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Atomic Timekeeping and the Statistics of Precision Signal Generators

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Abstract—Since most systems that generate atomic time employ quartz crystal oscillators to improve reliability, it is essential to determine the effect on the precision of time measurements that these oscillators introduce. A detailed analysis of the calibration procedure shows that the third finite difference of the phase is closely related to the clock errors. It was also found, in agreement with others, that quartz crystal oscillators exhibit a "flicker" or $|\omega|^{-1}$ type of noise modulating the frequency of the oscillator.

The method of finite differences of the phase is shown to be a powerful means of classifying the statistical fluctuations of the phase and frequency for signal generators in general. By employing finite differences it is possible to avoid divergences normally associated with flicker noise spectra. Analysis of several cesium beam frequency standards have shown a complete lack of the $|\omega|^{-1}$ type of noise modulation.

INTRODUCTION

AN ORDINARY clock consists of two basic systems: a periodic phenomenon (pendulum), and a counter (gears, clock face, etc.) to count the periodic events. An atomic clock differs from this only in that the frequency of the periodic phenomenon is, in some sense, controlled by an atomic transition (atomic frequency standard). Since microwave spectroscopic techniques allow frequencies to be measured with a relative precision far better than any other quantity, the desirability of extending this precision to the domain of time measurement has long been recognized [1].

From the standpoint of precision, it would be desirable to run the clock (counter) directly from the atomic frequency standard. However, atomic frequency standards in general are sufficiently complex that reliable operation over very extended periods becomes somewhat doubtful (to say nothing of the cost involved). For this reason, a quartz crystal oscillator is often used as the source of the "periodic" events to run a synchronous clock (or its electronic equivalent). The frequency of this oscillator is then regularly checked by the atomic frequency standard and corrections are made.

These corrections can usually take on any of three forms: 1) correction of the oscillator frequency, 2) correction of the indicated time, or 3) an accumulating

record of the difference from atomic time of the apparent or indicated time shown by the clock. Both methods 1) and 2) require a calculation of the time difference, and it is sufficient to consider only the last method and the errors inherent in it.

A careful consideration of the calibration procedure leads to the development of certain functionals of the phase which have a very important property—existence of the variance even in the presence of a flicker $(1/|\omega|)$ type of frequency noise. The simplest of these functionals, the second and third finite differences of the phase, turn out to be stationary, random variables whose auto-covariance function is sufficiently peaked to insure rapid convergence of the variance of a finite sample toward the true (infinite sample) variance. These functionals of the phase have the added features of being closely related to the errors of a clock run from the oscillator as well as being a useful measure of oscillator stability.

With the aid of these functionals, it is possible to classify the statistical fluctuations observed in various signal sources. In agreement with work of others [2]–[5], a flicker noise frequency modulation was observed for all quartz crystal oscillators tested. Similar studies on several commercial rubidium gas cells gave uniform indications of flicker noise modulation of levels comparable to those of the better quartz crystal oscillators.

In Section I, the effects on the precision of a time scale due entirely to the calibration procedure of the quartz crystal oscillator and the oscillator's inherent frequency instability are considered.

In Section II, the experimental results of Section I are used as the basis for a theoretical model of oscillator frequency fluctuations, and the results are compared to those of other experimenters.

In Section III, the statistics of an atomic frequency standard of the passive type (e.g., Cs-beam or Rb-gas cell) are considered, and the composite Clock system is treated. Section IV is devoted to a brief discussion of stability measures for signal sources.

I. Quartz Crystal Oscillator Phase Fluctuations

Typical Gross Behavior

Figure 1 shows a typical aging curve for a fairly good quartz crystal oscillator. The oscillator had been operating for a few months prior to the date shown in the graph. On May 1, 1963, the frequency of the oscillator was reset in order to maintain relatively small corrections.

Least square fits of straight lines to the two parts of Fig. 1 yield aging rates of 0.536×10^{-10} per day and 0.515×10^{-10} per day, respectively. This difference in aging rates could be explained by an acceleration of the frequency of about -9×10^{-15} per day per day. This acceleration of the frequency is sufficiently small over periods of a few days when compared to other sources of error that it can be safely ignored. Thus, the frequency of conventional quartz oscillators can be written in the form

$$\Omega(t) = \Omega_0[1 + \alpha t + \dot{\epsilon}(t)] \qquad (1)$$

where α is the aging rate, $\dot{\epsilon}(t)$ is a variation of the frequency probably caused by noise processes in the oscillator itself, and t can be considered to be some rather gross measure of the time (since α and $\dot{\epsilon}$ are quite small corrections).
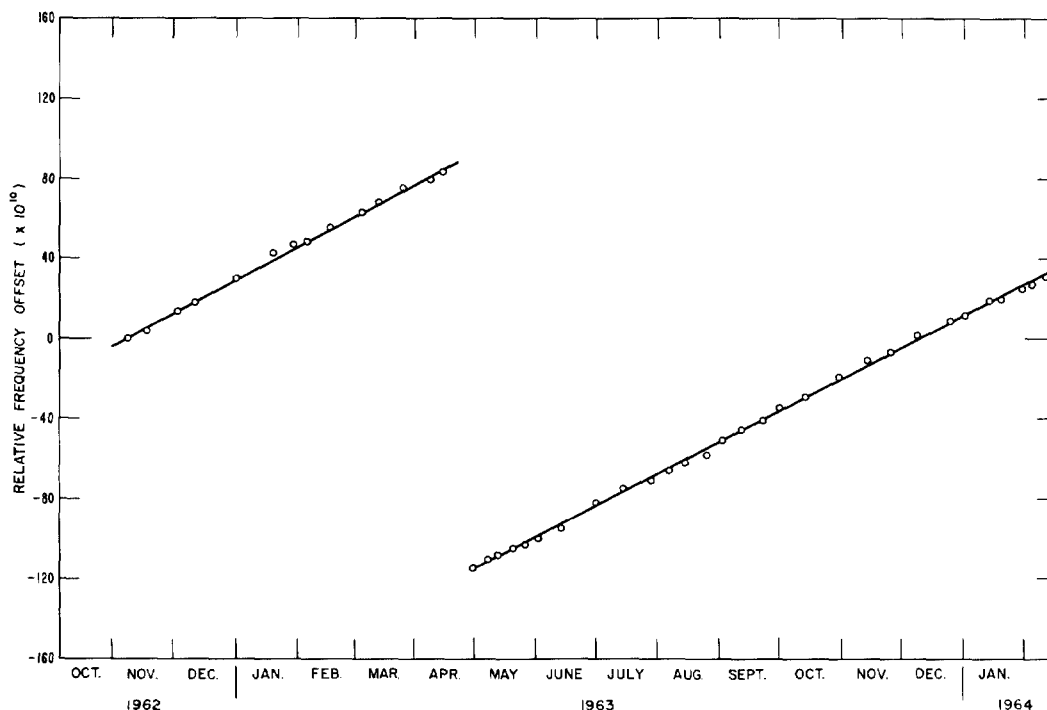

Fig. 1. Frequency aging of a good quartz crystal oscillator.

Actual Calibration Procedure of a Clock System

The fundamental equation for atomic timekeeping is [6]

$$\Omega = \frac{d\phi}{dt},\qquad(2)$$

where Ω is the instantaneous frequency of the oscillator as measured by an atomic frequency standard, $d\phi$ is the differential phase change, and dt is an increment of time as generated by this clock system. Since *time* is to be generated by this system, and ϕ and Ω are the directly measured quantities, it is of convenience to assume that $\Omega = \Omega(\phi)$ and to write the solution of (2) in the form,

$$\Delta t = \int_{\phi_1}^{\phi_2} \frac{d\phi}{\Omega(\phi)}.\qquad(3)$$

If one divides the output phase of the oscillator by Ω_0, and defines the apparent or indicated time t_A to be

$$t_A \equiv \frac{\phi}{\Omega_0},\qquad(4)$$

(1), (3), and (4) can be combined to give

$$\Delta t = \int_{t_{A1}}^{t_{A2}} \frac{dt_A}{1 + \alpha t_A + \dot\epsilon(t_A)}.\qquad(5)$$

As it is indicated in Fig. 1, it is possible to maintain the magnitude of the relative frequency offset $|\alpha t_A + \dot\epsilon|$ within fixed bounds of 10^{-8}. Expanding (5) to first order in this relative frequency offset yields

$$\Delta t \simeq \Delta t_A - \int_{t_A}^{t_{A2}} [\alpha t_A + \dot\epsilon(t_A)]\, dt_A.\qquad(6)$$

Equation (6) should then be valid to about one part in 10^{16}.

Normally the frequency of the oscillator is measured over some period of time (usually a few minutes) at regular intervals (usually a few days). At this point, it is desirable to restrict the discussion to the case where the calibration is periodic (i.e., period T, determined by t_A) and then generalize to other situations later. One period of the calibration is as follows:

t_A start of calibration interval
$t_A + \frac{1}{2}(T-\tau)$ start of frequency measurement $(\tau < T)$
$t_A + \frac{1}{2}(T+\tau)$ end of frequency measurement
$t_A + T$ end of calibration interval,

where τ is the frequency measurement interval. If $\dot\epsilon$ were constant in time, the frequency measured during the interval $t_A + \frac{1}{2}(T-\tau)$ to $t_A + \frac{1}{2}(T+\tau)$ would be just the average frequency during the complete measurement interval T since the oscillator would have an exactly linear drift in frequency. Also, if $\dot\epsilon$ were constant, (6) could be written as

$$\Delta t = \Delta t_A - T\left\langle \frac{\delta\Omega}{\Omega_0} \right\rangle_\tau,\qquad(7)$$

where $\langle \delta\Omega/\Omega_0 \rangle_\tau$ is the average relative frequency offset during the interval $t_A + \frac{1}{2}(T-\tau)$ to $t_A + \frac{1}{2}(T+\tau)$.

Even though, in general, $\dot\epsilon$ is not constant, ϵ and $\dot\epsilon$ are not knowable, and thus one is usually reduced to using (7) anyway. The problem, then, is to determine how much error is introduced by using (7).

The time error δt accumulated over an interval T committed by using (7) can be expressed in the form,

$$\delta t = \int_{t_A}^{t_A+T} [\alpha t_A + \dot\epsilon]\, dt_A - T\left\langle \frac{\delta\Omega}{\Omega_0} \right\rangle,\qquad(8)$$

where the quantity $\langle \delta\Omega/\Omega_0 \rangle_\tau$ is given by

$$\left\langle \frac{\delta\Omega}{\Omega_0} \right\rangle_\tau = \frac{1}{\tau} \int_{t_A+1/2(T-\tau)}^{t_A+1/2(T+\tau)} [\alpha t_A + \dot\epsilon]\, dt_A.\qquad(9)$$

Equations (8) and (9) can be combined to give

$$\delta t = \epsilon(t_A + T) - \epsilon(t_A)$$
$$- \frac{T}{\tau}\left[\epsilon\left(t_A + \frac{T+\tau}{2}\right) - \epsilon\left(t_A + \frac{T-\tau}{2}\right) \right].\qquad(10)$$

It is this equation which relates the random phase fluctuations with the corresponding errors in the time determination.

Meaningful Quantities

It is again of value to further restrict the discussion to a particular situation and generalize at a later point. In particular, let $T = 3\tau$, then (10) becomes

$$\delta t = \epsilon(t_A + 3\tau) - \epsilon(t_A) - 3[\epsilon(t_A + 2\tau) - \epsilon(t_A + \tau)].\qquad(11)$$

It is now possible to define the discrete variable ϵ_n by the relation

$$\epsilon_n \equiv \epsilon(t_A + n\tau),\qquad n = 0, 1, 2, \cdots$$

and rewrite (11) in the simpler form (see Table I)

$$\delta t \equiv \Delta^3\epsilon_n,\qquad(12)$$

where $\Delta^3\epsilon_n$ is the third finite difference of the discrete variable ϵ_n.

TABLE I

FINITE DIFFERENCES
$\epsilon_n \equiv \epsilon(t_A + n\tau)$

Discrete variable		Definition
ϵ_n	ϵ_n	$\epsilon(t_A + n\tau)$
$\Delta\epsilon_n$	$\epsilon_{n-1} - \epsilon_n$	$\epsilon(t_A + \tau) - \epsilon(t_A)$
$\Delta^2\epsilon_n$	$\Delta\epsilon_{n+1} - \Delta\epsilon_n$	$\epsilon(t_A + 2\tau) - 2\epsilon(t_A + \tau) + \epsilon(t_A)$
$\Delta^3\epsilon_n$	$\Delta^2\epsilon_{n+1} - \Delta^2\epsilon_n$	$\epsilon(t_A + 3\tau) - 3[\epsilon(t_A + 2\tau) - \epsilon(t_A + \tau)] - \epsilon(t_A)$
$\Delta^4\epsilon_n$	$\Delta^3\epsilon_{n+1} - \Delta^3\epsilon_n$	$\epsilon(t_A + 4\tau) - 4[\epsilon(t_A + 3\tau) + \epsilon(t_A + \tau)]$ $+ 6\epsilon(t_A + 2\tau) + \epsilon(t_A)$

Similarly, (2) may be integrated directly using (1) to obtain

$$\phi(t_A) = \Omega_0\left[t_A + \frac{\alpha}{2} t_A{}^2 + \epsilon(t_A) \right] + \phi(0). \qquad (13)$$

Thus, by defining another discrete variable ϕ_n, the third difference of (13) yields the relation

$$\Delta^3\phi_n = \Omega_0\Delta^3\epsilon_n, \qquad (14)$$

or equivalently,

$$\delta t = \left(\frac{1}{\Omega_0}\right)\Delta^3\phi_n. \qquad (15)$$

It is now possible to set up a table of meaningful quantities for time measurement (see Table II).

If three oscillators are used, it is possible to independently measure the three quantities σ_{12}, σ_{13}, and σ_{23}. Thus there exist three independent equations:

$$\left.\begin{array}{l} \sigma_{12}{}^2 = \sigma_1{}^2 + \sigma_2{}^2 \\ \sigma_{13}{}^2 = \sigma_1{}^2 + \sigma_3{}^2 \\ \sigma_{23}{}^2 = \sigma_2{}^2 + \sigma_3{}^2 \end{array}\right\}. \qquad (18)$$

While the three equations,

$$\delta t_{ij} = \Delta^3\epsilon_n{}^{(i)} - \Delta^3\epsilon_n{}^{(j)} \qquad \begin{cases} i, j = 1, 2, 3 \\ i < j \end{cases},$$

are not linearly independent, the standard deviations $\sigma_{ij}{}^2$ given by (18), in fact, form linearly independent equations (subject only to certain conditions analogous

TABLE II
MEANINGFUL QUANTITIES

Quantity		Discussion
Phase	ϕ_n	Must exist for all times of interest and be measurable.
First difference of the phase	$\Delta\phi_n$	Proportional to the average frequency in the interval τ. Must exist for all time.
Second difference of the phase	$\Delta^2\phi_n$	Related to the drift of the oscillator frequency. Must exist for all time.
Variance of the second difference	$\langle[\Delta^2\phi_n - \langle\Delta^2\phi_n\rangle]^2\rangle$	(It is possible to construct a time scale even if this does not exist. However, experiment indicates that it is probably finite.)
Third difference of the phase	$\Delta^3\phi_n$	Proportional to the clock error in the time interval $T = 3\tau$. Must exist if clock is to be of value.
Mean square third difference	$\langle(\Delta^3\phi_n)\rangle^2$	Proportional to the precision of time interval measurements. Must exist if clock is to be of value.

Where all averages are defined by the relation:

$$\langle f(t)\rangle = \mathop{\mathrm{Lim}}_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} f(t)dt.$$

Experimental Determination of Phase Fluctuations

If one measures the phase difference between two oscillators, (15) applies to both, and hence the difference phase $\theta_n = \phi_n{}^{(1)} - \phi_n{}^{(2)}$ is related to the difference time $\delta t_{12} = \delta t_1 - \delta t_2$ by the relations

$$\delta t_{12} = \frac{1}{\Omega_0}\Delta^3\theta_n = \Delta^3\epsilon_n{}^{(1)} - \Delta^3\epsilon_n{}^{(2)}. \qquad (16)$$

Provided that the cross-correlation coefficient $\langle(\Delta^3\epsilon_n{}^{(1)}) (\Delta^3\epsilon_n{}^{(2)})\rangle$ is zero (i.e., the $\epsilon_n{}^{(i)}$ are noncorrelated), the variance of δt_{12} becomes

$$\sigma_{12}{}^2 \equiv \langle(\delta t_{12})^2\rangle = \langle(\Delta^3\epsilon_n{}^{(1)})^2\rangle + \langle(\Delta^3\epsilon_n{}^{(2)})^2\rangle, \qquad (17)$$

since it is assumed that $\langle\Delta^3\epsilon_n\rangle = 0$. The assumption that the cross-correlation coefficients vanish is equivalent to postulating an absence of linear coupling between the oscillators either electrically or through their environment. It is of value to develop a scheme which is capable of classifying individual oscillators rather than treating ensembles of assumed identical members. Thus, this development is restricted to time averages of individual oscillators rather than ensemble averages.

to the triangle inequalities). Thus, the systems of (18) are solvable for the three quantities $\sigma_i{}^2 = \langle(\delta t_i)^2\rangle$. It is thus possible to estimate the statistical behavior of each individual oscillator.

Apparatus

A phase meter was used similar to one described in Cutler and Searle [5]. The basic system was aligned with an electrical-to-mechanical angle tolerance of about ± 0.25 percent of one complete cycle, and since the phase meter is operated at 10 Mc/s, this implies the possibility of measuring the time difference to ± 0.25 nanoseconds. The output shaft, in turn, drives a digital encoder with one hundred counts per revolution and a total accumulation (before starting over) of one million. Thus the digital information is accurate to within one ns and accumulates up to one ms. Since measurements are made relatively often, it is easy for a computer to spot when the digital encoder has passed one ms on the data, and this restriction on the data format in no way hampers the total length of the data handled.

Since the oscillator is assumed to have a linear drift in frequency with time, the variance (about the mean)

of the second difference of the phase should depend only on the $\epsilon_n^{(i)}$. For the evaluation of clocks, it is the mean square time error which is important and, thus, the mean square (rather than variance) of the third difference of the phase is the quantity of importance. Calculations of these quantities from the phase θ_n were accomplished on a digital computer. Normally the phase difference is printed every hour for at least 200 hours and the computer program computes the $\Delta^2\theta_n$ and $\Delta^3\theta_n$ for τ equal to 2, 4, 6, 8, etc., hours. Thus it is possible to plot the square root of the variance of the second difference and the root mean square third difference of the phase as a function of the variable τ. A typical plot of these two quantities appears in Fig. 2.
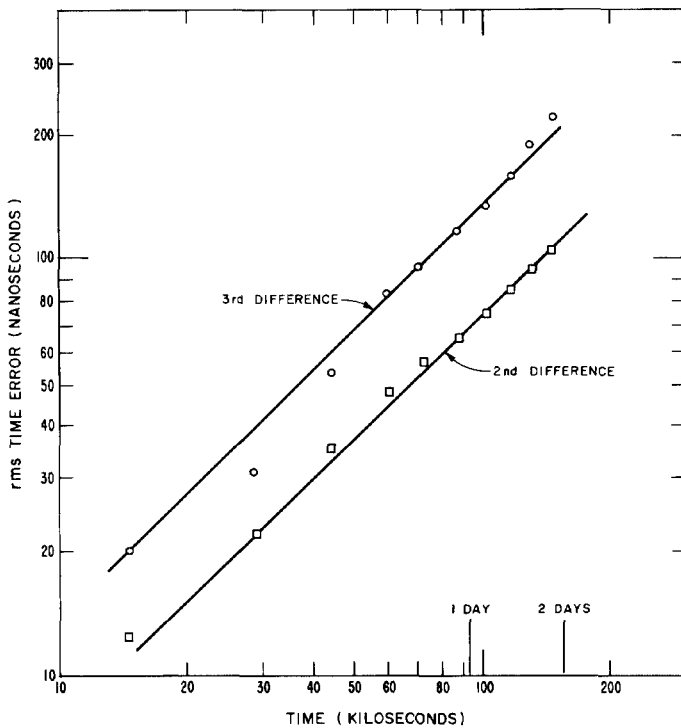

Fig. 2. Variance of second and third difference as a function of τ.

Since many revolutions of the phase meter are normally encountered between data points, the combined effects of nonperfect electrical-to-mechanical phase and rounding errors of the digital encoder can be combined into one stationary, statistical quantity γ, that can be assumed to have delta-function auto-correlation. Let χ represent the rounding errors of the digital encoder defined by the difference between the encoded number and the actual angular position of the shaft. Then these rounding errors form a rectangular distribution from -0.5 ns to $+0.5$ ns, and, thus, contribute an amount

$$\frac{\int_{-1/2}^{1/2} \chi^2 \, d\chi}{\int_{-1/2}^{1/2} d\chi} = 0.08 \text{ ns}^2$$

to $\langle\gamma^2\rangle$. The nonperfect electrical-to-mechanical phase conversion is sinusoidal in nature and, thus, contributes an amount equal to

$$\tfrac{1}{2}(0.25)^2 \simeq 0.03 \text{ ns}^2.$$

The final value for $\langle\gamma^2\rangle$ should then be about 0.11 ns^2.

This was checked by taking two very good oscillators operating at a rather large difference frequency ($\sim5\times10^{-9}$), and printing the phase difference every 10 seconds for several minutes. Since frequency fluctuations of the oscillators are small compared to 1×10^{-10}, the resultant scatter can be attributed to the measuring system. The results of this experiment gave the value 0.19 ns^2 for $\langle\gamma^2\rangle$.

Since γ is assumed to have a delta-function auto-correlation, reference to Table I shows that $\langle(\Delta^3\gamma)^2\rangle = 20\langle\gamma^2\rangle$, and thus (18) may be more precisely written,

$$\left. \begin{aligned} \sigma_{12}{}^2 &= \sigma_1{}^2 + \sigma_2{}^2 + 20\langle\gamma^2\rangle \\ \sigma_{13} &= \sigma_1{}^2 + \sigma_3{}^2 + 20\langle\gamma^2\rangle \\ \sigma_{23} &= \sigma_2{}^2 + \sigma_3{}^2 + 20\langle\gamma^2\rangle \end{aligned} \right\}. \tag{18'}$$

For the best oscillators tested, $\sigma_{ij}{}^2$ became about $2\langle(\Delta^3\gamma)^2\rangle$ for $\tau = 10^3$ seconds, and, therefore, measurements were limited on the lower end to 20 minutes or 1200 seconds. The longest run made lasted a little over a month or about one thousand hours. Thus the largest value of τ which might have reasonable averaging is about one hundred hours or 3.6×10^5 seconds. This limits the results to about two orders of magnitude variation on τ.

While it is possible to build appropriate frequency multipliers and mixers to improve the resolution of the phase meter and reduce the lower limit on τ, it was considered that the longer time intervals are of greater interest because T ($=3\tau$) is normally between one day and one week.

II. THEORETICAL DEVELOPMENT

Introductory Remarks

In the development which follows, certain basic assumptions are made. It is assumed that the coefficients Ω_0 and α of (1) may be so chosen that the average value of $\epsilon(t)$ is zero; i.e.,

$$\langle\epsilon(t)\rangle = 0.$$

It is also assumed that a translation in the time axis, $t_A \to t_A + \zeta$, (stationarity) causes no change in the value of the auto-covariance function,

$$\begin{aligned} R_\epsilon(\tau) &= \langle\epsilon(t)\cdot\epsilon(t + \tau)\rangle \\ &\equiv \langle\epsilon(t + \xi)\cdot\epsilon(t + \xi + \tau)\rangle. \end{aligned} \tag{19}$$

The only justification of this assumption lies in the fact that the results of the analysis agree well with experiment and the results of others. In the development which follows, one *cannot* assume that

$$R_\epsilon(0) = \langle[\epsilon(t)]^2\rangle$$

exists (i.e., is finite) and, hence, Wiener [7] cannot guarantee that (19) is valid. While it may be that $R_\epsilon(\tau)$ does not exist, quantities such as

$$U(\tau) = 2[R_\epsilon(0) - R_\epsilon(\tau)] \qquad (20)$$

may exist and be meaningful if limits are approached properly. It is, thus, assumed that relations such as (20) have meaning and may be handled by conventional means.

Development of Experimental Results

All oscillator pairs tested, which exhibited a stable drift rate as indicated in Fig. 1 and did not have obvious diurnal fluctuations in frequency, showed a definite, very nearly linear dependence on τ for both the standard deviation of the second difference and the root mean square third difference of the phase. It was observed that if an oscillator were disturbed accidentally during a measurement, the plot would have a more nearly $\tau^{3/2}$ dependence. This is probably because the assumption of a negligible quadratic dependence of frequency with time is not valid when the oscillator is disturbed. All oscillators tested, therefore, were shock mounted and all load changes and physical conditions were changed as little as possible.

A least square fit of all reliable data to an equation of the form

$$\sqrt{\langle (\Delta^m \epsilon_n)^2 \rangle} = \sqrt{k_m} \tau^\mu \qquad m = 2, 3, 4 \qquad (21)$$

gave a value of 1.09 for the average of the μ's. The values of μ ranged from about 0.90 to 1.15 (and even to 1.5 when the oscillators were disturbed during the measurement). It is of interest to postulate that, for an "ideal," undisturbed quartz crystal oscillator, the value of μ is exactly the integer one, and to investigate the consequences of this assumption.

Because certain difficulties arise at the value $\mu = 1$, it is essential to calculate with a general μ and then pass to the limits $\mu \rightarrow 1^{(-)}$ for the quantities of interest. This is equivalent to considering a sequence of processes which approach, as a limit, the case of the "ideal" crystal oscillator. Thus (21) may be rewritten for $m = 2$ in the form

$$\langle (\Delta^2 \epsilon_n)^2 \rangle = k_2 |\tau|^{2\mu}. \qquad (22)$$

Using Table I, (22) may then be rewritten as

$$6\langle [\epsilon(t_A)]^2 \rangle - 8\langle \epsilon(t_A) \cdot \epsilon(t_A + \tau) \rangle + 2\langle \epsilon(t_A) \cdot \epsilon(t_A + 2\tau) \rangle = k_2 |\tau|^{2\mu}$$

or, equivalently,

$$6R_\epsilon(0) - 8R_\epsilon(\tau) + 2R_\epsilon(2\tau) = k_2 |\tau|^{2\mu}. \qquad (23)$$

As mentioned above, the function $U(\tau)$ is defined by the relation

$$U(\tau) \equiv \langle [\epsilon(t_A) - \epsilon(t_A + \tau)]^2 \rangle$$
$$= 2[R_\epsilon(0) - R_\epsilon(\tau)] \qquad (24)$$

and is assumed to exist. Equations (23) and (24) may be combined to give

$$4U(\tau) - U(2\tau) = k_2 |\tau|^{2\mu}. \qquad (25)$$

If a trial solution of the form

$$U(\tau) = A(\beta) \cdot |\tau|^\beta$$

is used, one obtains

$$A(\beta) \cdot |\tau|^\beta [4 - 2^\beta] = k_2 |\tau|^{2\mu}$$

from which one concludes that

$$\beta = 2\mu \leq 2,$$
$$(4 - 2^\beta) A(\beta) = k_2,$$

and

$$U(\tau) = \frac{2}{4 - 2^{2\mu}} |\tau|^{2\mu}, \qquad (26)$$

where $\mu \leq 1$ since $U(\tau)$ and k_2 are non-negative. Equation (21) may, thus, be satisfied if

$$\underset{\mu \rightarrow 1^{(-)}}{\text{Lim}} (4 - 2^{2\mu}) A(2\mu) = k_2, \qquad (27)$$

which implies that $A(2)$ is infinite. It is for this reason that the limiting process must be employed.

It is not necessary, however, to assume a particular form for $A(2\mu)$ because (27) is sufficient for the purposes of this development.

It is now possible to determine the mean square third difference of ϵ_n; i.e., from Table I,

$$\langle (\Delta^3 \epsilon_n)^2 \rangle = 20 \langle [\epsilon(t_A)]^2 \rangle - 30 \langle [\epsilon(t_A) \cdot \epsilon(t_A + \tau)] \rangle$$
$$+ 12 \langle [\epsilon(t_A) \cdot \epsilon(t_A + 2\tau)] \rangle - 2 \langle [\epsilon(t_A) \cdot \epsilon(t_A + 3\tau)] \rangle, \qquad (28)$$

where use has again been made of (19). Equation (28) may equivalently be written in the form

$$\langle (\Delta^3 \epsilon_n)^2 \rangle = 15 U(\tau) - 6 U(2\tau) + U(3\tau), \qquad (29)$$

which may be combined with (26) to yield

$$\langle (\Delta^3 \epsilon_n)^2 \rangle = \frac{k_2 |\tau|^{2\mu}}{(4 - 2^{2\mu})} [15 + 6(2^{2\mu}) + 3^{2\mu}]. \qquad (30)$$

If one now passes to the limit $\mu \rightarrow 1^{(-)}$, (30) becomes

$$\langle (\Delta^3 \epsilon_n)^2 \rangle = \frac{k_2 \tau^2 (24 \ln 2 - 9 \ln 3)}{4 \ln 2}. \qquad (31)$$

Thus, a quadratic dependence of the variance of the second difference of the phase implies a quadratic dependence of the mean square third difference, and the ratio

$$\sqrt{\frac{\langle (\Delta^3 \epsilon_n)^2 \rangle}{\langle (\Delta^2 \epsilon_n)^2 \rangle}} = 1.5601 \cdots \qquad (32)$$

is independent of τ. The average ratio of the points plotted in Fig. 2 is 1.65. Values ranging from 1.4 to 1.7 were observed for various runs on different oscillator pairs. Average value for all reliable data taken is 1.52.
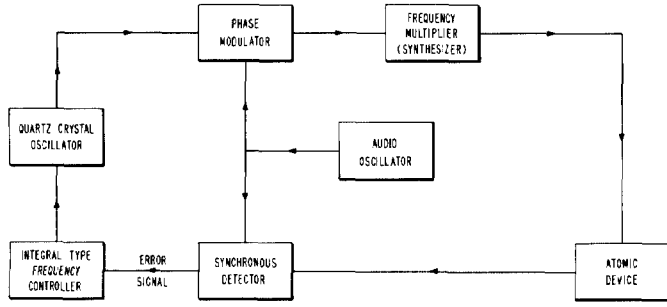
Fig. 3. Typical control loop for an atomically controlled oscillator.

Generalization of the Time Error Problem

The average frequency of an oscillator over an interval of time is just the total elapsed phase in the interval divided by the time interval. Since errors of the frequency standard are not presently being considered, the calibration interval of Section I gives rise to an error time δt which could be expressed as a sum

$$\delta t = \frac{1}{\Omega_0} \sum_{n=0}^{m} a_n \phi(t_A + b_n \tau), \qquad (33)$$

where $m+1$ is the total number of terms and the set $\{a_n, b_n\}$ are chosen to fit the particular calibration procedure. Indeed, any calibration procedure must give rise to an error time which is expressible in the form of (33).

There are, however, certain restrictions on the $\{a_n, b_n\}$ which are of importance. First, it is a matter of convenience to require that $b_l > b_n$ for $l > n$. Also, if the oscillator were absolutely perfect, and $\epsilon(t_A)$ were identically zero, one should logically require that the error time δt be identically zero, independent of t_A, the drift rate α, and the basic time interval τ. That is, from (13)

$$\delta t = \sum_{n=0}^{m} a_n \cdot (t_A + b_n \tau) + \frac{\alpha}{2} \sum_{n=0}^{m} a_n \cdot (t_A + b_n \tau)^2 \equiv 0 \quad (34)$$

for all t_A, α, and τ. One is thus led to the three conditions:

$$\sum_{n=0}^{m} a_n = 0, \qquad (35)$$

$$\sum_{n=0}^{m} a_n b_n = 0, \qquad (36)$$

$$\sum_{n=0}^{m} a_n b_n^2 = 0. \qquad (37)$$

It is of interest to form the quantity

$$\left(\sum_{n=0}^{m} a_n b_n^2 \right)\left(\sum_{n=0}^{m} a_l \right) = \sum_{n=0}^{m} a_n^2 b_n^2 + \sum_{n<l} a_n a_l b_n^2$$

$$+ \sum_{l<n}^{m} a_n a_l b_n^2, \qquad (38)$$

and it is now possible to interchange the subscripts n and l in the last term of (38) and write the equation in the form

$$\sum_{n=0}^{m} a_n^2 b_n^2 + \sum_{n<l}^{m} a_n a_l (b_n^2 + b_l^2) = 0. \qquad (39)$$

Subtracting the square of (36),

$$\left(\sum_{n=0}^{m} a_n b_n \right)\left(\sum_{l=0}^{m} a_l b_l \right) = \sum_{n=0}^{m} a_n^2 b_n^2 + 2 \sum_{n<l}^{m} a_n a_l b_n b_l,$$

one obtains

$$\sum_{n<l}^{m} a_n a_l (b_l - b_n)^2 = 0 \qquad (40)$$

as another condition on the $\{a_n, b_n\}$. Equation (40) is, of course, not independent of (35)–(37).

For the actual situation where $\epsilon(t_A)$ is not identically zero, one is reduced, as before [see (7)], to using conditions (35)–(37) since $\epsilon(t_A)$ is not knowable. The time error then becomes

$$\delta t = \sum_{n=0}^{m} a_n \cdot \epsilon(t_A + b_n \tau) \qquad (41)$$

where use has been made of (13), (33), and the restrictions (35)–(37). Note that (41) is the generalization of (10). The square of (41) can be written in the form

$$(\delta t)^2 = \sum_{n=0}^{m} a_n^2 \epsilon^2(t_A + b_n \tau)$$

$$+ 2 \sum_{n<l}^{m} a_n a_l [\epsilon(t_A + b_n \tau) \cdot \epsilon(t_A + b_l \tau)], \qquad (42)$$

which may be averaged over t_A to yield

$$\langle (\delta t)^2 \rangle = \sum_{n=0}^{m} a_n^2 \langle (\epsilon(t_A))^2 \rangle$$

$$+ 2 \sum_{n<l}^{m} a_n a_l \langle [\epsilon(t_A) \cdot \epsilon(t_A + (b_l - b_n)\tau)] \rangle, \qquad (43)$$

where use has again been made of (19).

The square of (35) may be written as

$$\sum_{n=0}^{m} a_n^2 + 2 \sum_{n<l}^{m} a_n a_l = 0,$$

and (43) then becomes

$$\langle (\delta t)^2 \rangle = - \sum_{n<l}^{m} a_n a_l \{ \langle \epsilon^2(t_A) \{ 2 \langle \epsilon^2(t_A) \rangle$$

$$- 2 \langle [\epsilon(t_A) \cdot \epsilon(t_A + (b_l - b_n)\tau)] \rangle \}. \qquad (44)$$

Combining (44) with (21) one obtains

$$\langle (\delta t)^2 \rangle = - \sum_{n<l}^{m} a_n a_l U((b_l - b_n)\tau). \qquad (45)$$

Substitution of (26) in (45) yields

$$\langle(\delta t)^2\rangle = -k_2\big|\tau\big|^{2\mu}\frac{\sum\limits_{n<l}^{m} a_n a_l(b_l - b_n)^{2\mu}}{4 - 2^{2\mu}}, \qquad (46)$$

which is indeterminate of the form $(0/0)$ for $\mu=1$ because of (40). Making use of L'Hospital's rule, the limit of this expression as $\mu \to 1^{(-)}$ is

$$\langle(\delta t)^2\rangle = \frac{k_2\tau^2}{4\ln 2}\sum_{n<l}^{m} a_n a_l(b_l - b_n)^2 \ln(b_l - b_n). \quad (47)$$

Equation (47) is thus the generalization of (31). Since many terms appear in the summation in (47), a computer program was written for evaluation with particular sets of $\{a_n, b_n\}$.

Comparison With Others

If one were to assume that (20) could be solved for $R_\epsilon(\tau)$, one would obtain

$$R_\epsilon(\tau) = R_\epsilon(0) - \tfrac{1}{2}U(\tau) \qquad (48)$$

which expresses the auto-covariance function in terms of $U(\tau)$. If one assumes still further that the Wiener-Khinchin theorem applies to (48), the power spectral density of $\epsilon(t)$ is then the Fourier transform of $R_\epsilon(\tau)$; i.e.,

$$S^2(\omega) = \text{F.T.}\left[R_\epsilon(0) - \frac{k_2\big|\tau\big|^{2\mu}}{2(4 - 2^{2\mu})}\right].$$

Fortunately, Fourier transforms of functions like $\big|\tau\big|^{2\mu}$ have been worked out in Lighthill [8]. The result is

$$S_\epsilon(\omega) = \frac{R_\epsilon(0)}{2\pi}\delta(\omega)$$
$$-\frac{k_2\left[\cos\dfrac{\pi(2\mu + 1)}{2}\right][(2\mu)!]\big|\omega\big|^{-2\mu-1}}{2\pi(4 - 2^{2\mu})}. \quad (49)$$

The factor of $(1/2\pi)$ occurs because $S_\epsilon(\omega)$ is assumed to be a density relative to an angular frequency ω, rather than a cycle frequency $(f=\omega/2\pi)$ as is used in [8]. The first term on the right of (49) indicates an infinite density of power at zero frequency, i.e., a nonzero average. The zero frequency components are not measurable experimentally, and hence this term will be dropped as not being significant to these discussions. The second term on the right of (49) is indeterminate for $\mu=1$. As before, the limit as $\mu \to 1^{(-)}$ may be obtained, yielding

$$S_\epsilon(\omega) = \frac{k_2}{8\ln 2}\big|\omega\big|^{-3} \qquad (50)$$

for the final result.

The assumptions needed to arrive at (50) are not wholly satisfying, and it is of value to show that (50)

implies that $\langle(\Delta^2\epsilon_n)^2\rangle$ is indeed given by (21) for $m=2$. Only one additional assumption is needed, the Wiener-Khinchin theorem. Since $R_\epsilon(\tau)$ and $S_\epsilon(\omega)$ are real quantities, this theorem may be written in the form

$$R_\epsilon(\tau) = 2\int_0^\infty S_\epsilon(\omega)\cos\omega\tau\, d\omega. \qquad (51)$$

From Table I, one may obtain (after squaring and averaging) the expression

$$\langle(\Delta^2\epsilon_n)^2\rangle = 6R_\epsilon(0) - 8R_\epsilon(\tau) + 2R_\epsilon(2\tau). \qquad (52)$$

Substitution of (51) in (52) yields

$$\langle(\Delta^2\epsilon_n)^2\rangle = 4\int_0^\infty S_\epsilon(\omega)[3 - 4\cos(\omega\tau) + \cos(2\omega\tau)]\, d\omega. \quad (53)$$

Using (50) for $S_\epsilon(\omega)$ is a lengthy but straightforward process to evaluate the integral in (53). The result is, in fact,

$$\langle(\Delta^2\epsilon_n)^2\rangle = k_2\tau^2,$$

in agreement with (21).

If $g_\epsilon(\omega)$ is the Fourier transform of $\epsilon(t)$, it is shown, for example, in Lighthill ([8] p. 20), that

$$g_{\dot\epsilon}(\omega) = i\omega g_\epsilon(\omega)$$

is the Fourier transform of $\dot\epsilon(t)$. Thus, the power spectral density of $\dot\epsilon(t)$ is related to the power spectral density of $\epsilon(t)$ by the familiar relation

$$S_{\dot\epsilon}(\omega) = \omega^2 S_\epsilon(\omega).$$

Thus, the power spectral density of the *frequency* fluctuations of an "ideal" oscillator may be given as

$$S_{\dot\epsilon}(\omega) = \frac{k_2}{8\ln 2}\big|\omega\big|^{-1},$$

that is, flicker noise frequency modulation. The existence of this type of noise modulating the frequency of good quartz crystal oscillators has been reported by several others [2]–[5], [9].

Comments on the "Ideal" Oscillator

It has thus been shown that the assumptions of stationarity and "ideal" behavior form a basis for a mathematical model of a quartz crystal oscillator which is in quite good agreement with several experiments and experimenters. On the basis of this model, it is now possible to predict the behavior of systems employing "nearly ideal" oscillators with the hope of committing no great errors. There are compelling reasons to believe that $U(\tau)$ actually exists (see Section IV) for real oscillators in spite of (26). Such conditions require that the $\big|\omega\big|^{-1}$ behavior for the power spectral density of the frequency fluctuations cut off at some small, nonzero frequency. From some experiments [2] conducted, however, this cutoff frequency is probably much smaller than one cycle per year. Such small differences from

zero frequency are of essentially academic interest to the manufacturer and user of oscillators. Quantities which may be expressed in the form of (41), however, where the coefficients satisfy conditions (35)-(37), have finite averages even in the limit as flicker noise behavior approaches zero modulation frequency. Such quantities are called cutoff *independent* in contrast to quantities like $U(\tau)$ which will exist only if the flicker noise cuts off at some nonzero frequency.

III. ATOMIC FREQUENCY STANDARDS

Passive Devices

There are in use today two general types of atomic frequency standards: 1) the active device such as a maser whose atoms actually generate a coherent signal whose frequency *is* the standard, and 2) the passive type such as a cesium beam or rubidium gas cell. In the passive type, a microwave signal irradiates the atoms and some means is employed to detect any change in the atom's energy state. This paper is restricted to the passive type of frequency standard. Some experiments are in progress, however, to determine the statistical behavior of a maser type oscillator.

It is first of value to discuss in what way a "standard" can have fluctuations or errors. Consider the cesium beam. *Ideally* the standard would be the exact frequency of the photons emitted or absorbed at zero magnetic field in the $(F=4, m_f=0)\leftrightarrow(F=3, m_f=0)$ transition of cesium[133] in the ground electronic state for an infinite interaction time. This is, of course, impossible. This means that the standard is at least less than ideal and one is, thus, led to speak, in some sense of the word, about "errors" or even "fluctuations" of the standard.

Figure 3 shows a block diagram of a typical standard of the passive type. An equivalent diagram of this frequency-lock servo is shown in Fig. 4, where $V_1(\omega)$ is the Fourier Transform of the noise generated in the detectors, associated demodulating circuitry, and the frequency multipliers of Fig. 3. $V_2(\omega)$ is an equivalent noise voltage driving the reactance tube in the oscillator to produce the $\epsilon(t_A)$ term in the unlocked oscillator, such that flicker noise FM results. The power spectrum of $V_2(\omega)$, then, is given by

$$C^2 S_{V_2} = \frac{h_s}{|\omega|},$$

(54)

where $h_s/|\omega|$ is the power spectral density of the *frequency* fluctuations of the unlocked oscillator.

It is easiest to treat the servo equations by the use of the variable $\dot{\gamma}$, defined to be the difference between the output frequency of the multiplier and the "ideal" frequency of the atomic transition (the output of the "atomic device," as shown in Fig. 4, is then assumed to be the constant zero). In order to preserve the dimensions of voltage for the addition networks, it is convenient to assume that the output of the subtraction net-

work is $-\beta\dot{\gamma}$ where β has the dimensions of volt-seconds. Thus, the equation governing the operation of the servo can be expressed, in the frequency domain, as

$$\left[V_2(\omega) + \frac{V_1(\omega) - \beta\dot{\gamma}(\omega)}{j\omega\tau_1} \right] C N = \dot{\gamma}(\omega).$$

(55)

This leads to a power spectral density for $\dot{\gamma}$ given by

$$S_{\dot{\gamma}}(\omega) = \frac{(NC)^2 S_{V_1}(\omega) + N^2\tau_1^2 h_s |\omega|}{\omega^2\tau_1^2 + (\beta NC)^2}$$

(56)

where use has been made of (54).

Normally, $\omega\tau_1$ becomes of the order of (βNC) for ω of the order of 10 s^{-1}. Thus, for small ω, (56) becomes

$$S_{\dot{\gamma}}(\omega) \approx \frac{1}{\beta^2} S_{V_1}(\omega)$$

(57)

for $|\omega| < 1$ s^{-1}.

V₁ (ω)   NOISE GENERATED IN DETECTOR
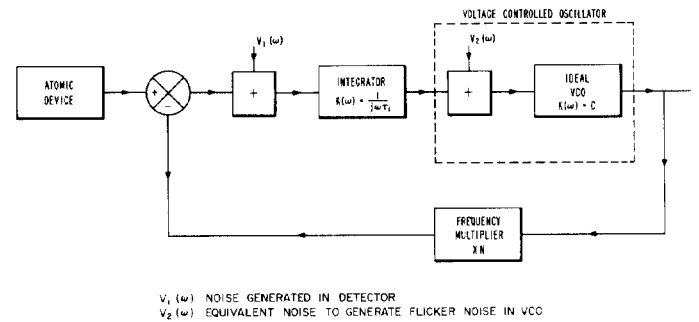V₂ (ω)   EQUIVALENT NOISE TO GENERATE FLICKER NOISE IN VCO

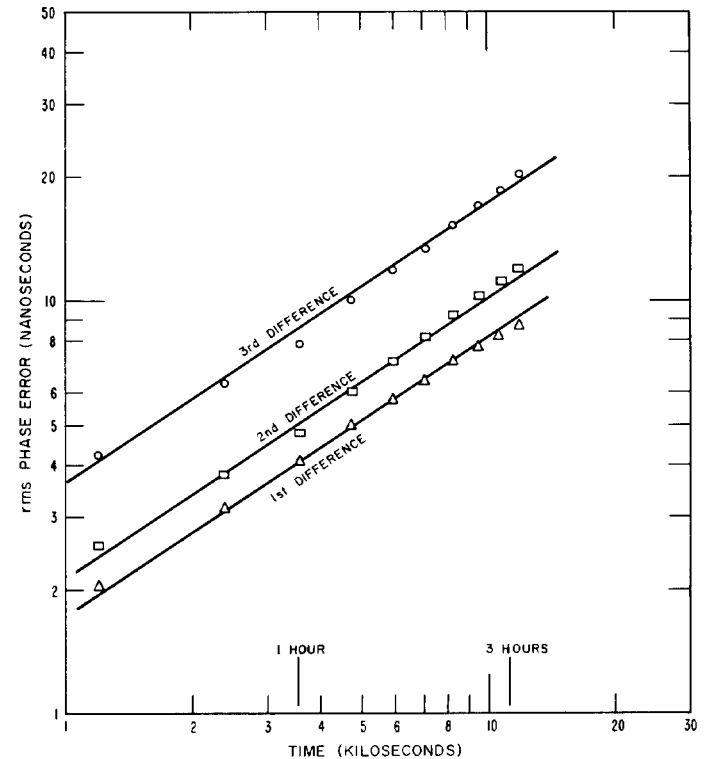Fig. 4. Equivalent servo diagram of passive type frequency standard.

Fig. 5. Variance of differences of the relative phase difference between two atomic standards.

By applying the techniques of Section I to the difference phase of two cesium beams, the curves shown in Fig. 5 were obtained. Through least square fits to the data and comparing the ratios of the variances of differences (see Appendix), the result was obtained that the data fit curves of the form

$$\langle (\Delta^m \gamma_n)^2 \rangle = B_m \mid \tau \mid^\eta$$

for $\eta = 1.34$, with an uncertainty (standard deviation) of about ± 0.04. Again using [8] as before, this leads to the result that

$$S_\gamma(\omega) = \frac{g}{\mid \omega \mid^\mu}, \qquad g \approx B_1 \frac{\sqrt{3}}{4\pi} \Gamma\left(\frac{7}{3}\right) \qquad (58)$$

where $\mu = 0.34 \pm 0.04$. One is, thus, led to the very strange conclusion that the spectral distribution of the detector and multiplier noise varies as $\sim \mid \omega \mid^{-1/3}$. The source of this noise was later traced to a faulty preamplifier. With a proper amplifier used, the spectral distribution appears white as other papers [5], [10], [11] indicate should be the case.

Ideally, for measurements over times large compared to the servo time constant, the error accumulated during one measurement interval should be independent of errors accumulated during nonoverlapping intervals, i.e., mathematically analogous to Brownian Motion [12]. This implies that $S_\gamma(\omega)$ should be constant for $\mid \omega \mid < 1 \text{ s}^{-1}$.

The Composite Clock System

If an oscillator's frequency is measured by an atomic standard, the error in measurement of the frequency is given by

$$\delta f = \frac{1}{2\pi\tau} \left[\gamma(t_A + \tfrac{1}{2}(T + \tau)) - \gamma(t_A + \tfrac{1}{2}(T - \tau)) \right] \qquad (59)$$

for a calibration interval as given in Section I. Also, if f_s is the defined output frequency of the frequency standard, the total error time for the entire calibration interval T, due to the Standard, is given by $T\delta f/f_s$, and if the clocks contribute an additional error (uncorrelated to the standard's error), of δt, the total error δT is given by

$$\delta T = \delta t + \frac{T \delta f}{f_s}. \qquad (60)$$

It is of value to explore the dependence of $\langle (\delta t)^2 \rangle$ on the ratio of T/τ. In particular, let $T = (2n+1)\tau$ for (10). This can be put in the form of (41) with the following assignments:

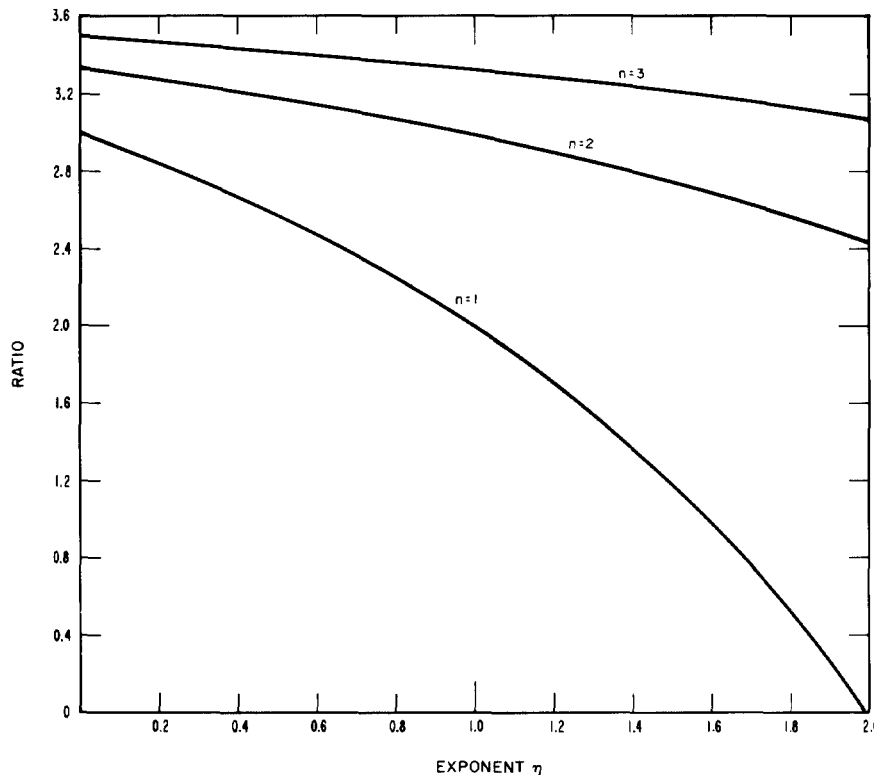| n	a_n	b_n
0	-1	0
1	$2n + 1$	n
2	$-2n - 1$	$n + 1$
3	1	$2n + 1$

Fig. 6. Ratio of variances, $\langle (\Delta^{n+1} \epsilon m)^2 \rangle, / \langle (\Delta^n \epsilon m)^2 \rangle$, as a function of the exponent η.

It is easy to show that these coefficients satisfy (35)–(37).

Substitution of these coefficients into (47) yields

$$\langle (\delta t)^2 \rangle = 2h\tau^2 \{ -2n^2(2n + 1) \ln (n) \\ + 2(n + 1)^2(2n + 1) \ln (n + 1) \\ - (2n + 1)^2 \ln (2n + 1) \}, \qquad (61)$$

where $h = k_2/8 \ln 2$.

Since τ is normally small compared to T, it is reasonable to approximate (61) for large n. That is, the relative time error is approximately given by

$$\frac{\sqrt{\langle (\delta t)^2 \rangle}}{T} \approx \sqrt{2h \ln (n)}, \qquad (62)$$

where the approximations

$$n \gg 1, \ln (n + 1) \approx \ln (n) + \frac{1}{n}$$

have been used. It is apparent from (62) that for a given interval T, the errors accumulated by the clock are not *critically* dependent on the measuring time τ, since $\sqrt{\ln (n)}$ is a very slowly changing quantity.

The mean square error associated with the standard, however, can be written in the form

$$T^2\langle (\delta f)^2 \rangle = \left(\frac{T}{\tau} \right)^2 \langle (\Delta \gamma_n)^2 \rangle, \qquad (63)$$

since (59) may be written as the first difference of γ_n. From (57) it is apparent that the relative time error is then given by

$$\frac{T^2\langle (\delta f)^2 \rangle}{T^2 f_s^2} = \frac{B_1}{(2\pi \tau f_s)^2} \tau^\eta, \qquad (64)$$

where $\eta = 1$ ("white" noise). Or, combining (59), (62), and (64),

$$\frac{\sqrt{\langle (\delta T)^2 \rangle}}{T} \approx \sqrt{2h \ln (n) + \frac{B_1}{(2\pi f_s)^2 \tau}}. \qquad (65)$$

As one should expect, for a given T, the errors get less for larger τ but not rapidly. In the limit of $\tau = T$ [not using the approximate (65)], the errors are those of the standard alone, i.e., $\sqrt{B_1 \tau / 2\pi f_s}$. Also, as $\tau \rightarrow 0$, the clock errors are unbounded.

Compounding Time Errors

Equation (65) represents a reasonable approximation to the time errors of a clock system after one calibration interval. The next question is: how do the errors of many calibrations compound to give a total error $\delta(NT)$ after N calibrations? Returning to (59), one may write

$$\delta(NT) = \sum_{n=1}^{N} \delta T_n, \qquad (66)$$

where δT_n is the time error associated with the nth calibration interval. There have been papers published [13] that assume that the errors of one calibration interval are not correlated to the errors of any other calibration interval. It is now possible to investigate this assumption more precisely.

In particular, the clock errors (not including the frequency standard errors) δt_n compound to give $\delta(Nt)$, given by

$$\delta(Nt) = \sum_{n=1}^{N} \delta t_n, \qquad (67)$$

which can obviously be put in the form of (41). For N equal to any number larger than one, the algebra becomes much too lengthy for actual calculation by hand and it is desirable to make use of a digital computer. Table III shows the results of this calculation for compounding several third-difference-type calibrations. It is interesting to note that, in fact, the total mean square error after N calibrations is very nearly equal to N-times the mean square error after one calibration. This is in quite good agreement with DePrins [13]. Thus, the rms relative eror may be approximated by the relation

$$\sqrt{\frac{\langle (\delta(NT))^2 \rangle}{NT}} \approx \sqrt{\frac{2h \ln (n) + B_1\tau}{N}}, \qquad (68)$$

where h, n, B_1, f_s, and τ have the same meanings as in (65).

TABLE III

THEORETICAL DETERMINATION OF TIME ERROR PROPAGATION

N	$\dfrac{\left\langle \left(\sum\limits_{n=1}^{N} \delta t_n \right)^2 \right\rangle}{\langle (\delta t_n^2) \rangle}$
2	2.054
3	3.109
4	4.165
5	5.220
6	6.276
7	7.332

The conclusions which can now be drawn are that: 1) the total rms time error of this clock system from an "ideal" atomic clock is unbounded as time increases, and 2) the relative rms error time to total elapsed time (68) approaches zero about as fast as $N^{-1/2}$. It should be mentioned here that systematic errors in the atomic standard have not been considered. While this is a very important problem, it has been treated rather thoroughly elsewhere [10], [11], [14]–[16].

IV. MEASURES OF FREQUENCY STABILITY

General Restrictions

It is of value to consider the problem of establishing a stability measure in a very general sense. Consider some functional of the phase

$$\chi = \chi(\phi(t)),$$

from which the stability measure Ψ is obtained according to the relation

$$\Psi^2 = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |\chi|^2 \, dt,$$

provided this limit exists.

In practice it is not possible to pass to the limit $T \to \infty$, and, thus, one measures for some fixed time, T; i.e.,

$$\Psi_T^2 = \frac{1}{T} \int_{-T/2}^{T/2} |\chi|^2 \, dt.$$

Under favorable conditions, Ψ_T may be a reasonable approximation to Ψ. Unfortunately, this may not always be the case.

The frequency emitted by any physically realizable device must be bounded by some upper bound, say B. The following inequalities must, then, be valid:

$$|\Omega(t)| \le B, \quad \text{for some } B > 0;$$

$$\lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} (\Omega(t))^2 \, dt \le B^2.$$

With $S_\phi(\omega)$ being the power spectral density of $\Omega(t)$, it follows from the definition of power spectra that

$$\lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} (\Omega(t))^2 \, dt = 2 \int_0^\infty S_{\dot\phi}(\omega) \, d\omega$$

for real $\Omega(t)$, and thus

$$2 \int_0^\infty S_{\dot\phi}(\omega) \, d\omega \le B^2. \tag{69}$$

If $S_\phi(\omega)$ has a flicker noise spectrum for small ω, it is apparent that this $1/|\omega|$ type of noise cannot persist to absolute zero frequency or the inequality (69) would be violated. It is, thus, reasonable to postulate the existence of a lower cutoff frequency ω_L for the flicker noise modulation.

It is apparent from the preceding considerations that stability measures may exist for which Ψ_T begins to approach Ψ only after T is several times larger than $1/\omega_L$. From some of the experiments on crystal oscillators [2], this may require T to exceed several years in duration. This is quite inconvenient from a manufacturing or experimental standpoint. The logical conclusion is to consider only those stability measures Ψ which are "cutoff independent," that is, those measures of stability which would be valid even in the limit $\omega_L \to 0^+$.

Finite Differences

It was shown in Section II that an expression of the form

$$\delta t = \frac{1}{\Omega_0} \sum_{n=0}^{m} a_n \phi(t + b_n \tau) \tag{70}$$

will have a finite variance if the $\{a_n, b_n\}$ satisfy conditions (35)–(37). It is easy to show that the first difference of the phase (i.e., frequency) cannot be put in the form of (70) with the coefficients satisfying conditions (35)–(37). Indeed, the limit

$$\lim_{\mu \to 1(-)} U(\tau) = \lim_{\mu \to 1(-)} \frac{k_2 |\tau|^{2\mu}}{4 - 2^{2\mu}}$$

does not exist, and, hence, $U(\tau)$ is not a good measure of stability.

The variances of the second and third finite differences, however, are convergent. It is of interest to note that the first line of Table III may be expressed in the form

$$\frac{\langle |\Delta^3 \phi_{n+3} + \Delta^3 \phi_n| \rangle}{\langle (\Delta^3 \phi_n)^2 \rangle} = 2.054,$$

which may be simplified to the form

$$\frac{\langle (\Delta^3 \phi_{n+3})(\Delta^3 \phi_n) \rangle}{\langle (\Delta^3 \phi_n)^2 \rangle} = 0.027.$$

This equation expresses the fact that a third difference has a very small correlation ($\sim$3 percent) to an adjacent, nonoverlapping third difference. By extending this procedure with the other values given in Table III, it is found that the correlation of one third difference with a nonoverlapping third difference becomes small very rapidly as the interval between these differences becomes large. This is sufficient to insure that the variance of a finite sample of third differences will approach the "true" variance (infinite average) in a well-behaved and reasonable fashion as the sample gets larger.

Variance of Frequency Fluctuations for Finite Averaging Times

Even though the variance of the first difference of the phase does not satisfy the condition of being cutoff independent, it is possible (by specifying both the sample time and the total averaging time) to construct a cutoff independent measure of the frequency fluctuations. Instead of Ψ_T, the variance of N adjacent samples of the frequency will be denoted by $\sigma^2(\tau, N)$ where τ is the sample time for each of the N measurements of frequency. The variance is given by the conventional formula

$$\sigma^2(\tau, N) = \frac{1}{N-1} \sum_{i=1}^{N} \left[\frac{\Delta\phi_i}{\tau} - \frac{\phi_{N+1} - \phi_1}{N\tau} \right]^2.$$

If one neglects the drift rate α, which is essentially equivalent to obtaining the standard deviation around a linear drift, one obtains

$$\langle \sigma^2(\tau, N) \rangle = \frac{1}{(N-1)\tau^2} \left\{ N \langle [\epsilon(t+\tau) - \epsilon(t)]^2 \rangle \right.$$
$$\left. - \frac{1}{N} \langle [\epsilon(t+N\tau) - \epsilon(t)]^2 \rangle \right\}. \quad (71)$$

For the case of an "ideal" crystal oscillator, (19), (24), and (26) allow (71) to be simplified (after passing to the limit $\mu \to 1^{(-)}$ to the form

$$\langle \sigma^2(\tau, N) \rangle = \frac{2hN \ln (N)}{N-1}.$$

Thus, as N increases, the expected value of $\sigma^2(\tau, N)$ increases without bound (at least until $N\tau \sim 1/\omega_L$). It is interesting to note that for a given oscillator, $\langle \sigma^2(\tau, N) \rangle$ has a minimum value for $N=2$. Obviously one would have to average several experimental determinations of $\sigma^2(\tau, 2)$ in order to have a reasonable approximation to $\langle \sigma^2(\tau, 2) \rangle$.

While $\sigma(\tau, N)$ is a cutoff independent measure of frequency stability, it has the significant disadvantage of being a function of two variables. Indeed, in order to compare the stability of two oscillators, both the τ's and the N's should have nearly corresponding values.

Delayed Frequency Comparison

In radar work, often the frequency of a signal is compared to the frequency of the same source after it has been delayed in traversing some distance—often a very great distance. One might, thus, be interested in defining a stability measure in an analogous fashion:

$$\Psi^2(\tau, T) = \left\langle \left[\frac{\phi(t+T+\tau) - \phi(t+T)}{\tau} \right.\right.$$
$$\left.\left. - \frac{\phi(t+\tau) - \phi(t)}{\tau} \right]^2 \right\rangle. \quad (72)$$

Again neglecting the drift rate α of the oscillator, (24) and (72) combine to yield

$$\Psi^2(\tau, T) = \frac{1}{\tau^2} [2U(\tau) - U(T-\tau) + 2U(T) - U(T+\tau)]. \quad (73)$$

After substitution of (26) into (73) the equation can be rearranged to give (again passing to the limit, $\mu \to 1^{(-)}$)

$$\Psi^2(\tau, T) = -2h[\rho^2 \ln \rho - (1+\rho)^2 \ln (1+\rho)$$
$$- (\rho - 1)^2 \ln (\rho - 1)],$$

where $\rho \equiv T/\tau$. Although this is a rather complicated expression, it may be simplified with the approximation $\rho \equiv T/\tau \gg 1$. The result is

$$\Psi^2(\tau, T) = 4h \left(2 + \ln \frac{T}{\tau} \right), \quad \text{for } \frac{T}{\tau} \gg 1 \quad (74)$$

for an "ideal" oscillator. It is interesting to note here that even when considering only $1/|\omega|$ type of noise, one cannot pass to the limit $\tau = 0$ for this problem. In the limit as $\tau \to 0^+$, the expression

$$\operatorname*{Lim}_{\tau \to 0^+} \left[\frac{\phi(t+\tau) - \phi(t)}{\tau} \right] \equiv \dot{\phi}(t) \equiv \Omega(t)$$

and thus

$$\operatorname*{Lim}_{\tau \to 0^+} \Psi^2(\tau, T) = \langle [\Omega(t+T) - \Omega(t)]^2 \rangle \to \infty$$

from (74), even though $T \ll 1/\omega_L$. The source of this difficulty is the high-frequency divergence of the flicker noise spectrum. If the system is limited at the high frequency by ω_H, then one should pass to the limit $\tau \to 1/\omega_H$.

Again, $\psi(\tau, T)$ is a function of two variables with all of the associated annoyances. It may, however, be useful in certain applications.

CONCLUSION

The assumptions of stationarity and "ideal" behavior for a quartz crystal oscillator lead to a statistical model which agrees well with many different experiments. One finds, however, that certain quantities are unbounded as averaging times are extended and it is important to consider only those quantities which have reasonable hope of converging toward a good value in reasonable time. Thus, the concepts of cutoff dependent and cutoff independent measures of frequency stability form a natural classification for all possible frequency stability measures.

On the basis of "ideal" behavior, it has been shown that the errors of a clock, run from a quartz crystal oscillator and periodically referenced to an atomic frequency standard, accumulate error at a probable rate proportional to the square root of the number of calibrations. That is, the errors of one calibration interval are essentially uncorrelated to errors of nonoverlapping intervals in spite of the fact that "ideal" behavior is highly correlated for long periods of time.

It has also been shown that the method of finite differences can be a useful method of determining spectral distributions of noise, as well as being a possible measure of frequency stability. By using higher order finite differences, phase fluctuations with even a higher order pole at zero-modulation frequency can similarly be treated. The need for higher than second or third differences, however, has not yet been demonstrated.

It should be noted that the existence of higher-frequency modulation noise of different origin also has significant affect on stability measures. In general, the factors which limit the system to a finite bandpass are sufficient to insure convergence of the stability measures as $\omega \to \infty$. If it is primarily the measuring system which limits the system bandpass, however, the results may be significantly altered by the measuring system itself.

Appendix

Ratio of Variances

Let $\epsilon(t)$ be a real generalized function such that $\langle \epsilon(t) \rangle = 0$ and define the discrete variable ϵ_m by the relation

$$\epsilon_m \equiv \epsilon(t + m\tau). \tag{75}$$

Also, let the auto-covariance function of $\epsilon(t)$ be, as before, independent of a simple time transition. One may now write (see Table I)

$$\langle (\Delta \epsilon_m)^2 \rangle = 2[\langle (\epsilon_n)^2 \rangle - \langle [\epsilon(t + \tau)\epsilon(t)] \rangle] \tag{76}$$

and assume that

$$\langle (\Delta \epsilon_m)^2 \rangle = k_1 \tau^\eta, \tag{77}$$

where k_1 is a constant for a given η. It is also possible to obtain the variance of the second difference:

$$\langle (\Delta^2 \epsilon_m)^2 \rangle = 6\langle (\epsilon_m)^2 \rangle - 8\langle [\epsilon(t + \tau) \cdot \epsilon(t)] \rangle$$
$$+ \langle [\epsilon(t + 2\tau)\epsilon(t)] \rangle. \tag{78}$$

Using (77) and (78), one may obtain

$$\frac{\langle (\Delta^2 \epsilon_m)^2 \rangle}{\langle (\Delta \epsilon_m)^2 \rangle} = \frac{4 k_1 \tau^\eta - k_1 (2\tau)^\eta}{k_1 \tau^\eta} = 4 - (2)^\eta. \tag{79}$$

Since (79) must be non-negative (ϵ is real), the exponent is restricted to the range $\eta \leq 2$.

Similarly, one may obtain the variance of the third difference

$$\langle (\Delta^3 \epsilon_m)^2 \rangle = 15 k_1 \tau^\eta - 6 k_1 (2\tau)^\eta + k_1 (3\tau)^\eta, \tag{80}$$

and hence the ratio

$$\frac{\langle (\Delta^3 \epsilon_m)^2 \rangle}{\langle (\Delta^2 \epsilon_m)^2 \rangle} = \frac{15 - 6 \cdot (2)^\eta + 3^\eta}{4 - (2)^\eta}. \tag{81}$$

Similarly,

$$\frac{\langle (\Delta^4 \epsilon_m)^2 \rangle}{\langle (\Delta^3 \epsilon_m)^2 \rangle} = \frac{56 - 28 \cdot (2)^\eta + 8 \cdot (3)^\eta - (4)^\eta}{15 - 6 \cdot (2)^\eta + (3)^\eta}. \tag{82}$$

Equations (79), (81), and (82) are plotted in Fig. 6 as a function of the exponent η.

For $\eta = 4/3$, as in Fig. 6, the theoretical ratios,

$$\frac{\langle (\Delta^{n+1} \epsilon_m)^2 \rangle}{\langle (\Delta^n \epsilon_m)^2 \rangle}$$

for $n = 1$ and 2 are 1.48 and 2.84, respectively. The straight lines drawn in Fig. 5 were made to have these ratios and slope 2/3 (the square root of the variances).

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