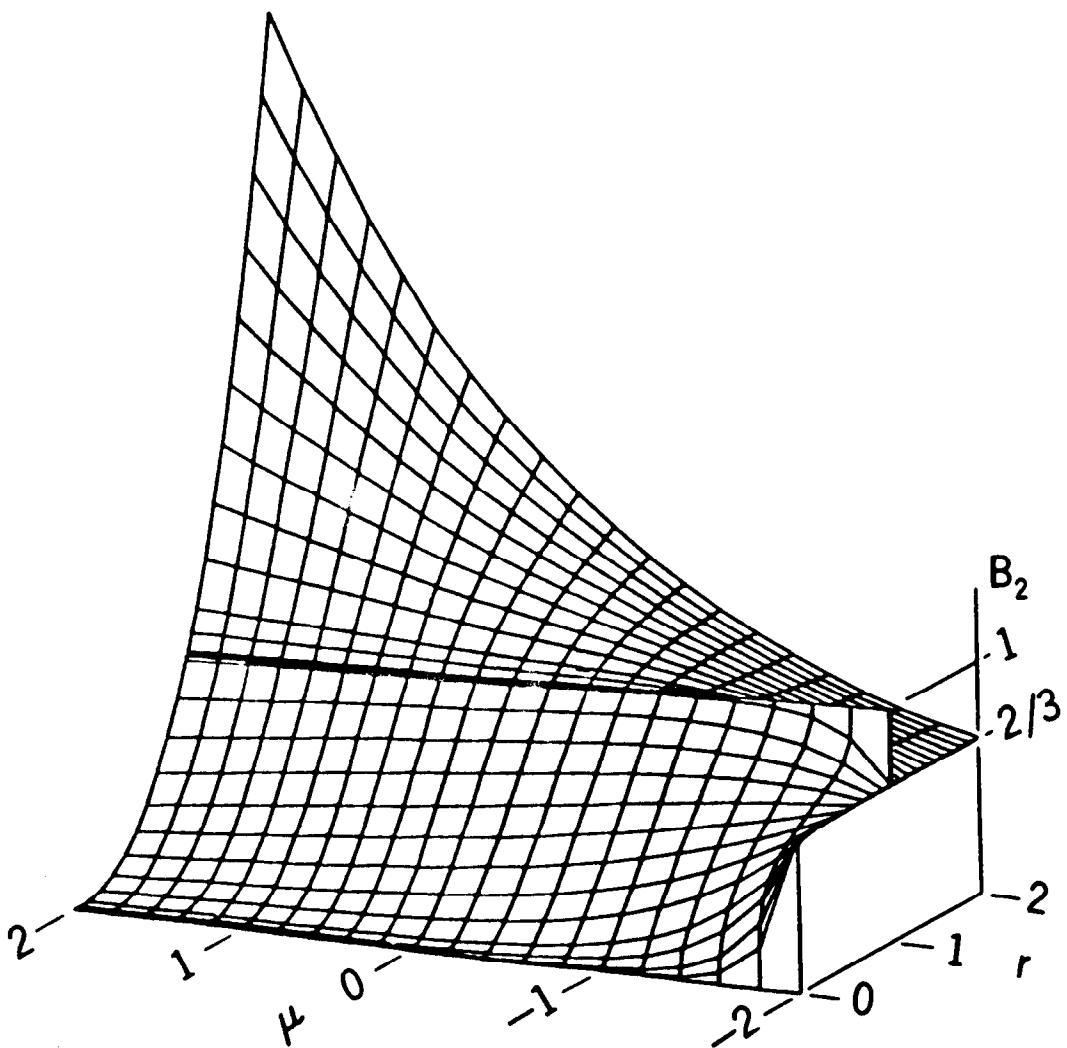


Figure 2. A plot of the relationship between the frequency-domain power-law spectral-density exponent α and the time-domain two-sample Allan variance exponent μ ($\alpha = -\mu - 1$, $-2 \leq \mu < 2$ and $\alpha \geq 1$ for $\mu = -2$). Also shown is the similar relationship between α and the modified Allan variance with exponent on τ of μ' ($\alpha = -\mu' - 1$, $-4 \leq \mu' < 2$). The pointing arrows indicate the mu-alpha relationship (α vs. μ or μ') for which the particular variance applies.



THE BIAS FUNCTION, $B_2(r, \mu)$

Figure 3. A three dimensional plot of the bias function $B_2(r, \mu)$, where $r = T/\tau$, and the dead time is $T - \tau$. The "fin" at $r = 1$ and $\mu = -2$ approaches zero width as the measurement bandwidth approaches infinity (see appendix ref. [3]).

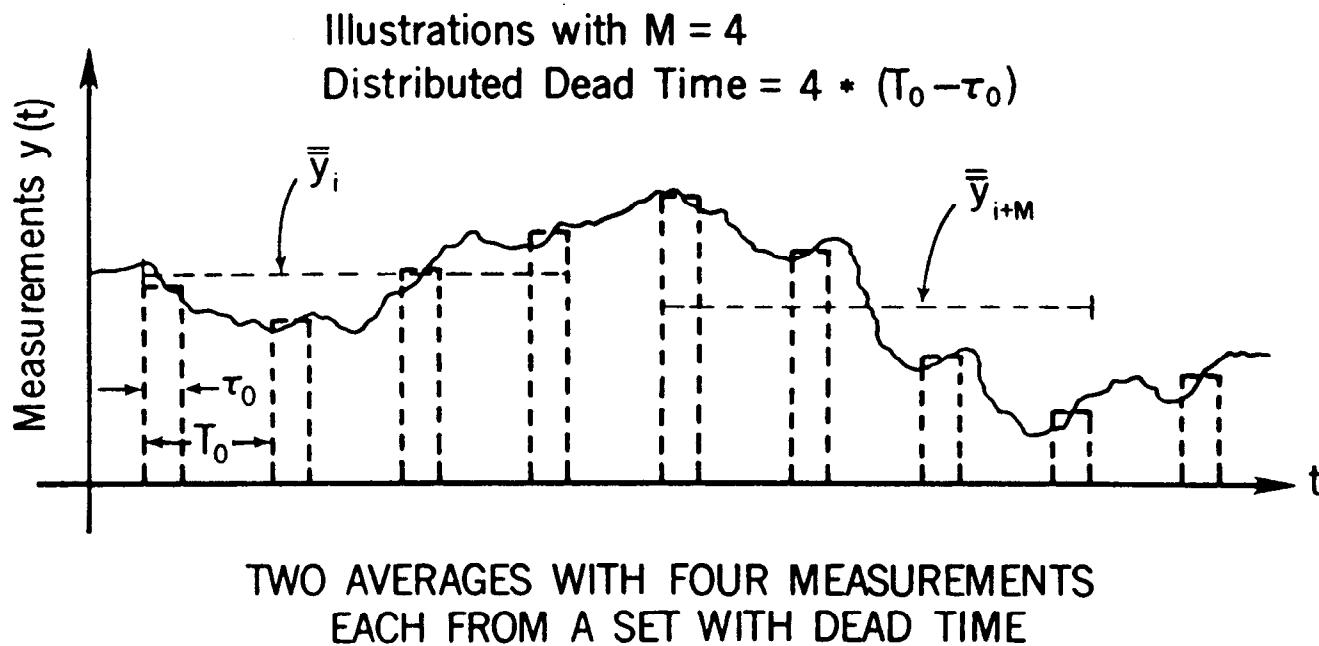


Figure 4. An illustration of determining two averages, each taken over four measurements. In this illustration the original data have a sample time τ_0 and a dead time $T_0 - \tau_0$. The two averages each have a distributed sample time $4\tau_0 = \tau$, and a distributed dead time $4(T_0 - \tau_0) = T - \tau$, since in this case $M = 4$. These two averages are two of a set similar averages with distributed sample times and dead times needed to calculate a two-sample distributed sample-time and dead-time variance -- the numerator for the bias function B_3 . The denominator for B_3 is a two-sample variance with no distributed sample time or dead time but with $T = MT_0$ and $\tau = M\tau_0$ and dead time $T - \tau$.

Appendix

With reference to figure 1, the frequency sampling window has an equivalent phase sampling window. The intent is to evaluate the variance, $S(M)$, of the sampled phase function in terms of the phase autocorrelation function, $R(r)$. The process here is to correctly account for terms and cross-terms coming from squaring and averaging the samples for each M . The $B_3(2,M,r,\mu)$ function can then be obtained from the relation,

$$B_3(2,M,r,\mu) = \frac{S(M)}{S(1) \cdot M^{\mu+2}},$$

for appropriate M , r , and μ . The denominator is just the two-sample variance with dead time for MT and $M\tau$ (in accordance with the definition of $B_3(2,M,r,\mu)$). The factors common to the numerator and denominator are ignored in the following.

For $M = 1$, the variance $S(1)$ is just

$$S(1) = 4 \cdot R(0) - 4 \cdot R(\tau) - 4 \cdot R(T) + 2 \cdot R(T+\tau) + 2R(T-\tau),$$

where use has been made of the definition of the autocorrelation function,

$$R(T) = E[\phi(t) \cdot \phi(t+T)].$$

It is convenient to define a function $G(T)$ as

$$G(T) = 2 \cdot R(T) - R(T+\tau) - R(T-\tau).$$

Similarly, $S(2)$ can now be written in the form,

$$S(2) = 8 \cdot R(0) - 8 \cdot R(\tau) + 2 \cdot G(T) - 4 \cdot G(2T) - 2 \cdot G(3T).$$

Following this procedure, we can verify that the general $S(M)$ is just

$$S(M) = 4 \cdot M \cdot R(0) - 4 \cdot M \cdot R(r) - 2 \cdot M \cdot G(MT)$$

$$+ 2 \sum_{n=1}^{M-1} (M-n) [2 \cdot G(nT) - G((M+n)T) - G((M-n)T)].$$

Following the work of Barnes and Allan [2,3], we can define the function $U(r)$ by the relation,

$$U(r) = 2 \cdot R(0) - 2 \cdot R(r),$$

and also define

$$F(nr) = G(nT)/U(r),$$

where $r = T/\tau$. The function $U(r)$ for power-law power spectral densities has the form,

$$U(r) = \frac{|r|^{\mu+2}}{4-2^{\mu+2}},$$

which yields

$$F(nr) = 2 \cdot (nr)^{\mu+2} - (nr+1)^{\mu+2} - |nr-1|^{\mu+2}.$$

Finally, the working relation can be written as

$$B_3(2, M, r, \mu) = \frac{2 \cdot M + M \cdot F(Mr) - \sum_{n=1}^{M-1} (M-n) [2 \cdot F(nr) - F((M+n)r) - F((M-n)r)]}{[2 + F(r)] \cdot M^{\mu+2}}.$$

B1(N,r,mu) for r = 64

Mu \ N=	4	8	16	32	64	128	256	512	1024	INF
-2 :	1.000E+00									
-1.8 :	1.000E+00									
-1.6 :	9.999E-01									
-1.4 :	9.999E-01	9.998E-01	9.997E-01	9.997E-01	9.997E-01	9.997E-01	9.997E-01	9.997E-01	9.996E-01	9.996E-01
-1.2 :	9.998E-01	9.997E-01	9.996E-01	9.995E-01						
-1 :	1.000E+00									
-0.8 :	1.001E+00	1.002E+00	1.003E+00	1.004E+00						
-0.6 :	1.005E+00	1.009E+00	1.013E+00	1.016E+00	1.018E+00	1.020E+00	1.021E+00	1.022E+00	1.022E+00	1.024E+00
-0.4 :	1.014E+00	1.028E+00	1.042E+00	1.054E+00	1.064E+00	1.072E+00	1.078E+00	1.083E+00	1.087E+00	1.100E+00
-0.2 :	1.035E+00	1.073E+00	1.112E+00	1.151E+00	1.187E+00	1.220E+00	1.249E+00	1.275E+00	1.298E+00	1.456E+00
0 :	1.073E+00	1.161E+00	1.261E+00	1.369E+00	1.482E+00	1.600E+00	1.719E+00	1.840E+00	1.961E+00	
.2 :	1.135E+00	1.313E+00	1.536E+00	1.808E+00	2.129E+00	2.506E+00	2.944E+00	3.450E+00	4.033E+00	
.4 :	1.224E+00	1.548E+00	2.003E+00	2.624E+00	3.461E+00	4.578E+00	6.060E+00	8.029E+00	1.062E+01	
.6 :	1.341E+00	1.889E+00	2.749E+00	4.080E+00	6.119E+00	9.229E+00	1.396E+01	2.114E+01	3.203E+01	
.8 :	1.489E+00	2.363E+00	3.909E+00	6.624E+00	1.137E+01	1.966E+01	3.411E+01	5.929E+01	1.031E+02	
1 :	1.670E+00	3.010E+00	5.691E+00	1.105E+01	2.177E+01	4.322E+01	8.611E+01	1.719E+02	3.435E+02	
1.2 :	1.890E+00	3.891E+00	8.429E+00	1.879E+01	4.251E+01	9.690E+01	2.218E+02	5.085E+02	1.167E+03	
1.4 :	2.156E+00	5.087E+00	1.265E+01	3.238E+01	8.415E+01	2.204E+02	5.793E+02	1.526E+03	4.022E+03	
1.6 :	2.477E+00	6.716E+00	1.919E+01	5.644E+01	1.685E+02	5.067E+02	1.530E+03	4.630E+03	1.402E+04	
1.8 :	2.865E+00	8.943E+00	2.938E+01	9.927E+01	3.404E+02	1.176E+03	4.080E+03	1.418E+04	4.933E+04	
2 :	3.333E+00	1.200E+01	4.533E+01	1.760E+02	6.933E+02	2.752E+03	1.097E+04	4.378E+04	1.749E+05	

B1(N,r,mu) for r = 128

Mu \ N=	4	8	16	32	64	128	256	512	1024	INF
-2 :	1.000E+00									
-1.8 :	1.000E+00									
-1.6 :	1.000E+00									
-1.4 :	1.000E+00	9.999E-01								
-1.2 :	9.999E-01	9.999E-01	9.998E-01							
-1 :	1.000E+00									
-0.8 :	1.001E+00	1.001E+00	1.002E+00							
-0.6 :	1.003E+00	1.006E+00	1.008E+00	1.010E+00	1.012E+00	1.013E+00	1.014E+00	1.014E+00	1.015E+00	1.015E+00
-0.4 :	1.010E+00	1.021E+00	1.031E+00	1.040E+00	1.047E+00	1.053E+00	1.058E+00	1.062E+00	1.065E+00	1.074E+00
-0.2 :	1.029E+00	1.060E+00	1.092E+00	1.124E+00	1.153E+00	1.181E+00	1.205E+00	1.226E+00	1.245E+00	1.375E+00
0 :	1.065E+00	1.144E+00	1.232E+00	1.329E+00	1.430E+00	1.534E+00	1.640E+00	1.748E+00	1.856E+00	
.2 :	1.127E+00	1.294E+00	1.504E+00	1.759E+00	2.062E+00	2.416E+00	2.827E+00	3.303E+00	3.851E+00	
.4 :	1.217E+00	1.532E+00	1.973E+00	2.576E+00	3.388E+00	4.471E+00	5.909E+00	7.813E+00	1.033E+01	
.6 :	1.336E+00	1.877E+00	2.725E+00	4.037E+00	6.048E+00	9.115E+00	1.378E+01	2.086E+01	3.160E+01	
.8 :	1.486E+00	2.354E+00	3.891E+00	6.589E+00	1.131E+01	1.955E+01	3.391E+01	5.893E+01	1.025E+02	
1 :	1.668E+00	3.005E+00	5.679E+00	1.103E+01	2.172E+01	4.311E+01	8.589E+01	1.714E+02	3.426E+02	
1.2 :	1.889E+00	3.887E+00	8.421E+00	1.877E+01	4.246E+01	9.680E+01	2.215E+02	5.079E+02	1.166E+03	
1.4 :	2.156E+00	5.085E+00	1.265E+01	3.237E+01	8.411E+01	2.203E+02	5.790E+02	1.525E+03	4.021E+03	
1.6 :	2.477E+00	6.715E+00	1.919E+01	5.643E+01	1.684E+02	5.067E+02	1.530E+03	4.629E+03	1.402E+04	
1.8 :	2.865E+00	8.943E+00	2.938E+01	9.926E+01	3.404E+02	1.176E+03	4.080E+03	1.418E+04	4.932E+04	
2 :	3.333E+00	1.200E+01	4.533E+01	1.760E+02	6.933E+02	2.752E+03	1.097E+04	4.378E+04	1.749E+05	

B1(N,r,mu) for r = INFINITY

Mu \ N=	4	8	16	32	64	128	256	512	1024	INF
-2 :	1.000E+00									
-1.8 :	1.000E+00									
-1.6 :	1.000E+00									
-1.4 :	1.000E+00									
-1.2 :	1.000E+00									
-1 :	1.000E+00									
-0.8 :	1.000E+00									
-0.6 :	1.000E+00									
-0.4 :	1.000E+00									
-0.2 :	1.000E+00									
0 :	1.000E+00									
.2 :	1.091E+00	1.210E+00	1.360E+00	1.541E+00	1.757E+00	2.009E+00	2.303E+00	2.642E+00	3.033E+00	
.4 :	1.199E+00	1.487E+00	1.890E+00	2.441E+00	3.184E+00	4.174E+00	5.489E+00	7.231E+00	9.533E+00	
.6 :	1.327E+00	1.854E+00	2.680E+00	3.958E+00	5.916E+00	8.903E+00	1.344E+01	2.034E+01	3.080E+01	
.8 :	1.482E+00	2.343E+00	3.867E+00	6.543E+00	1.123E+01	1.940E+01	3.364E+01	5.846E+01	1.017E+02	
1 :	1.667E+00	3.000E+00	5.667E+00	1.100E+01	2.167E+01	4.300E+01	8.567E+01	1.710E+02	3.417E+02	
1.2 :	1.889E+00	3.885E+00	8.415E+00	1.875E+01	4.243E+01	9.672E+01	2.213E+02	5.075E+02	1.165E+03	
1.4 :	2.156E+00	5.084E+00	1.264E+01	3.236E+01	8.409E+01	2.202E+02	5.789E+02	1.525E+03	4.020E+03	
1.6 :	2.477E+00	6.714E+00	1.919E+01	5.642E+01	1.684E+02	5.066E+02	1.530E+03	4.628E+03	1.402E+04	
1.8 :	2.865E+00	8.943E+00	2.938E+01	9.926E+01	3.404E+02	1.176E+03	4.080E+03	1.418E+04	4.932E+04	
2 :	3.333E+00	1.200E+01	4.533E+01	1.760E+02	6.933E+02	2.752E+03	1.097E+04	4.378E+04	1.749E+05	

MU \ r	.0001	.0003	.001	.003	.01	.03	.1	.3	.5	.7
-2 :	6.667E-01									
-1.8 :	1.112E-01	1.365E-01	1.762E-01	2.195E-01	2.793E-01	3.479E-01	4.431E-01	5.566E-01	6.264E-01	6.869E-01
-1.6 :	1.874E-02	2.908E-02	4.709E-02	7.304E-02	1.183E-01	1.836E-01	2.979E-01	4.693E-01	5.901E-01	7.013E-01
-1.4 :	3.205E-03	6.196E-03	1.276E-02	2.467E-02	5.081E-02	9.826E-02	2.032E-01	3.999E-01	5.572E-01	7.061E-01
-1.2 :	5.586E-04	1.345E-03	3.525E-03	8.489E-03	2.225E-02	5.362E-02	1.410E-01	3.444E-01	5.273E-01	7.052E-01
-1 :	1.000E-04	3.000E-04	1.000E-03	3.000E-03	1.000E-02	3.000E-02	1.000E-01	3.000E-01	5.000E-01	7.000E-01
-.8 :	1.862E-05	6.956E-05	2.949E-04	1.101E-03	4.662E-03	1.735E-02	7.271E-02	2.642E-01	4.749E-01	6.916E-01
-.6 :	3.687E-06	1.715E-05	9.231E-05	4.280E-04	2.298E-03	1.047E-02	5.439E-02	2.351E-01	4.517E-01	6.908E-01
-.4 :	8.121E-07	4.678E-06	3.174E-05	1.809E-04	1.204E-03	6.664E-03	4.195E-02	2.112E-01	4.303E-01	6.683E-01
-.2 :	2.159E-07	1.510E-06	1.260E-05	8.606E-05	6.921E-04	4.507E-03	3.340E-02	1.915E-01	4.103E-01	6.546E-01
0 :	7.726E-08	6.240E-07	6.045E-06	4.745E-05	4.404E-04	3.250E-03	2.742E-02	1.750E-01	3.915E-01	6.401E-01
.2 :	3.906E-08	3.397E-07	3.394E-06	3.049E-05	3.100E-04	2.494E-03	2.316E-02	1.611E-01	3.740E-01	6.251E-01
.4 :	2.590E-08	2.311E-07	2.530E-06	2.228E-05	2.381E-04	2.020E-03	2.006E-02	1.492E-01	3.574E-01	6.097E-01
.6 :	2.013E-08	1.808E-07	2.001E-06	1.788E-05	1.955E-04	1.708E-03	1.773E-02	1.388E-01	3.416E-01	5.943E-01
.8 :	1.700E-08	1.529E-07	1.697E-06	1.524E-05	1.683E-04	1.493E-03	1.593E-02	1.297E-01	3.267E-01	5.789E-01
1 :	1.500E-08	1.350E-07	1.500E-06	1.349E-05	1.495E-04	1.337E-03	1.450E-02	1.215E-01	3.125E-01	5.635E-01
1.2 :	1.357E-08	1.221E-07	1.356E-06	1.221E-05	1.355E-04	1.216E-03	1.333E-02	1.141E-01	2.989E-01	5.483E-01
1.4 :	1.245E-08	1.120E-07	1.245E-06	1.120E-05	1.244E-04	1.118E-03	1.233E-02	1.074E-01	2.859E-01	5.333E-01
1.6 :	1.152E-08	1.037E-07	1.152E-06	1.037E-05	1.152E-04	1.036E-03	1.147E-02	1.012E-01	2.735E-01	5.186E-01
1.8 :	1.072E-08	9.645E-08	1.072E-06	9.645E-06	1.072E-04	9.642E-04	1.070E-02	9.541E-02	2.615E-01	5.042E-01
2 :	1.000E-08	9.000E-08	1.000E-06	9.000E-06	1.000E-04	9.000E-04	1.000E-02	9.000E-02	2.500E-01	4.900E-01

