

**A NEW "FILTERED ALLAN VARIANCE" AND ITS APPLICATION TO THE IDENTIFICATION OF PHASE AND FREQUENCY NOISE SOURCES**

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**Abstract**

In part one different digital noise generators are described. They use

1) a recursive equation known for its properties of successive bifurcations leading to a chaotic behavior.

2) a second method is presented based on the theory of fractals. The noise is generated from a recurrent fractal curve with a given fractal dimension, the slope of the noise spectrum being linked to the fractal dimension.

3) a third method consists in applying the mean theorem to the random number series given by RDN function of a calculator. In this manner a sample series is obtained representing a white noise with a gaussian distribution. With three different numerical filters using the Z transform all the noise types can be generated.

In part two these noise sources are used to generate sample series which simulate frequency samples as given by a counter.

Applying the Allan variance to the samples of six generators (white phase, flicker phase, white frequency, flicker frequency, frequency random walk, filtered flicker frequency) yields the expected theoretical slopes have been observed.

It is well known that both white phase and flicker phase noises give in time domain when using Allan variance the same characteristic in log-log plot slope, which corresponds to  $\tau^{-2}$ . It will be interesting to separate the two contributions, this is one of the properties of the Modified Allan variance.

In the present work a new method is used. It consists in filtering the  $f^{-1}$  and  $f^0$  phase noise (simulated by the previous noise generators) by means of a digital filter in time domain (using the Z transform), which yields  $f^{-3}$  and  $f^{-2}$  noises, which thus can be identified by Allan variance.

The combination of the digital filter and the Allan variance corresponds to a new variance, which is described and compared to the well known "Modified Allan Variance".

**Introduction**

The "Allan Variance" which is used for the characterizations of a signal, gives the possibility to identify in time domain the different types of noises present in a frequency source ; their respective levels in frequency domain can be calculated by Fourier transform, but for white phase noise and flicker phase noise, this transformation is not possible because they exhibit an Allan variance with the same power law  $\tau^{-2}$  and consequently they cannot be separated.

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The "Modified Allan" variance [1] proposed by Allan in 1981 gives the possibility to solve the ambiguity.

For the white phase noise this variance exhibits a  $\tau^{-3}$  slope which is slightly different from the  $\tau^{-2}$  slope obtained for the flicker phase noise : however, on a log-log plot it is difficult to find the respective asymptotes. Therefore, it can be interesting to use a new variance which offers a better separation of the white and flicker phase noises. This is the purpose of the present paper.

The variances are tested on a computer by means of series of well identified samples, which first of all must be generated. If it is simple to achieve a noise generator exhibiting a noise spectrum in  $1/f^\alpha$  when  $\alpha$  is an even integer by using a white noise filtered by a first order filter, it is more difficult to do for the odd values of  $\alpha$ , (or when  $\alpha$  is not an integer).

In fact all the noise spectra showing an odd value of  $\alpha$  can be obtained from the flicker noise corresponding to  $\alpha = 1$ . Two methods can be used to obtain this flicker noise, either by superposition of weighted Lorentzian spectra or by filtering the white noise with a cascade of first order filters in series with cut-off frequencies distributed as a geometrical series.

These noise generators will be used for comparing the properties of different kinds of variances : Allan variance, Modified Allan variance and Filtered Allan variance.

**White noise generators**

Recursive equation

The first method which was tested is based on the use of a recursive equation known for its properties of successive bifurcations leading to a chaotic behavior (Fig. 1)

$$x_{n+1} = 4 \lambda x_n (1 - x_n)$$

- For  $\lambda < .8$  there is only one solution obtained by solving the equation  $x_{n+1} = x_n$
- For  $.8 < \lambda < .88$  there are two solutions corresponding to  $x_{n+2} = x_n$ . Then the number of possible solutions increases drastically with  $\lambda$ .
- and
- For  $\lambda = 1$  the number of solutions is so large that the behavior is chaotic, and the power spectral density of the corresponding values appears as a white spectrum.

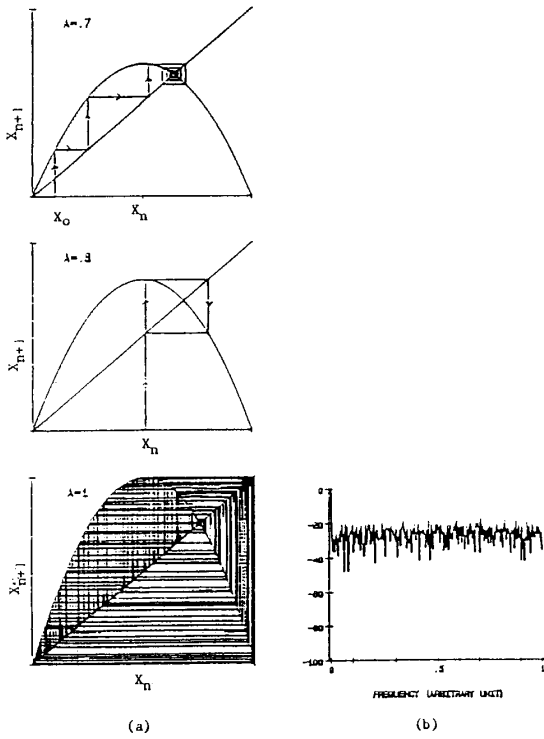


Fig. 1 : Recursive equation  $x_{n+1} = 4\lambda X_n(1 - X_n)$  leading to successive bifurcations and chaotic behavior (a) white spectrum corresponding to the chaotic behavior (b)

This is a very easy method to generate random numbers with white spectrum. But it must be noticed that these distribution functions is not gaussian.

Using fractal

The second method uses the theory of fractals. A deterministic fractal signal, looking like a noisy signal can be easily fabricated by means of a broken curve drawn in a rectangle lattice with  $m$  times  $n$  elementary rectangles as illustrated by figure 2.

Then by successive iterations of each elementary segment a fractal curve is obtained with a resolution depending on the number of iterations, and with a fractal dimension  $D$  which is simply given by

$$D = 2 - \frac{\log n}{\log m}$$

A curve with a given fractal dimension can be easily obtained by simply choosing the number of the elementary rectangles.

The fractal dimension of various curves was plotted as a function of the coefficient  $\alpha$  of the measured  $1/f^\alpha$  power spectral densities (Fig. 3).

Therefore, it is possible to generate a noise with a given power spectral density by choosing the appropriate fractal dimension, following Fig. 3. But this curve depends on the frequency cut-off of the signal. On the same figure is indicated the limit corresponding to frequency cut-offs and zero and infinity.

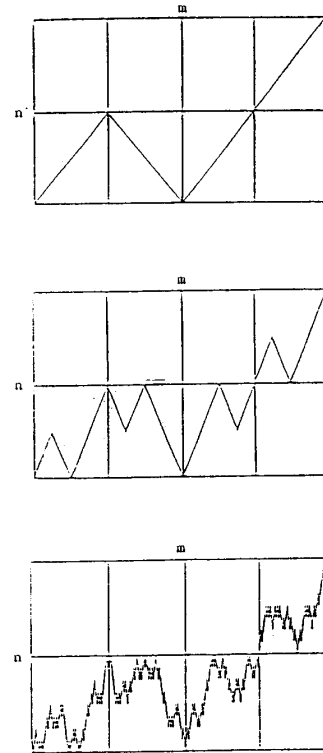


Fig. 2 : Construction of a fractal curve with a given fractal dimension by successive iterations

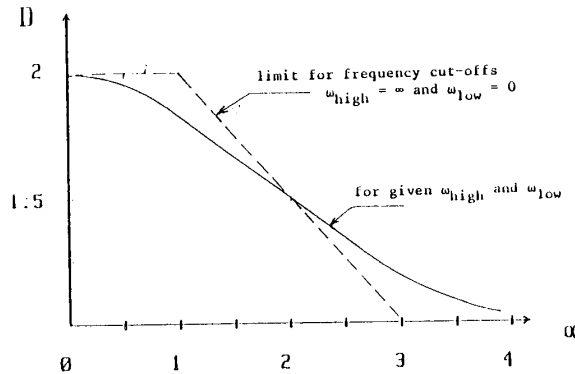


Fig. 3 : Relation between fractal dimension  $D$  and power spectral density coefficient  $\alpha$

From a computer random number generator

This other method uses a generator of random number  $x_i$ , corresponding to the "RND" functions of a computer, whose values are between 0 and 1 with a white distribution function. This white distribution function can be transformed to a gaussian distribution with a zero mean value

by applying on the  $x_i$  samples the mean theorem in order to obtain the  $\bar{x}_j$  samples defined by

$$\bar{x}_j = \frac{1}{N} \sum_{i=1}^N (x_i - 0,5)$$

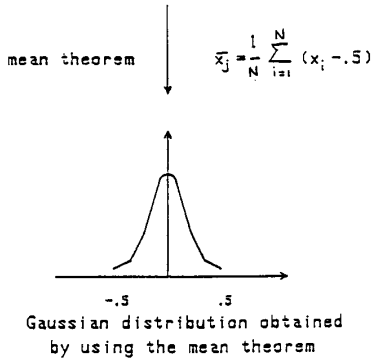
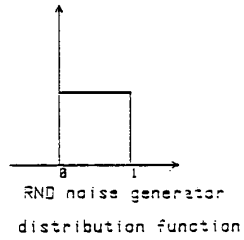


Fig. 4

Distribution function of a RND generator and application of the mean theorem to obtain a gaussian distribution

For  $N = 1$  we obtain the original shifted white distributions (Fig. 5) calculated on 8192 successive generated numbers.

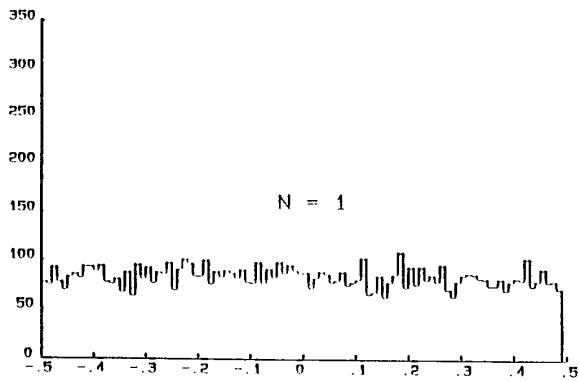


Fig. 5 : Distribution function of the RND number generator after application of the mean theorem for  $N = 1$

Figs. 6 and 7 show the gaussian distribution for  $N = 3$  and  $N = 8$ .

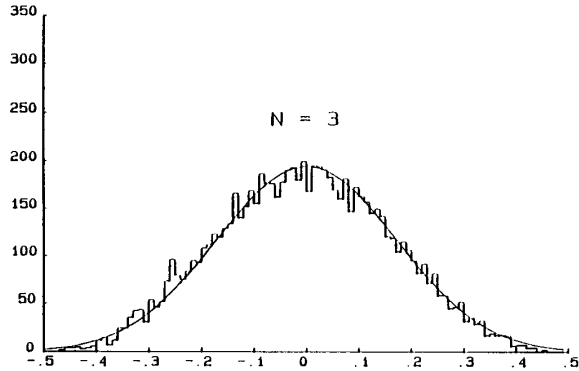


Fig. 6 : Distribution function of the RND number generator after application of the mean theorem for  $N = 3$

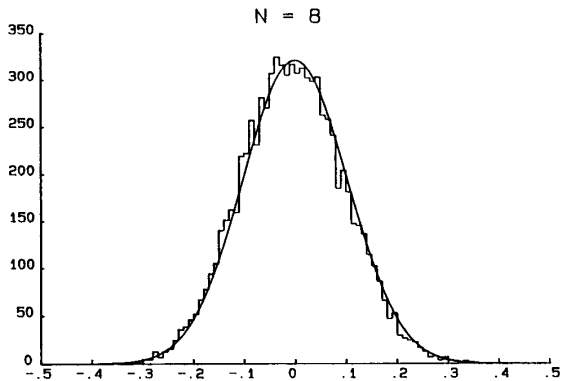


Fig. 7 : Distribution function of the RND number generator after application of the mean theorem for  $N = 8$

This series of samples is considered as a series of time samples integrated over a time interval  $\tau$  similar to frequency samples furnished by a counter. The corresponding Allan variance  $\sigma^2(\tau)$  exhibits a slope of -1 on a log-log plot as expected for white noise (Fig. 8).

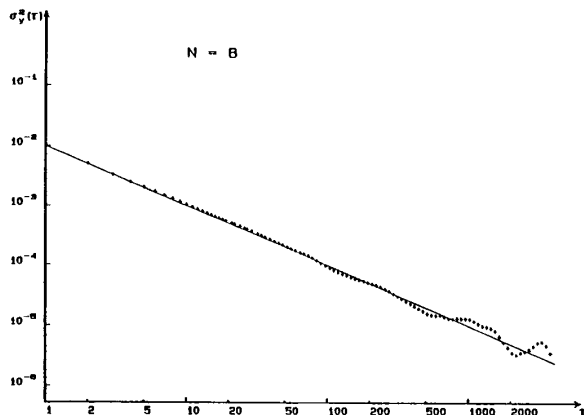


Fig. 8 : Allan variance obtained for a white noise and  $N = 8$

### Generation of 1/f<sup>a</sup> noises

The different noises are obtained by filtering. Only three different types of filters (derivative filter, integrator filter and 1/f filter) are necessary to obtain from white noise all the kinds of noises found in the frequency noise spectrum of oscillators.

$f_0 \times f^2$	→	white phase noise
$f_0 \times f^1 \times f^2$	→	flicker phase noise
$f_0$	→	white frequency noise
$f_0 \times f^1$	→	flicker frequency noise
$f_0 \times f^2$	→	random walk frequency noise.

Two processes can be used, digital filtering or a fast Fourier transform.

#### Digital filtering

A general transfer function for a filter can be written as

$$\frac{S(p)}{E(p)} = \frac{1 + p k_2}{1 + p k_1}$$

By applying the complex homographic transform corresponding to the well known Z transform, which links Z to p

$$Z = \frac{1+p}{1-p} \text{ or } p = \frac{Z-1}{Z+1}$$

The output samples  $S(x_j)$  are expressed in terms of the output samples  $E(x_j)$  by the relation

$$S(x_j) = \frac{1+k_2}{1+k_1} E(x_j) + \frac{1-k_2}{1+k_1} E(x_{j-1}) - \frac{1-k_1}{1+k_1} S(x_{j-1})$$

Four types of filters can be obtained :

- for  $k_1 = 0$  and  $k_2 \neq 0$ , the corresponding transfer function  $|H(f)|^2$  is proportional to  $f^2$ . This is a derivative filter.
- for  $k_2 = 0$  and  $k_1 \neq 1$ , the corresponding transfer function  $|H(f)|^2$  is proportional to  $f^{-2}$ . This is an integrator filter.
- for  $k_2 = ak_1$  with  $k_1 < k_2$ , the transfer function corresponds to a pseudo-derivative filter.
- for  $k_2 = bk_1$ , with  $k_1 > k_2$  the transfer function corresponds to a pseudo-integrator filter.

The association of n pseudo-integrator filters with cut-off frequencies distributed by a geometrical series gives an equivalent 1/f transfer function.

For n filters in series

$$|H(f)|^2 = \left| \prod_{i=1}^n \left( \frac{1+a^i f}{1+a^{i+1} f} \right) \right|^2 \sim \frac{1}{f}$$

#### FFT method

In this method the sequence of random numbers, given by the white generator, is first Fast Fourier Transformed, to obtain the components in the frequency domain, then these components are weighted by the corresponding coefficient of the H(f) transfer function of the desired filter and finally

an Inverse Fourier Transform is applied to the filtered components in order to obtain a sequence of random number with the desired 1/f<sup>a</sup> spectrum.

$$x_j(t) \xrightarrow{\text{FFT}} E_n(f) \xrightarrow{H(f)} E_{n \text{ filt}}(f) \xrightarrow{\text{IFT}} x_{j \text{ filt}}(t)$$

Then these noise generators are used to test the Allan variance, Modified Allan variance, and the new Filtered Allan variances.

#### Comparison of variances

For each type of variance a double check is made (Fig. 9). In the first one the variances are obtained in the frequency domain by means of a FFT of the random signal, then by filtering with the transfer function  $H_i(f)$  of the corresponding variance and by integrating where :

- $H_1(f)$  corresponds to the Allan variance
- $H_2(f)$  corresponds to Modified Allan variance
- $H_3(f)$  corresponds to the Filtered Allan variance with  $(\pi f^2)$  filter
- $H_4(f)$  corresponds to the Filtered Allan variance with  $f^2$  filter

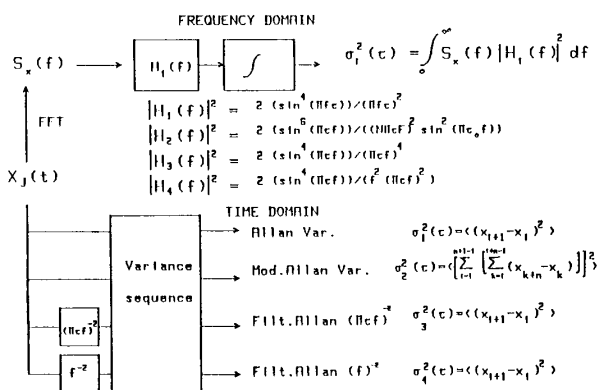


Fig. 9 : Definition of the Filtered Allan variance and schematic diagram of the double check of the four variances in the frequency domain and in the time domain

Simultaneously the second one is the simulation in the time domain performed by calculating the variance with the corresponding algorithm. For the Filtered Allan variance two types of digital filters are applied. The first has a transfer function proportional to  $(\pi f)^2$  and the second has a transfer function proportional to  $f^2$ . These filters are the digital filters defined in the last paragraph.

An example of this double check for white phase noise is shown on Fig. 10 for a Filtered Allan variance obtained with a  $(f^2)$  type filter calculated with 8192 samples.

These curves exhibit a -1 slope as expected. For large values of  $\tau$  the number of averages decreases and the accuracy of the curve corresponding to the time domain is worse.

To increase the number of averages we can use the Allan variance with overlapping averages. The result of this calculation gives a curve (Fig. 10) which is closer to the curve calculated by transformation from the frequency domain. This means that information on the signal is lost with the Allan variance when the number of averages is too small.

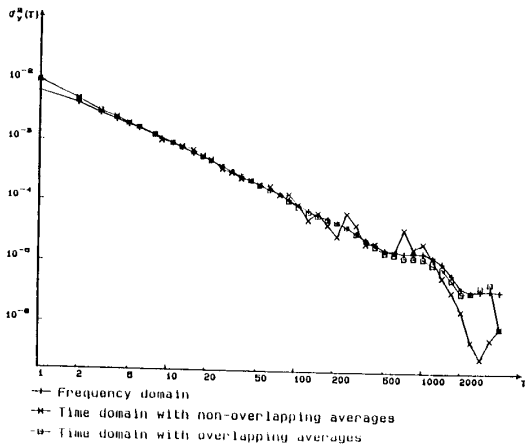


Fig. 10 : Results of the double check for a Filtered Allan Variance obtained with a  $f^2$  type filter calculated with 8192 samples

These double checks have been made for the four variances in the case of white phase noise, flicker phase noise and white frequency noise. Table I gives the characteristic slopes of the four variances.

$S_y(f)$	Allan variance	Modified Allan variance	Filtered variance	
			$(\pi\tau f)^2$	$f^2$
2	-2	-3	-3	-1
1	-2	-2	-2	0
0	-1	-1	-1	1

Table I  
Characteristic slopes of the different variances

This shows that for white phase noise and flicker phase noise, Allan variance gives the same -2 slope as it well known.

The Modified Allan variance distinguishes these two noises with -3 and -2 slopes. The Filtered Allan variance gives -3 and -2 slopes for filtered transfer function in  $(\pi\tau f)^2$  and -1 and 0 for filtered transfer function in  $f^2$ .

However if we compare the variances in a log-log plot (Fig. 11) it is clear that the distinction between the two noises (flicker phase noise and white phase noise) is even easier with the Filtered variance since it exhibits an angle of  $45^\circ$  between the asymptot rather than  $8^\circ$  as the Modified variance does.

In Fig. 12 is given the stability curve in the time domain of a 5 MHz quartz oscillator measured with the Filtered ( $f^2$ ) variance. The value of the transfer function of the digital filter being equal to 1 at 1 Hz, this filter transforms the frequency fluctuations into phase fluctuations and the well known relations between frequency domain and time domain must be shifted by a factor of  $f^2$ .

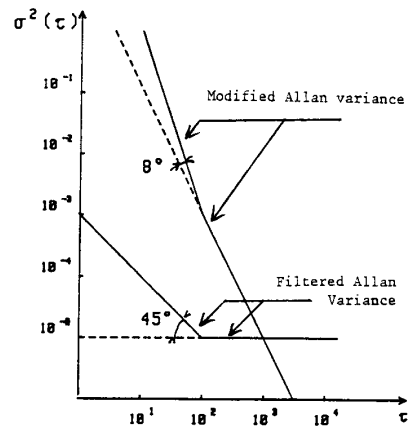


Fig. 11 : Angular difference between the asymptotes of the Modified Allan Variance and the Filtered Allan variance

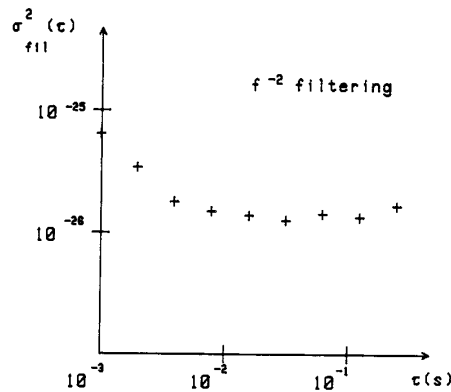


Fig. 12 : Filtered Allan variance of a 5 MHz oscillator showing white phase noise and flicker phase noise

In other words a white phase noise spectrum will be treated as a white frequency spectrum, a flicker phase spectrum as a flicker frequency spectrum and so on.

For a white phase noise spectrum  $S_y(f)$  the Filtered Allan Variance becomes

$$\sigma_{filt}^2(\tau) = h_2 / 2\tau$$

and for flicker phase noise spectrum  $S_y(f)$

$$\sigma_{filt}^2(\tau) = h_1 \cdot 2 \ln 2$$

The  $h_1$  and  $h_2$  values of the signal spectrum can be deduced from the last relations

$$h_1 \approx 10^{-26}$$

$$h_2 \approx 10^{-28}$$

### References

- [1] David W. Allan and James A. Barnes, Proc. 35th Ann. Symp. on Freq. Cont. (1981), p. 470
- [2] J.C. Radix, Introduction au Filtrage Numérique, 1970, Eyrolles.