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TIME SCALE STABILITIES BASED ON TIME AND FREQUENCY KALMAN FILTERS

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Abstract

The use of Kalman filters to generate time scales has been well documented in the literature. The typical "Time" Kalman model used for the commercial cesium beam standards is the superposition of white noise frequency modulation (FM) and random walk noise FM. These processes are considered to be continuous and usually sampled at regular intervals. The sample data are the differences in clock readings between a reference clock and each of the other clocks in the time scale system. A Kalman filter can estimate both the time and frequency corrections for each clock in the scale.

There are, however, other options for time scale operation. One can integrate the frequency correction element of the Kalman state vector for a clock in the scale to obtain a new and different time scale. Also, one can re-cast the entire Kalman model in terms of frequency rather than time. The input (measurement) data for this Kalman filter, then, is exactly the first (time) difference of the data discussed above for the "Time" Kalman. As one might expect, however, each of these options provides different performance in the resulting time scale. While this has been pointed out before, the present paper details the various scale performances between measurements and provides an insight into the different performances based on computer simulation studies. For example, the "Time" Kalman filter displays discrete steps in the time corrections where the "Frequency" Kalman filters are continuous (being the integral of a bounded process). Depending on whether one is most interested in minimizing the RMS time error or minimizing the Allan Variance, one chooses the one time scale over the other.

In a more fundamental sense, FREQUENCY is the basic quantity which is measured in the laboratory while TIME is subject to many conventions and exhibits unbounded errors. In fact, as realized today, time is a defined quantity (dependent upon algorithms, definitions and procedures) and not intrinsic to the atomic clocks used to generate time. For these reasons the frequency Kalman algorithms should be used for the realization of primary time scales since it is frequency, not time, which has a physical basis.

I. Introduction

The literature contains several papers on the use of Kalman Filters in the establishment of time and frequency standards [1,2,3,4]. The typical "time" Kalman model is based on commercial cesium beam frequency standards. Specifically the models are the superposition of white noise frequency modulation (FM) and random walk noise FM.

While several researchers have reported the existence of linear frequency drifts, typically the establishment of a statistically significant drift requires measurements extending over an appreciable fraction of the clock's life expectancy. This paper ignores linear frequency drift. For times shorter than 100 seconds or so other model elements become important, but we ignore them also since most time scales sample data at longer intervals.

II. The Time-Kalman Model

Jones and Tryon [1,2] have shown that the individual Kalman noise models can be written in the form:

$$\begin{pmatrix} X(t+\tau) \\ Y(t+\tau) \end{pmatrix} = \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} + \begin{pmatrix} \varepsilon\,(t+\tau) \\ \eta\,(t+\tau) \end{pmatrix}$$

where X(t) and Y(t) are the continuous time and frequency (approximately) errors of a clock, and $\varepsilon(t)$ and $\eta(t)$ are independent (band limited) white noises. Jones and Tryon also pointed out that there are actually correlations between these noise terms which can become significant if the time interval, τ, becomes too large. For the clocks considered here, these correlations can be ignored. The idea here is that X(t) and Y(t) are continuous random processes which can be sampled at arbitrary intervals, t and $t+\tau$. (For a more complete discussion see Jones and Tryon [1,2].)

The covariance matrix of the driving noise terms is [1]

$$Q(\tau) = \begin{pmatrix} \sigma_\varepsilon^2 & 0 \\ 0 & \sigma_\eta^2 \end{pmatrix} \tau$$

for sufficiently small τ.

The state vector for an ensemble of M clocks is obtained by appending the two state elements of each clock into a column vector of length 2M. The state transition matrix is a 2M by 2M square matrix consisting of the 2 by 2 blocks for the state transition matrices along the main diagonal, with off-diagonal blocks being zeros. Similarly, the ensemble covariance matrix of the driving noise terms is a 2M by 2M square matrix formed from the M 2 by 2 individual blocks, with the off-diagonal blocks being zeros.

The measurements consist of time differences between clock #1 (the reference clock) and each of the other clocks in the ensemble. Although each "measurement" in the simulation studies was treated separately, an equivalent procedure [5] would be to define the H(t)-matrix in the form:

$$H(t) = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & . & . \\ & & & & & & & & \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & . & . \\ & & & & & & & & \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & . & . \\ & & & & & & & & \\ . & . & . & . & . & . & . & . & . \\ & & & & & & & & \\ . & . & . & . & . & . & . & . & . \end{pmatrix}$$

which is an M-1 by 2M matrix for the M-1 independent measurements. All primary time scales have one more unknown than they have independent measurements since absolute time accuracy is impossible.

III. Estimated State Vector for the Time-Kalman Filter

We assume that every τ seconds a complete measurement of M-1 clock comparisons is performed. These comparisons allow an up-date of the estimated Kalman state vector. Of course, the time and frequency errors are thought to evolve continuously even if they were only observed every τ seconds. Kalman theory allows one to estimate (i.e., forecast) these subsequent states even though the comparisons between clocks were not performed continuously. Figure 1 displays the estimated Kalman state vector for clock #1 and the "actual" clock #1 error for a simulated time scale.

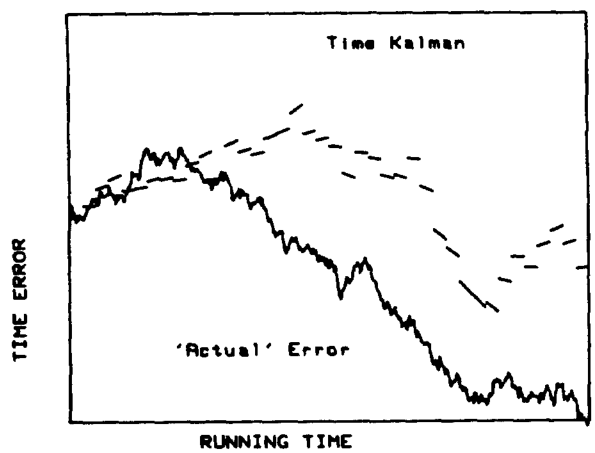

FIG. 1, SIMULATED CLOCK ERROR AND KALMAN TIME STATE

The simulated time scale consists of five "clocks" perturbed by various levels of white noise FM and random walk noise FM. The clocks with the lowest white FM had the highest random walk FM, and conversely. The reference clock, clock #1, had the lowest level of white FM so that it would be the most stable clock between measurements. Figure 2 displays the theoretical Allan variances for the simulated clock data. Of course, it is impossible to generate continuous noise on a digital computer, but the noise was generated every 1/10th τ spacing (ten points per clock per measurement). Hence, the Allan Variances extend to 0.1τ units on the plots.

Each measurement cycle provides new information for the estimated Kalman state vector, and hence abrupt

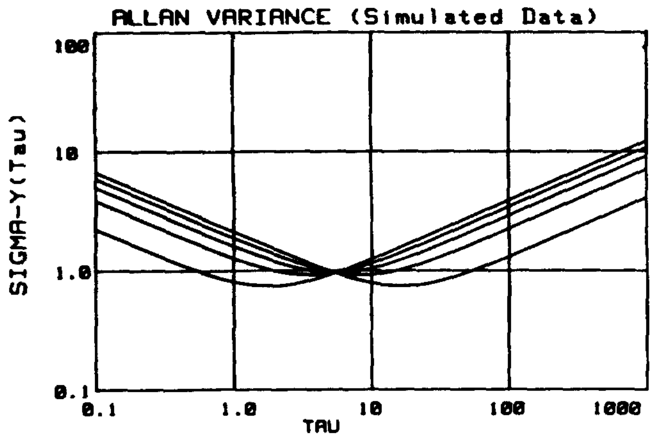

FIG. 2, THEORETICAL ALLAN VARIANCES OF CLOCKS

changes occur (see Fig. 1). Forecasts for the state vector prior to the next measurement are obtained from the frequency element of the estimated state vector. Since the forecast frequency elements of the state vector are constant between measurements, the forecasts in Fig. 1 are segments of straight lines. Of course, the simulation procedure allows one to observe the individual "clock" error, uncontaminated by the instabilities of the other clocks. The absolute time scale error for Fig. 1 is just the difference between the two curves. Clearly, for forecasts following the clock comparisons, the other clocks have essentially no affect on short term variations, and individual clock instabilities are whatever they are. Hence for best time scale resolution, the best reference clock is the most stable clock in short term.

It is not surprising that the abrupt changes in the estimated time error cause observable effects in the Allan Variances (see Fig. 3). Comparing Figs. 1 and 3 reveals that in short term clock #1 by itself is about 8 dB better than the Time-Kalman time scale. This analysis subsumes that the time scale should be based on all of the available data from the clock intercomparisons as provided by the Kalman Filter and its extrapolation up to the next measurement. In effect, this analysis assumes that TIME errors should be minimized regardless of the consequences on FREQUENCY stability.

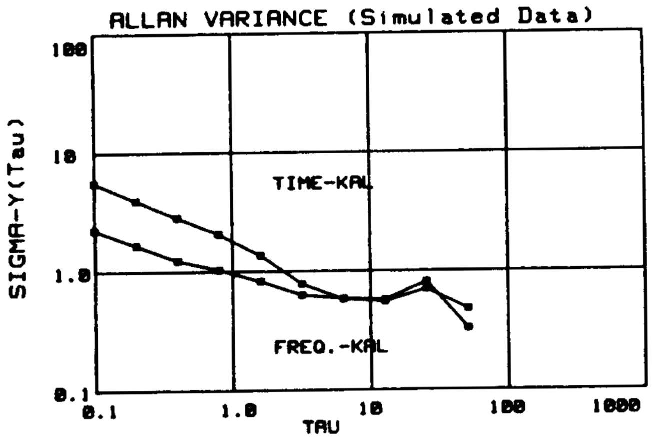

FIG. 3, ALLAN VARIANCES OF TIME SCALES

IV. Time Kalman vs. Frequency Kalman

There is a very important, fundamental and philosophical distinction to be made between time and frequency as it relates to atomic time scales and atomic clocks. Frequency is the fundamental quantity in atomic clocks. It can be well modeled and described by quantum mechanical processes, probability theory, and is the intrinsic natural process upon which the SI second is based as given by the definition for the cesium 133 atom. The energy state selection technique, signal-to-noise and other relevant electronic servo conditions, for example, determine the statistical properties and the averaging time necessary to measure a particular quantum state with a certain level of precision. In addition, because of the good models that have been developed one can evaluate the systematic uncertainties and develop a list of uncertainties associated with the realization of an absolute frequency per some natural resonance phenomenon in an atom. Because of the above, the General Conference of Weights and Measures (CGPM) in 1967 developed the current definition for the second -- as mentioned above. Independently different laboratories can realize the SI second without communication with one another strictly based on fundamental physical processes. When these clocks are compared they should agree within the uncertainty error budgets associated with each experimental determination of the SI second. This is currently the case between the four fundamental primary frequency standards which contribute to the determination of the rate of TAI. These standards are located at NBS, NRC, PTB and RRL, and are independent and agree to within their accuracies (less than or equal to about 1 part in 10^{13}).

In contrast it is impossible to do the same with time. The time from a time scale is typically dependent on three things: 1) an arbitrary origin (date) from whence seconds are accumulated; 2) the particular algorithm used to average a set of clocks used for this accumulation of seconds; and 3) a periodic frequency or rate calibration with either a primary standard or with a commercial standard that may be used in the composition of the scale. Therefore, the time so generated is an artifact of some arbitrary epoch (beginning point), plus some arbitrary algorithm and plus the integrated time errors from the frequency calibration inaccuracies. Whereas independent frequency standards will agree within some error budget, the time difference between independent time scales will be unbounded. We can speak of the natural resonance frequency of quantum mechanical transition, but the natural time of same has no meaning!

Therefore, as one applies Kalman filter theory given these philosophical and physical differences, the idea of optimizing the Kalman parameters around some natural resonance frequency has both intuitive and strong physical meaning; in contrast setting optimum Kalman parameters to minimize the time error is an artifact, is artificial and has no analogous physical meaning. Since the frequency Kalman gives better short-term stability for an arbitrary set of clocks and is apparently comparable to the time Kalman in long term, and since the frequency Kalman has a more sound physical basis the frequency Kalman would appear to be the better approach in the generation of atomic time scales. To date, no one is using this approach. The National Bureau of Standards is in the development process evaluating such an algorithm and doing experimentation to compare it with the current methods of generating time. Should the experiment corroborate the theory and above argument, the plans are to change the official algorithm for the generation of atomic time, TA(NBS), at the National Bureau of Standards to a frequency Kalman approach.

If a person's task were to synchronize a secondary clock to some primary clock, then the Time-Kalman approach would be better [6]. If frequency stability were the principal concern, then the Frequency-Kalman approach may be better.

V. Alternatives

Including the "TIME" Kalman filter discussed above, this paper considers four alternatives:

1. Time-Kalman, discussed above.

2. Linear slewing of the scale time from forecast to forecast as obtained using the Time-Kalman.

3. Forming a new time scale as the (discrete) integral of the frequency elements obtained from the Time-Kalman.

4. A complete re-casting of the Kalman model in terms of frequency rather than time. The underlying model of white FM and random walk FM would still be retained. The time scale, then, would be realized by integration of the estimated frequency elements of the new Frequency-Kalman filter. The measurement data would be exactly the first difference of the same time data used in the Time-Kalman (divided by τ).

Figure 1 depicts the type of instabilities one should expect from option 1, the simple Time-Kalman. Figure 4 uses the same data as for Fig. 1, but slews the correction to clock #1 from the forecast made at the previous measurement to the forecast for the next measurement, τ seconds in advance. That is, after completing a current measurement, one can forecast the time element of the clock #1 state vector for the next measurement, τ seconds hence. Knowing what the last forecast was for the current point allows one to calculate a slewing rate to apply to the clock #1 data to reach the new correction value.

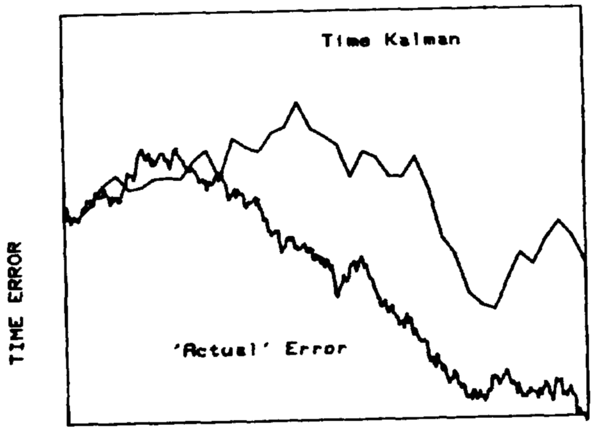

FIG. 4

SIMULATED CLOCK ERROR SLEWED TO NEXT FORECAST (Option 2)

The state vector for the ensemble of clocks includes the frequency element in addition to the time element discussed in the first two options. One can sum up these frequency corrections (and multiply by τ) for the clock with the best short-term stability (clock #1, for the current example). Figure 5 displays the results of these calculations for the same exact data as used for Figs. 1 and 4. Note that the Time-Kalman data and computations are all the same for options 1, 2, and 3. The differences are in what one does with the various elements of the state vector and the transition matrix, in making forecasts.

Option 4, above, is a more substantive change. Starting with the same clock model of white FM and random walk FM, one can show that a Frequency-Kalman model can be written in the form:

$$\begin{pmatrix} Y(t+\tau) \\ Z(t+\tau) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} Y(t) \\ Z(t) \end{pmatrix} + \begin{pmatrix} \varepsilon(t) \\ \eta(t) \end{pmatrix}$$

where Y(t) is the frequency element and Z(t) is a dummy variable needed for the model but without any obvious interpretation. Z(t) is essential for the generation of the random walk FM component.

The H(t)-matrix is the same as before since the first element is now frequency rather than time.

The measurement input data is now frequency rather than time. Each of the M-1 measurements is of the form:

$$W_n(t) = (X_n(t+\tau) - X_n(t)) / \tau$$

where $W_n(t)$ is the FREQUENCY measurement for the n-th clock deduced from the exact same data as used in the Time-Kalman. The Q(t) matrix is also unchanged except the factor τ is replaced by $1/\tau$.

As in option 3, option 4 uses the discrete summation of the frequency forecast of the state vector (multiplied by τ) to obtain an estimate of the clock's time error. Figure 6 displays the results of the Frequency-Kalman and integrator using the first difference of the same input data as in Figs. 1, 4, and 5. Figure 7 is an overlay on Fig. 2 of the frequency stability obtained using options 1 and 4 (Time-Kalman and Frequency-Kalman).

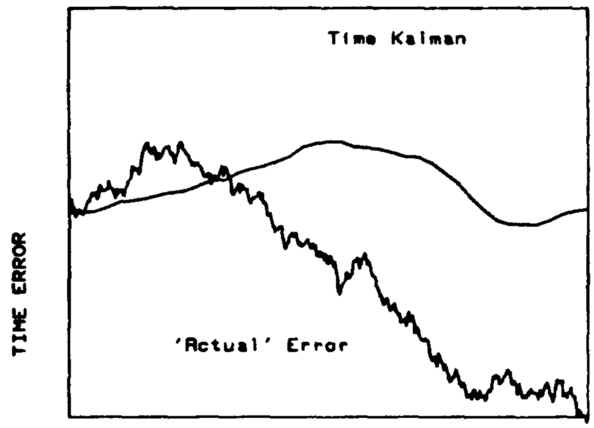

FIG. 5, SIMULATED CLOCK ERROR INTEGRATED FREQUENCY (Option 3)

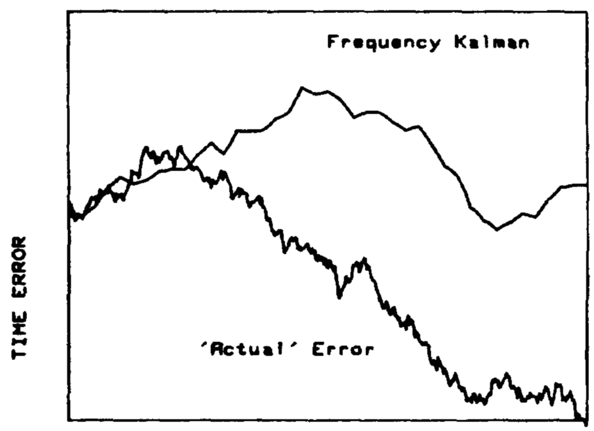

FIG. 6, SIMULATED CLOCK ERROR FOR FREQUENCY KALMAN (OPTION 4)

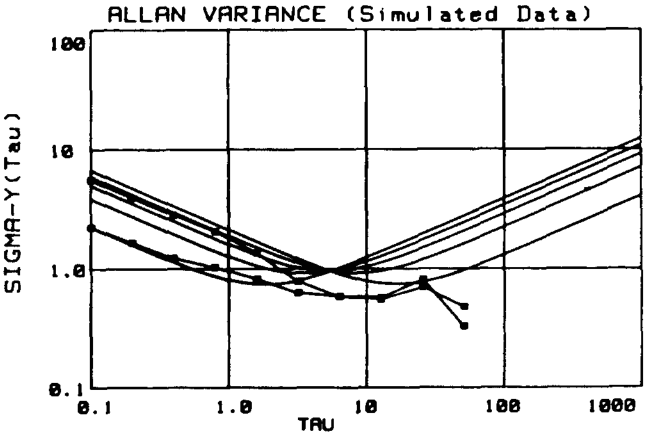

FIG. 7, THEORETICAL AND EXPERIMENTAL VARIANCES

VI. Comparisons of Time Scale Algorithms

In long-term there seems little to differentiate between the four algorithms, although there is a slight indication that the Frequency-Kalman (option 4) might be a fraction of a dB better than the others. It is clear that the various scales could depart asymptotically as $\tau^{3/2}$ from each other, but the same is true for any one of the time scales relative to some "ideal" scale. For short times (< τ)the four options have about the relationships in Table 1:

TABLE 1.

Option	Short-Term Noise Level Above Frequency-Kalman
1. Time Kalman	+8 dB
2. Time-Kal (slew time)	+6.5 dB
3. Time-Kal (freq. integ)	-0 dB
4. Frequency-Kalman	0 dB

As a more important application of these Kalman algorithms, we applied them to a simulation of the NBS time scale. From other studies [Jones, et al.], we have good estimates of the white FM and random walk FM components of each clock. Table 2 lists standard deviations (σ's) for both noise types for each clock. The best clock in short-term was chosen as the reference clock, clock number 1. These values are in units of nanoseconds and correspond to a time interval of one day. To obtain values corresponding to one second, the sigmas for white FM should be divided by $(86400)^{1/2}$, and the random walk sigmas by $(86400)^{3/2}$.

TABLE 2. NBS Time Scale Parameters (Units ns, Daily Basis)

Clock No.	White FM	Random Walk FM
1	0.5	0.55
2	2.8	0.84
3	0.6	0.83
4	9.1	3.0
5	9.9	1.7
6	9.4	1.9
7	14.3	0.86
8	11.4	2.0
9	4.7	0.55
10	2.3	0.55
11	11.4	2.1

Figure 8 represents the square root of the theoretical Allan variances corresponding to the parameters listed in Table 2. We simulated the noises for each of the eleven clocks and formed the differences between the reference clock (No. 1) and each of the other clocks corresponding to typical time scale data. This simulated data was used by the Time-Kalman and the Frequency-Kalman to define two (simulated) time scales. Since the data of each simulated clock was known, the absolute scale error for each simulated scale was calculated. (This is the value of simulation.) Figure 9 results from the calculated Allan variances for each simulated scale. Surprisingly, the turn-on transients for the time Kalman persisted for about one (simulated) month. The data for Figs. 9 and 10 come from sufficiently long averages to minimize the transients.

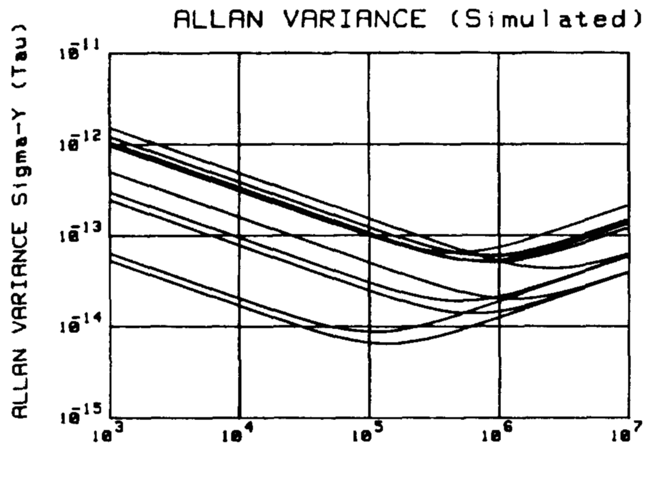

FIG. 8

THEORETICAL ALLAN VARIANCES BASED ON NBS CLOCKS

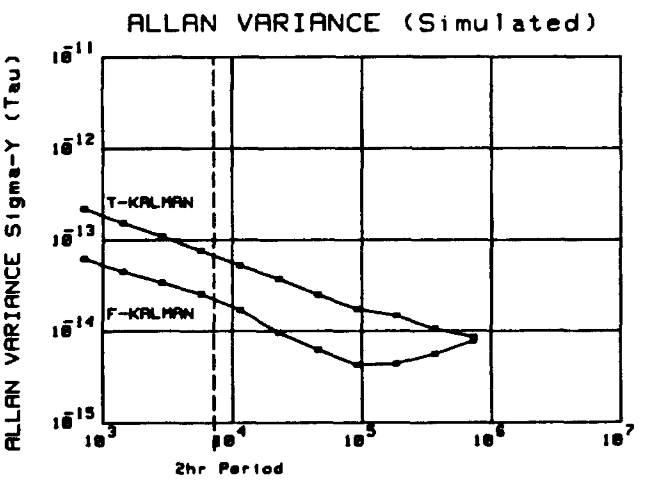

FIG. 9

OBSERVED VARIANCES FOR NBS TIME SCALE SIMULATION

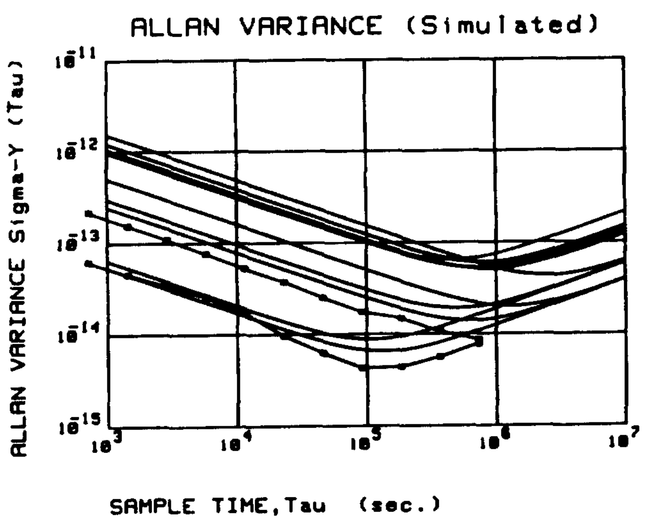

FIG. 10, NBS TIME SCALE SIMULATION

The upper curve in Fig. 9 corresponds to the typical time Kalman (option 1, above), while the lower curve corresponds to the frequency Kalman (option 4). Figure 10 superimposes Figs. 8 and 9. The integral of the frequency state of the Time-Kalman (option 3) is indistinguishable from the Frequency-Kalman (option 4).

The fundamental conclusion is that either of the Frequency-Kalman's (option 3 or 4) offers about a 10 dB improvement to the NBS time scale over the Time Kalman (option 1) for sample times shorter than a few days.

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