4 TH EUROPEAN FREQUENCY AND TIME FORUM NEUCHÂTEL - 13 - 14 - 15 MARCH 1990

Session PO-01 Number 11

TIME DOMAIN FREQUENCY STABILITY CALCULATED FROM THE FREQUENCY DOMAIN: AN UPDATE

F. L. Walls, John Gary, Abbie O'Gallagher, Linda Sweet, and Roland Sweet

National Institute of Standards and Technology
Boulder, Colorado 80303 USA

Abstract

In this paper we investigate the dependence of the fractional frequency stability measures $\sigma_y(\tau)$ and mod$\sigma_y(\tau)$ on the parameter $2\pi f_h\tau$ for common power-law noise types. Previous investigators have calculated the value of $\sigma_y(\tau)$ and mod$\sigma_y(\tau)$ in the limit that $2\pi f_h\tau$ is either small or large compared to 1. We have implemented the calculations using numerical techniques which make it possible to perform the calculations for virtually any value of $2\pi f_h\tau$ and a wide variety of high-frequency filters. We find significant differences for $\sigma_y(\tau)$ depending on the filter, when $2\pi f_h\tau \ll 1$. The values for $\sigma_y(\tau)$ are nearly independent of filter slope for $2\pi f_h\tau \gg 1$. Of particular importance is R(n), the ratio of $\sigma_y(\tau)$ to mod$\sigma_y(\tau)$, as a function of n, the number of measurements averaged together in the calculations of mod$\sigma_y(\tau)$. We find differences from the values of R(n) versus n given in the literature for both flicker-phase and flicker-frequency noise types.

Introduction

The calculation of fractional frequency stability expressed in terms of the two sample or Allan Variance, $\sigma_y^2(\tau)$, from frequency domain measures such as the spectral density of frequency fluctuations, $S_y(f)$, is well known [1-3]:

$$\sigma_y^2 = 2\int_0^{f_h} S_y(f)\frac{\sin^4(\pi f n\tau_0)}{(\pi f n\tau_0)^2}\,df.$$

$S_y(f)$, for most physical systems used in signal handling, is characterized by random noise whose power spectral density varies as a simple power law over wide ranges of Fourier frequencies, f. $\sigma_y(\tau)$ can be

calculated analytically under the condition that $2\pi f_h\tau$ is either very large or very small compared to 1. Here f_h is the equivalent noise bandwidth for the time domain characterization and τ is the measurement time [1-4]. When these conditions are not fulfilled there are corrections to $\sigma_y(\tau)$ that are not well known.

Numerical Techniques

The integral,

$$\sigma_y^2 = 2\int_0^{f_h} S_y(f)\frac{\sin^4(\pi f n\tau_0)}{(\pi f n\tau_0)^2}\,df,$$

can be treated as an oscillatory integral except in the neighborhood of the origin. τ_0 is the minimum measurement time. Therefore, the integral is computed using one approximation in the neighborhood of the origin and a second method away from the origin, that is

$$\sigma_y^2 = I_1 + I_2,$$

where

$$I_1 = \int_0^{\alpha} 2F_1(f)df,$$

$$I_2 = \int_{\alpha}^{f_h} 2F_2(f)\sin^4(\pi f n\tau_0)df,$$

$$F_1(f) = S_y(f)\,\frac{\sin^4(\pi f n\tau_0)}{(\pi(f n\tau_0)^2},$$

and

$$F_2(f) = \frac{S_y(f)}{(\pi f n\tau_0)^2},$$

with α chosen so that the first integral is taken over $\eta/2$ cycles of the sin function, provided $\tau = n\tau_0$ is large enough. Otherwise it is taken over 2/3 of the interval f_h. That is, α = minimum $(\frac{2f_h}{3}, \frac{\eta}{\pi})$. If the function F_1 grows more slowly than f^{-3} at f = 0, then the first integral exists. We assume that $S_y(f)$ grows no more rapidly than f^{-2}, and thus the integrand in this first integral has a removable singularity and can be computed using a general purpose quadrature routine.

The second integral is oscillatory. It can be transformed into canonical form by use of the trigonometric substitution,

$$\sin^4(\pi f\tau) = \frac{3}{8} - \frac{1}{2}\cos(2\pi f\tau) + \frac{1}{8}\cos(4\pi f\tau),$$

and can therefore be written as the sum of three integrals,

$$I_2 = I_{21} + I_{22} + I_{23}.$$

Since α is chosen large enough to avoid the singularity at the origin, the first of these three integrals can be computed with the standard quadrature routine. The remaining two integrals are oscillatory and are therefore computed using a routine, designed for integrals of the form $\int f(x)\sin(\omega x)dx$ or $\int f(x)\cos(\omega x)dx$ where ω can be very large. These integrals may have values of $n\tau_0$ which easily exceed 10^4 (here $n\tau_0$ corresponds to ω). Additional details can be found in [4].

The numerical code for $S_y(f)$ has the following form.

$$S_y(f) = \frac{C_1 f^{-2} + C_2 f^{-1} + C_3 + C_4 f + C_5 f^2}{K(f)M(f)} + C_6,$$

where K(f) is used to describe the offset on $S_y(f)$ due to locking to an external reference characterized by $S_y(f) = C_6$, and M(f) is used to provide high-frequency filtering. Alternately we can use an infinitely sharp cut-off, f_h to attenuate the high frequency noise. The values of the Cs are stored in an array. The filter function K(f) must be chosen from the following four functions.

$$K(f) = 1,$$

$$K(f) = \left[1 + \frac{1}{K_1 f}\right]^2,$$

$$K(f) = \left[\left[1 + \frac{1}{K_1 f}\right]\left[1 + \frac{1}{K_2 f}\right]\right]^2,$$

$$K(f) = \left[\left[1 + \frac{1}{K_1 f}\right]\left[1 + \frac{1}{K_2 f}\right]\left[1 + \frac{1}{K_3 f}\right]\right]^2.$$

$$M(f) = [(1 + M_1 f(1 + M_2 f)\ (1 + M_3 f)]^2,$$

where any or all of the M_i can be 0.

Results

Figure 1 shows the well known dependence of $\sigma_y(\tau)$ on τ for the common noise types when $2\pi f_h\tau \gg 1$ and an infinitely sharp filter at f_h is used to attenuate the high frequency noise.[1-3] The only difference between the results shown here, and that obtained for a single low pass filter is that the values for flicker phase are about 10% smaller. These calculations, performed using our numerical techniques, agree to within 1% with previous work. They serve to verify that the codes are operating correctly.

Figure 2 shows the results of our numerical calculation of $\sigma_y(\tau)$ for $\alpha = 2$ (white phase modulation noise) as a function of measurement time τ and three values of f_h. The calculations were made using both an infinitely sharp filter of width f_h and a single low pass filter of the form $M(f) = (1 + f/f_h)^2$. From the variations of $\sigma_y(\tau)$ from one f_h curve to the next we can uniquely determine the dependence of $\sigma_y(\tau)$ on f_h and on τ in the limits where $2\pi f_h\tau$ is either very small or large compared to 1. Once the dependence on f_h and τ has been determined, the numerical coefficient can be calculated. In the limit that $2\pi f_h\tau$ is either very small or large compared to 1 they generally agree to within 1% with those obtained from the analytical calculations [1-3]. The dependence of $\sigma_y(\tau)$ on τ and $2\pi f_h\tau$ near $2\pi f_h\tau = 1$ for the two filter shapes has not, to our knowledge, been investigated before. Figures 3 to 6 show similar results for the other common noise types. For a single noise type, the slope of $\sigma_y(\tau)$ versus τ, averaged over a decade in τ, can vary widely depending on the value of $2\pi f_h\tau$ and the type of filter used. The asymptotic expressions of $\sigma_y(\tau)$ for the various noise types with the infinitely sharp and single low-pass filters are given in Table 1. [1-3]

We have adapted the numerical techniques used to calculate $\sigma_y(\tau)$ to calculate $\mathrm{mod}\sigma_y(\tau)$ in the limit of an infinitely sharp filter of width f_h.

The mod sigma integral

$$\mathrm{mod}\sigma_y^2(n) = \frac{2}{n^4\pi^2\tau_0^2}\left\{n\int_0^{f_h}\frac{S_y(f)}{f^2}\sin^4(\pi f n\tau_0)df + 2\int_0^{f_h}\sum_{k=1}^{n-1}(n-k)\frac{S_y(f)}{f^2}\cos(2\pi f k\tau_0)\sin^4(\pi f n\tau_0)df\right\},$$

is more difficult to approximate because of the sum over k. Large values of n require computation of many integrals which takes a long time. The sum over k is eliminated by use of the following identity [5,6]

$$\sum_{k=1}^{n-1} (n-k)\cos\theta k = -\frac{n}{2} + \frac{1}{2}\frac{\sin^2\frac{\theta n}{2}}{\sin^2\frac{\theta}{2}}.$$

This identity is derived by looking at the real part of a sum of complex exponentials. Since cos is the real part of the complex exponential, this transforms the sum into a geometric series. Such a series can be summed explicitly yielding the final formula. Application of this identify to the original integral leads to the following form for the integral:

$$\mathrm{mod}\sigma_y^2(n) = \frac{2}{n^4\pi^2\tau_0^2}\int_0^{f_h} \frac{S_y(f)\sin^6(\pi\tau_0 nf)}{f^2\sin^2(\pi\tau_0 f)}df.$$

This integral can be evaluated in the same way as the $\sigma_y^2(n)$ integral, except that this integrand has a singularity for integral values of $\tau_0 f$, instead of the single singularity at $f = 0$. As in the case of the $\sigma_y^2(n)$ integral, it is necessary to use one routine near the singularities and another routine for those subintervals in which the integrand is highly oscillatory. It is therefore necessary to break up the interval of integration into $f_h\tau_0$ subintervals and sum the values of the integral over each of these smaller intervals. If f_h is large this involves a large number of subintervals. Often, later subintervals contribute negligibly to the value of the integral. For this reason an estimate of the error produced by ignoring further contributions is made every few subintervals. If this error is small relative to the computed value of the integral up to that point, then no further calculation is done and the value returned by the program as $\mathrm{mod}\sigma_y^2(n)$ is the sum of the integrations done over the subintervals up to that point. Additional details can be found in [4].

Rather than show the explicit values of $\mathrm{mod}\sigma_y(\tau)$, we have chosen to represent the results as

$$R(n) = \frac{[\mathrm{mod}\sigma_y(\tau)]^2}{[\sigma_y(\tau)]^2}.$$

This form clearly shows the differences between $\sigma_y(\tau)$ and $\mathrm{mod}\sigma_y(\tau)$ as a function of noise type. It is interesting that R(n) shows a dependence on $2\pi f_h\tau$ only for flicker phase noise ($\alpha = 1$) using an infinitely sharp filter. For flicker phase and flicker frequency noise types the results given in Figure 7 and Table 2 differ somewhat from these given by previous investigators [7,8].

Discussion

We believe that this numerical approach to the calculation of both $\sigma_y(\tau)$ and $\mathrm{mod}\sigma_y(\tau)$ will prove to be very useful. It offers more flexibility in the types of noise which can be considered and can easily accommodate various filter shapes. Such filter shapes can be used to limit the noise bandwidth or define the many types of servo systems found in modern frequency standards. Using this numerical approach, we have investigated the dependence of $\sigma_y(\tau)$ and $\mathrm{mod}\sigma_y(\tau)$ on measurement time as a function of noise type, the value of $2\pi f_h\tau$, and filter shape. We have shown that the slope of $\sigma_y(\tau)$ or $\mathrm{mod}\sigma_y(\tau)$, which is normally used to identify the noise type, depends strongly on the value of $2\pi f_h\tau$ and the high-frequency filter shape.

References

1. J.A. Barnes, A.R. Chi, L.S. Cutler, D.J. Healey, D.B. Leeson, T.E. McGunigal, J.A. Mullen, Jr., W.L. Smith, R.L. Sydnor, R.F.C. Vessot, G.M. Winkler, Characterization of Frequency Stability, Proc. IEEE Trans. on I&M 20, 105-120 (1971).
2. D.W. Allan, H. Hellwig, P. Kartaschoff, J. Vanier, J. Vig, G.M.R. Winkler, and N.F. Yannoni, Standard Terminology for Fundamental Frequency and Time Metrology, Proc. of the 42nd Ann. SFC, 419-425 (1988).
3. P. LeSage and C. Audoin, Characterization of Measurement of Time and Frequency Stability, Radio Science 14, 521-539 (1979).
4. F.L. Walls, J. Gary, A. O'Gallagher, R. Sweet and L. Sweet, Time Domain Frequency Stability Calculated from the Frequency Domain Description: Use of the SIGINT Software Package to Calculate Time Domain Frequency Stability from the Frequency Domain, NISTIR 89-3916.
5. V.F. Kroupa and L. Sojdr, Modified Allan Variance: Transfer Functions and Practical Applications, CPEM 1984 Digest.
6. P. Tremblay in Etude de l'Effet du Filliage du Signal sur la Caracterisation de la Stabilite de Frequence, These Universite Laval, Quebec 1985.
7. D.W. Allan and J.A. Barnes, A Modified "Allan Variance" with Increased Oscillator Characterization Ability, Proc. of the 35th Ann. SFC, 470-475 (1981).
8. P. Lesage and T. Ayi, Characterization of Frequency Stability: Analysis of the Modifed Allan Variance and Properties of Its Estimate, IEEE Trans. on I&M, 1984, IM-33, 332-337.

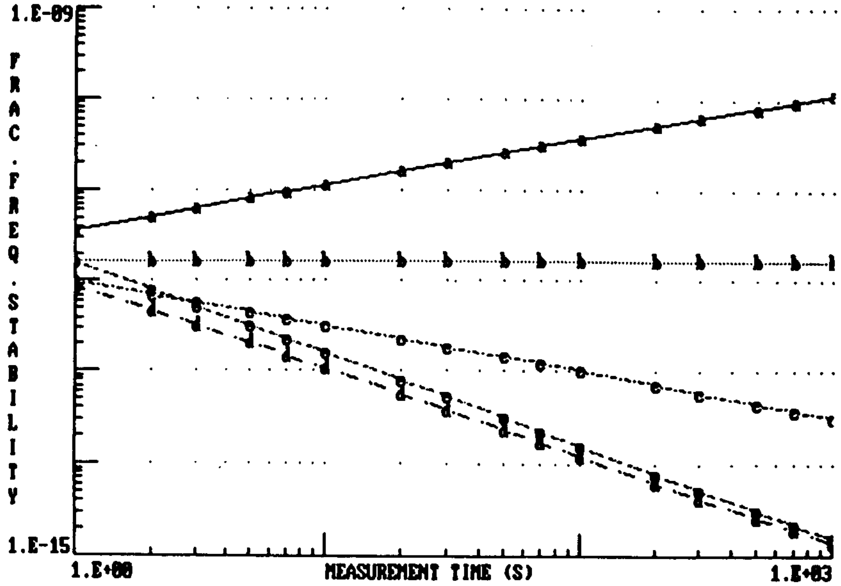

Figure 1. $\sigma_y(\tau)$ versus τ for the five common power-law noise types in the limit that $2\pi f_h \tau$ is large compared to 1 and an infinitely sharp filter is used. Curve a is for random-walk frequency modulation, $S_y(f) = h_{-2} f^{-2}$. Curve b is for flicker frequency modulation, $S_y(f) = h_{-1} f^{-1}$. Curve c is for white frequency modulation, $S_y(f) = h_0$. Curve d is for flicker phase modulation, $S_y(f) = h_1 f$. Curve e is for white phase modulation, $S_y(f) = h_2 f^2$.

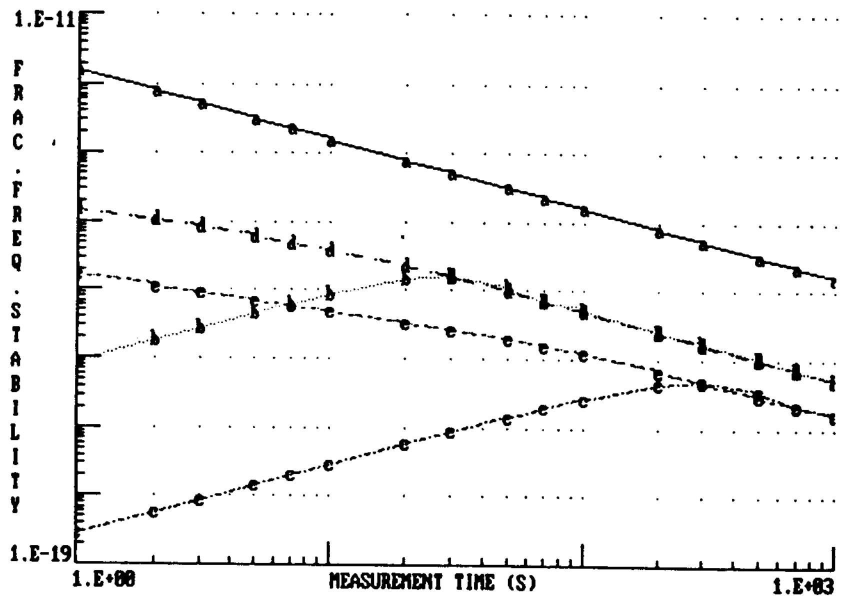

Figure 2. $\sigma_y(\tau)$ for white phase modulation ($\alpha = 2$) as a function of measurement time, τ, and measurement bandwidth, f_h. Curves a, b, and c have an infinitely sharp filter with width, $f_h = 16$ Hz, $f_h = 0.016$ Hz, $f_h = 0.0016$ Hz respectively. Curves d and e have a single pole filter width, $f_h = 0.016$ Hz and $f_h = 0.0016$ respectively.

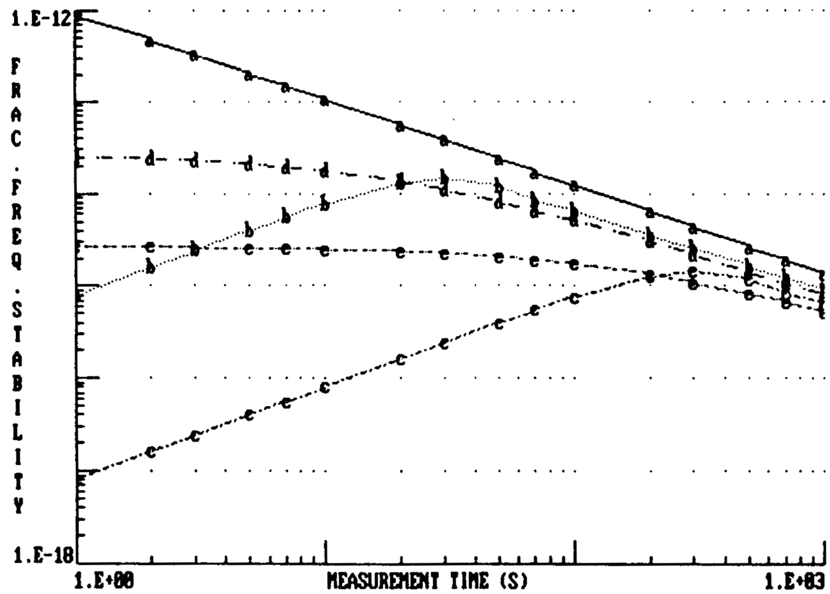

Figure 3. $\sigma_y(\tau)$ for flicker phase frequency modulation ($\alpha = 1$) as a function of measurement time, τ, and measurement bandwidth, f_h. Curves a, b, and c have an infinitely sharp filter with width, f_h = 16 Hz, f_h = 0.016 Hz, f_h = 0.0016 Hz respectively. Curves d and e have a single pole filter width, f_h = 0.016 Hz and f_h = 0.0016 respectively.

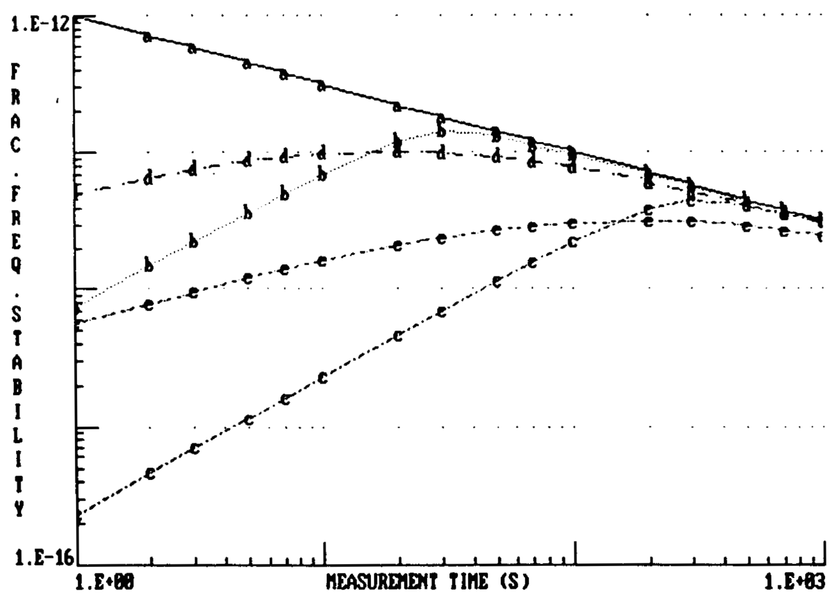

Figure 4. $\sigma_y(\tau)$ for white frequency modulation ($\alpha = 0$) as a function of measurement time, τ, and measurement bandwidth, f_h. Curves a, b, and c have an infinitely sharp filter with width, f_h = 16 Hz, f_h = 0.016 Hz, f_h = 0.0016 Hz respectively. Curves d and e have a single pole filter width, f_h = 0.016 Hz and f_h = 0.0016 respectively.

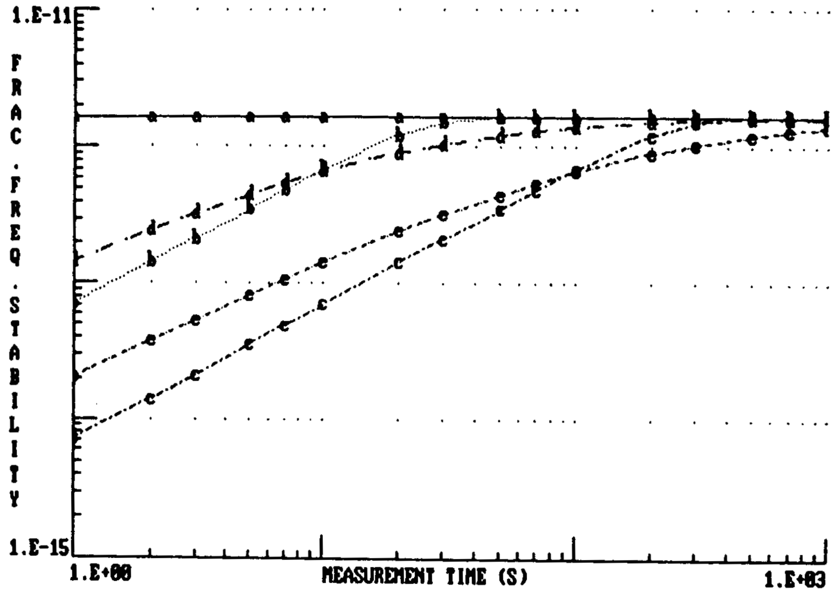

Figure 5. $\sigma_y(\tau)$ for flicker frequency modulation (α = -1) as a function of measurement time, τ, and measurement bandwidth, f_h. Curves a, b, and c have an infinitely sharp filter with width, f_h = 16 Hz, f_h = 0.016 Hz, f_h = 0.0016 Hz respectively. Curves d and e have a single pole filter width, f_h = 0.016 Hz and f_h = 0.0016 respectively.

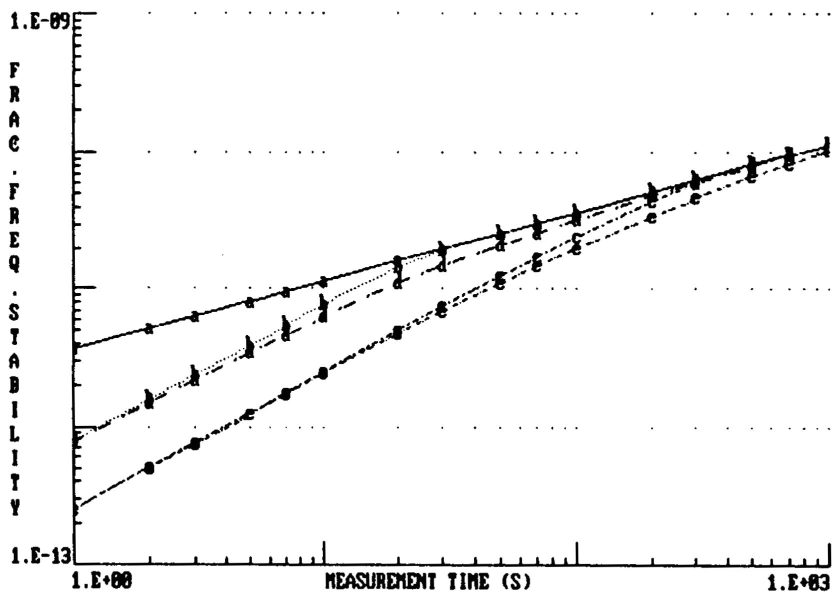

Figure 6. $\sigma_y(\tau)$ for random walk frequency modulation (α = -2) as a function of measurement time and measurement bandwidth, f_h for π. Curves a, b, and c have an infinitely sharp filter with width, f_h = 16 Hz, f_h = 0.016 Hz, f_h = 0.0016 Hz respectively. Curves d and e have a single pole filter width, f_h = 0.016 Hz and f_h = 0.0016 respectively.

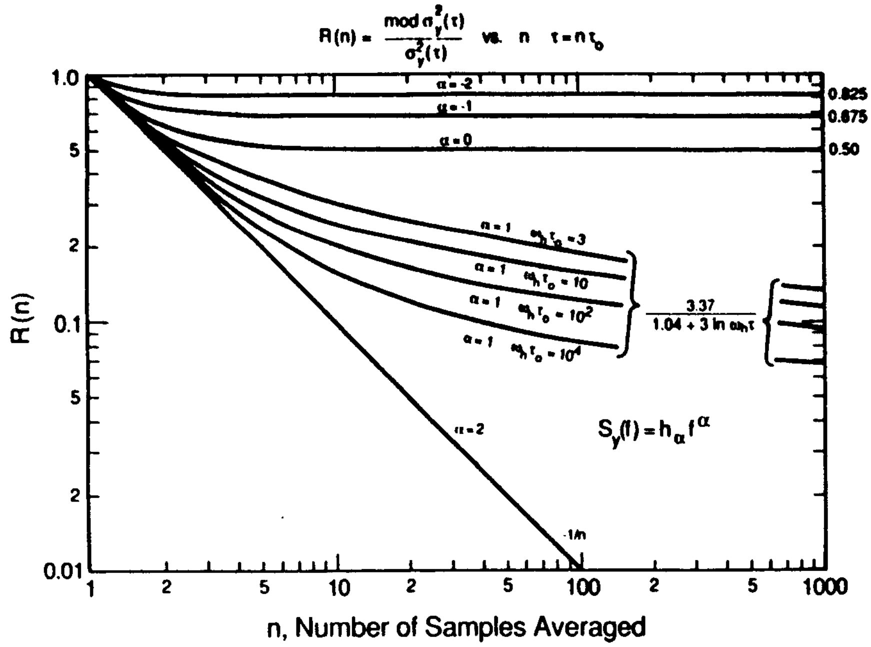

Figure 7. Ratio of mod$\sigma_y(\tau)^2$ to $\sigma_y(\tau)^2$ as a function of n, the number of points averaged to obtain mod$\sigma_y(\tau)$. The measurement time $\tau = n\tau_0$, where τ_0 is the minimum data interval.

TABLE I. Asymptotic forms of $\sigma_y^2(\tau)$ for various power-law noise types and two filter types. Note: $\omega_h/2\pi = f_h$ is the measurement system bandwidth--often called the high-frequency cutoff. ln ≡ $\log_e$.

Name of Noise	α	$S_y(f)$	$\sigma_y^2(\tau)$			
			$\omega_h\tau>>1$ Infinite Sharp Filter	$\omega_h\tau>>1$ Single Pole Filter	$\omega_h\tau<<1$ Infinite Sharp Filter	$\omega_h\tau<<1$ Single Pole Filter
White Phase	2	h_2f^2	$\frac{3f_hh_2}{(2\pi)^2\tau^2}$	$\frac{3f_hh_2}{(2\pi)^2\tau^2}$	$2/5\pi^2f_h^5\tau^2h_2$	$\frac{f_h^2h_2}{2\tau}$
Flicker Phase	1	h_1f	$\frac{(1.038 + 3\ln(\omega_h\tau))h_1}{(2\pi)^2\tau^2}$	$\frac{(3\ln(\omega_h\tau))h_1}{(2\pi)^2\tau^2}$	$\frac{1}{2}\pi^2f_h^4\tau^2h_1$	$2f_h^2(\ln(2))h_1$
White Frequency	0	h_o	$\frac{h_o}{2\tau}$	$\frac{h_o}{2\tau}$	$2/3\pi^2f_h^3\tau^2h_0$	$2/3\pi^2f_h^2\tau h_0$
Flicker Frequency	-1	$h_{-1}f^{-1}$	$2(\ln(2))h_{-1}$	$2(\ln(2))h_{-1}$	$\pi^2f_h^2\tau^2h_{-1}$	$8\pi^2f_h^2\tau^2h_{-1}$
Random-Walk Frequency	-2	$h_{-2}f^{-2}$	$\frac{2\pi^2\tau h_{-2}}{3}$	$\frac{2\pi^2\tau h_{-2}}{3}$	$2\pi^2f_h\tau^2h_{-2}$	$2\pi^2f_h\tau^2h_{-2}$

ABLE II. Ratio of $mod\sigma_y^2(\tau)$ to $\sigma_y^2(\tau)$ vs n, for common power-law noise types $S_y(f) = h_\alpha f^\alpha$. n is the number of ime or phase samples averaged to obtain $mod\sigma_y^2(\tau = n\tau_0)$ where τ_0 is the minimum sample time, and ω_h is 2π times he measurement bandwidth f_h.

n	α = -2	α = -1	α = 0	α = +1 $\omega_h\tau_0$ = 3	$\omega_h\tau_0$ = 10	$\omega_h\tau_0$ = 100	$\omega_h\tau_0$ = 10^4	α = +2
1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2	0.859	0.738	0.616	0.568	0.543	0.525	0.504	0.500
3	0.840	0.701	0.551	0.481	0.418	0.384	0.355	0.330
4	0.831	0.681	0.530	0.405	0.359	0.317	0.284	0.250
5	0.830	0.684	0.517	0.386	0.324	0.279	0.241	0.200
6	0.828	0.681	0.514	0.349	0.301	0.251	0.214	0.167
7	0.827	0.679	0.507	0.343	0.283	0.235	0.195	0.143
8	0.827	0.678	0.506	0.319	0.271	0.219	0.180	0.125
10	0.826	0.677	0.504	0.299	0.253	0.203	0.160	0.100
14	0.826	0.675	0.502	0.274	0.230	0.179	0.137	0.0714
20	0.825	0.675	0.501	0.253	0.210	0.163	0.119	0.0500
30	0.825	0.675	0.500	0.233	0.194	0.148	0.106	0.0333
50	0.825	0.675	0.500	0.210	0.176	0.134	0.0938	0.0200
100	0.825	0.675	0.500	0.186	0.159	0.121	0.0837	0.0100
Limit	0.825	0.675	0.500	⟵	$\left(\frac{3.37}{1.04 + 3\ \ln\omega_h\tau}\right)$		⟶	1/n