Proc. 35th Ann. Freq. Control Symposium, USAERADCOM, Ft. Monmouth, NJ 07703, May 1981

A MODIFIED "ALLAN VARIANCE" WITH INCREASED OSCILLATOR CHARACTERIZATION ABILITY

David W. Allan and James A. Barnes

Time and Frequency Division
National Bureau of Standards
Boulder, Colorado 80303

Summary

Heretofore, the "Allan Variance," $\sigma_y^2(\tau)$, has become the de facto standard for measuring oscillator instability in the time-domain. Often oscillator frequency instabilities are resonably modelable with a power law spectrum: $S_y(f) \sim f^\alpha$, where y is the normalized frequency, f is the Fourier frequency, and α is a constant over some range of Fourier frequencies. It has been shown that for power law spectrum $\sigma_y^2(\tau) \sim \tau^\mu$, and that $\mu = -\alpha-1$ for $-3<\alpha\leq +1$, where τ is the nominal sample time over which each value of y is measured. The modified "Allan Variance" developed in this paper yields $\mu \cong -\alpha-1$ for all α in the range $-3<\alpha$, which removes the previous ambiguity: $\mu = -2$ for $+1\leq\alpha$. In other words, with the modified "Allan Variance" one can easily distinguish between white phase noise ($\alpha = +2$) and flicker phase noise ($\alpha = +1$) -- commonly occurring for the short term instabilities of quartz crystal oscillators and active hydrogen masers.

Key Words. Flicker Noise; Frequency Stability; Oscillator Noise Modeling; Power Law Spectrum; Time-Domain Stability; White Noise.

Introduction

The random fluctuations in precision oscillators may often be characterized by a power law spectrum:

$$S_y(f) = h_\alpha f^\alpha , \tag{1}$$

where y is the normalized frequency deviation, f is the Fourier frequency, h_α is the intensity of the particular noise process, and α is constant over some range of f. The typical values of α are: +2 (white noise phase modulation, PM); +1 (flicker noise PM); 0 (white noise frequency modulation, FM); -1 (flicker noise FM); and -2 (random walk FM). The Allan variance, as it has come to be known,[1] has been demonstrated as a very useful statistical tool for characterizing these various random processes with the exception that if $\alpha = +1$ or +2, the dependence on τ is nominally the same, i.e., $\sim \tau^{-2}$. It is not at all uncommon for white PM and flicker PM to occur in precision oscillators for τ of the order of one second and shorter. The modified Allan variance, as developed in this paper, depends as τ^{-2} for $\alpha = +1$ and as τ^{-3} for $\alpha = +2$. This yields a clear distinction in the time domain between these heretofore somewhat ambiguous processes.

Definition of "Allan Variance" and Related Concepts

Define y, the normalized frequency deviation, as

$$y(t) = \frac{\nu(t) - \nu_0}{\nu_0} \tag{2}$$

where $\nu(t)$ is the output frequency of the oscillator being studied, and ν_0 is nominally the same frequency, but of a reference oscillator assumed for the moment without loss of generality to be better than the test oscillator. The time deviation from some arbitrary origin (t = 0) is the integral of the frequency deviations (from that origin):

$$x(t) = \int_0^t y(t') \cdot dt' \tag{3}$$

The i^{th} average frequency deviation over an interval, τ, is

$$\bar{y}_i = \frac{x_{i+1} - x_i}{\tau} \tag{4}$$

where the assumption is made that the time deviation measurements are nominally spaced τ apart.

TN-254

The "Allan Variance" is defined as:

$$\sigma_y^2(\tau) = \tfrac{1}{2} \langle(\bar{y}_{i+1} - \bar{y}_i)^2\rangle \ , \qquad (5)$$

where the brackets "< >" denote infinite time average. Using equation (4), one may write:

$$\sigma_y^2(\tau) = \frac{1}{2\tau^2} \langle(x_{i+2} - 2x_{i+1} + x_i)^2\rangle \ . \qquad (6)$$

It has been shown that typically $\sigma_y^2(\tau)$ varies as τ^μ, and that $\mu = -\alpha-1$ for $-3 \leq \alpha \lesssim +1$.[1,2] Hence, we see one of the dimensions of usefulness of $\sigma_y^2(\tau)$; i.e., ascertaining the dependence on τ allows an estimate of α (the power law spectral type of noise). However, if $\alpha \geq +1$, then $\mu \cong -2$, and the τ dependence becomes somewhat ambiguous as to the type of noise in this region. It is interesting to note that in the region $\alpha \geq +1$, $\sigma_y^2(\tau)$ is bandwidth (f_h) dependent; i.e., the bandwidth of the measurement system will affect the value of $\sigma_y(\tau)$, and furthermore, one may use the bandwidth dependence[3] to determine the value of α (see also Appendix Ref. 2).

Development of the Modified Allan Variance

One may also write $\sigma_y^2(\tau)$ in terms of a generalized autocovariance function:

$$\sigma_y^2(\tau) = \frac{1}{2\tau^2} [4U_x(\tau) - U_x(2\tau)] \ , \qquad (7)$$

where

$$U_x(\tau) = 2[R_x(0) - R_x(\tau)], \qquad (8)$$

and where

$$R_x(\tau) = \langle x(t+\tau) \cdot x(t)\rangle \ , \qquad (9)$$

the classical autocovariance function of x(t). Using the Fourier transforms of generalized functions, one may determine the coefficients relating the power spectral density to $\sigma_y^2(\tau)$. Ref. 1 gives these relationships. It is of interest to note that $U_x(\tau)$ has the following approximate form in the region $\alpha \geq +1$ (see Appendix Ref. 2):

$$U_x(\tau) \sim a(\alpha) \left[\left|\frac{1}{2\pi f_h}\right|^{-\alpha+1} - |\tau|^{-\alpha+1} \right] \qquad (10)$$

Hence, one notes that by changing the reciprocal bandwidth as well as τ, one affects $\sigma_y^2(\tau)$ in similar ways, depending on the value of α. From this, one should be able to deduce the value of α, since the bandwith dependence becomes stronger for α moving positive from +1, and the τ dependence becomes stronger as α moves negative from +1. One can change the bandwidth in the hardware or in the software. In the past, it has typically been done in the hardware.[3] James Snyder[4] has shown that it is relatively easy to change the bandwidth in the data processing by a clever technique and we have followed his lead. In particular, we have chosen a new variance analysis scheme which coincides with the Allan variance at the minimum sample time, τ_0, (i.e., minimum data spacing), but which changes the bandwidth in the software as the sample time, τ, is changed.

Each reading of the time deviation, x_i, has associated with it an intrinsic nominal (hardware) measurement system bandwidth, f_h. Define $\tau_h = \frac{1}{2\pi f_h}$; and similarly we may define a software bandwidth, $f_s = f_h/n$, which is 1/n times narrower than the hardware bandwidth. This software bandwidth can be realized by averaging n adjacent x_i's; $\tau_s = n\tau_h$, where $\tau_s = 1/f_s$. We have defined a modified Allan variance which allows the reciprocal software bandwidth to change linearly with the sample time, τ:

$$\text{Mod } \sigma_y^2(\tau) = \frac{1}{2\tau^2} \left\langle \left[\frac{1}{n} \sum_{i=1}^{n} (x_{i+2n} - 2x_{i+n} + x_i) \right]^2 \right\rangle \qquad (11)$$

where $\tau = n\tau_0$. Eq. 11 clearly coincides with Eq. 6 for n = 1. One can see that, in general, we have formed a second difference of three time readings with each of the three being an average of n of the x_i's (with non-overlapping averages). As n increases, the (software) bandwidth decreases and this bandwidth varies just as $f_s = f_h/n$.

For a finite data set of N readings of x_i (i = 1 to N), we may write an estimate:

$$\text{Mod}\ \sigma_y^2(\tau) = \frac{1}{2\tau^2 n^2 (N-3n+1)} \cdot$$

$$\sum_{j=1}^{N-3n+1} \left[\sum_{i=j}^{n+j-1} (x_{i+2n} - 2x_{i+n} + x_i) \right]^2 \qquad (12)$$

Eq. 12 is easy to program, but takes more time to compute than for $\sigma_y(\tau)$. This is only of significance for the smaller computer or handheld calculator.

Comparisons, Tests, and Examples of Usage of the Modified Allan Variance

We simulated various power law noise processes, and applied Eq. 12. Shown in Fig. 1 are the resulting τ-dependences of the modified Allan variances for $\alpha = -2, -1, 0, +1,$ and $+2$. The solid lines drawn are the anticipated or theoretical slopes for the particular noise process. One sees excellent agreement for white noise PM and for flicker noise PM, and nominal agreement for the others.

One can express Eq. 11 in terms of the generalized autocovariance function:

$$\begin{aligned}\text{Mod}\ \sigma_y^2(\tau) = \frac{1}{2\tau^2 n^2} \Big\{ & [4U_x(\tau) - U_x(2\tau)] \cdot n \\ & + \sum_{i-1}^{n-1} (n-i)[-6U_x(i\tau_o) + 4U_x((n+i)\tau_o) \\ & + 4U_x((n-i)\tau_o) - U_x((2n+i)\tau_o) \\ & - U_x((2n-i)\tau_o)] \Big\} \quad . \end{aligned} \qquad (13)$$

In the range $-3 \le \alpha \le +1$, one may write:

$$U_x(\tau) = a(\alpha) \cdot \tau^{-\alpha+1} \ , \qquad (14)$$

which when substituted in Eq. 13, and using Eq. 7, yields

$$\begin{aligned}\text{Mod}\ \sigma_y^2(\tau) = \sigma_y^2(\tau) \Big\{ & \frac{1}{n} + \frac{1}{n^2\ 4n^{-\alpha+1} - (2n)^{-\alpha+1}} \cdot \\ & \sum_{i=1}^{n-1} (n-i) \cdot \Big[-6i^{-\alpha+1} + 4(n+i)^{-\alpha+1} \\ & -(2n+i)^{-\alpha+1} + 4(n-i)^{-\alpha+1} - (2n-i)^{-\alpha+1} \Big] \Big\} \end{aligned} \qquad (15)$$

Since we know that $\sigma_y^2(\tau)$ is well behaved in this range and $\mu = -\alpha -1$, it is of interest to look at the ratio:

$$R(n) = \text{Mod}\ \sigma_y^2(\tau)/\sigma_y^2(\tau) \ .$$

As stated before, at $n = 1$ ($\tau = \tau_o$) the ratio is unity. One can evaluate Eq. 15 with a computer. A reasonable empirical fit may be formed, which is good to 0.5% or better of Eq. 15:

$$R(n) = \frac{q+pn^E - p}{qn^E} \qquad (16)$$

which approaches p/q asymptotically as n approaches infinity, and is within 1% of p/q for $n \ge 8$. Listed in Table 1 are the empirical values of p, q, and E and the quality of fit for the appropriate power law noise processes.

TABLE 1

Noise Type Name	α	p	q	E	fit	Mod $\sigma_y(\tau)/\sigma_y(\tau)$ $\tau >> \tau_o$
White FM	0	1	2	2	perfect	.707
Flicker FM	-1	99.9	148	2.35	½%	.821
Random Walk FM	-2	33	40	2.35	<½%	.908
Flicker Walk FM	-3	1	1	--	perfect	1

The results of Table 1 are in reasonable agreement with simulated results of Fig. 1(a) through 1(e). The last row in Table 1, "flicker walk" frequency modulation, is out of the range of applicability of α, but the ratio, R(n), is still convergent.

The $U_x(\tau)$ function for flicker noise PM is extremely complicated and has not been developed, but one can arrive at an empirical value for it. The $U_x(\tau)$ function is derivable for the other power law spectral processes. Table 2 gives the relationships between the time domain measure Mod $\sigma_y^2(\tau)$ and its power law spectral counterpart, given in Eq. 1. Also listed in the right hand column of Table 2 are the asymptotic values of $R(n)$:

TABLE 2

Noise Type: Name	α	Mod $\sigma_y^2(\tau)$	Comment	$R(n)$ $n \to \infty$
White PM	+2	$h_2 \cdot \frac{3 f_h}{(2\pi)^2 \, n \, \tau^2}$	Exact	1
Flicker PM	+1	$h_1 \cdot \frac{1.038 + 3\ln(\omega_h \tau)}{(2\pi)^2 \, \tau^2}$	Empirical	1
White FM	0	$h_0 \cdot \frac{R(n)}{2\tau}$	Exact	0.5
Flicker FM	-1	$h_{-1} \, 2\ln(2) \cdot R(n)$	Empirical; Exact Available	0.674
Random Walk FM	-2	$h_{-2} \cdot \frac{(2\pi)^2 \tau}{6} \cdot R(n)$	Empirical; Exact Available	0.824

It is clear from Table 2 that Mod $\sigma_y^2(\tau)$ is very useful for white PM and flicker noise PM, but for $\alpha < +1$ the conventional Allan variance, $\sigma_y^2(\tau)$, gives both an easier-to-interpret and an easier-to-calculate measure of stability.

It is interesting to make a graph of α versus μ for both the ordinary Allan variance and the modified Allan variance. Shown in Fig. 2 is such a graph. This graph allows one to determine power law spectra for non-interger as well as interger values of α. The dashed line for the modified Allan variance has been intentionally moved to the left in Fig. 2 because for small values of n the value of μ will appear to be slightly more negative that for $\sigma_y^2(\tau)$, even though for large n, they both approach the same slope (i.e., the same values of μ). In fact, in the asymtotic limit, the equation relating μ and α for the modified Allan variance is

$$\alpha = -\mu - 1, \text{ for } -3 < \alpha < +3 \,. \qquad (17)$$

The value of $\mu = -4$ for $\alpha = +3$ was verified empirically with simulated data, and it appears that for $\alpha > +3$, μ remains at -4.

A direct application for using the modified Allan variance recently arose in the analysis of atomic clock data as received from a Global Positioning System (GPS) satellite. We were interested in knowing the short-term characteristics of the newly developed, high-accuracy NBS/GPS receiver, as well as the propagation fluctuations. Fig. 3 shows both $\sigma_y^2(\tau)$ and Mod $\sigma_y^2(\tau)$ for comparison. Using Mod $\sigma_y^2(\tau)$, we can tell that the fundamental limiting noise process involved in the system is white noise PM with the exciting result that averaging for four minutes can allow one to * ascertain time difference to better than one nanosecond excluding other systematic effects.

Conclusion

We have developed a supplemental measure, the "Modified Allan Variance" (Mod $\sigma_y^2(\tau)$), which has very useful properties when analyzing oscillator or signal stability in the presence of white noise phase modulation or flicker noise phase modulation. It also works reasonably well as a stability measure for other commonly occuring noise processes in precision oscillators.

We would recommend that for most time domain analysis, $\sigma_y^2(\tau)$ should be the first choice. If $\sigma_y^2(\tau)$ depends on τ as τ^{-1}, then the modified Allan variance can be used as a substitute to help remove the ambiguity as to the noise processes.

Acknowledgments

The authors are deeply grateful for stimulating discussions with Dr. James J. Snyder, and for the useful comments of Dr. Robert Kamper and Mr. David A. Howe.

References

1. James A. Barnes, Andrew R. Chi, Leonard S. Cutler, Daniel J. Healey, David B. Leeson, Tomas E. McGunical, James A. Mullen, Jr., Warren L. Smith, Richard L. Sydnor, Robert F. C. Vessot, and Gernot M. R. Winkler, Proc. IEEE Trans. Instrum. Meas. IM-20, 105 (1971); also published as NBS Tech. Note 394 (1971).
2. David W. Allan, Proc. IEEE 54, 221 (1966).
3. R. F. C. Vessot, Proc. IEEE 54, 199 (1966).
4. James J. Snyder, Proc. 35th Annual Symp. on Freq. Control (1981), to be published.

* See Appendix Note # 34

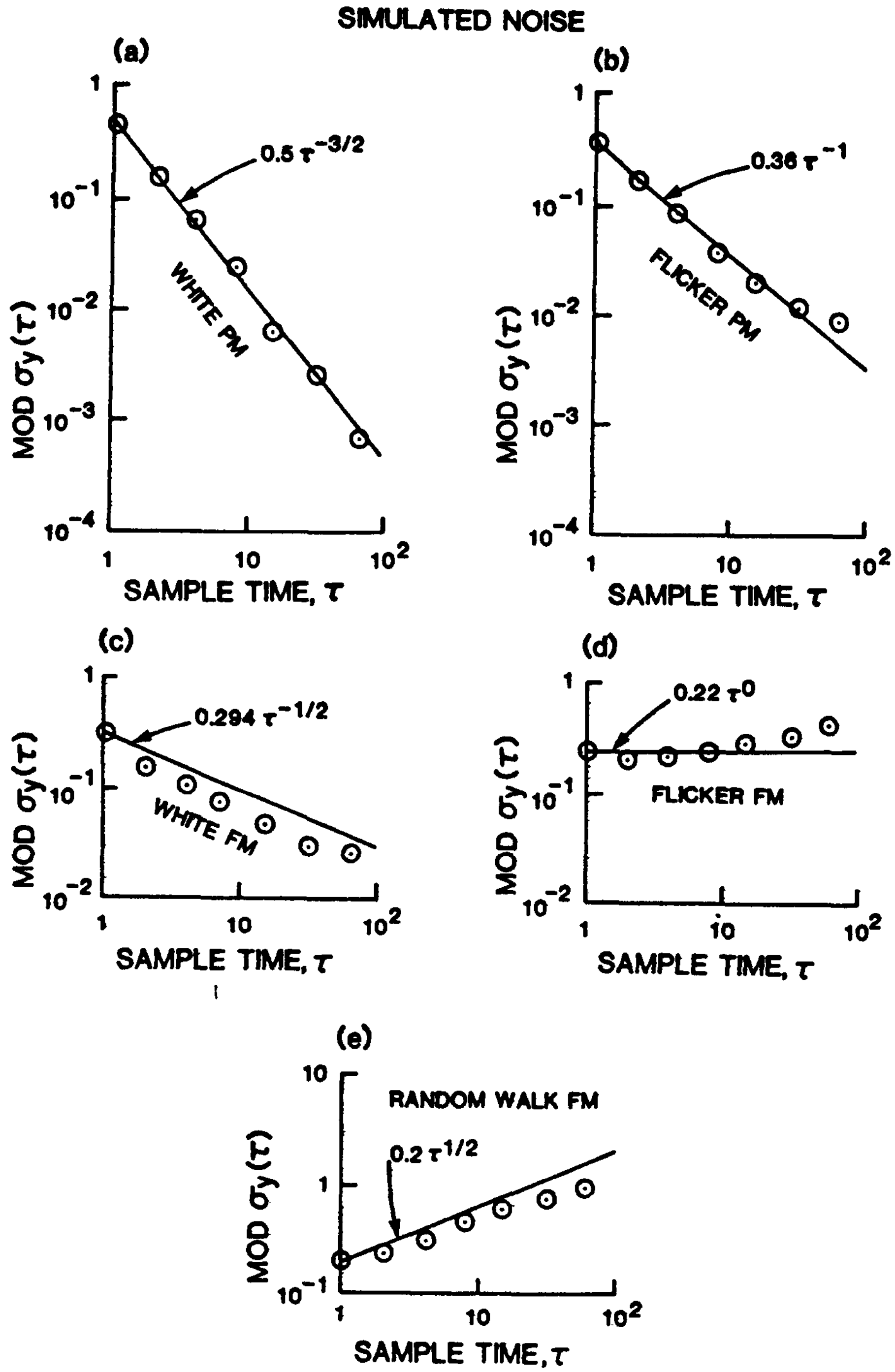

Fig 1a-e. Mod $\sigma_y(\tau)$ using Eq. 12 was calculated for different sample times for independently generated and simulated noise processes, which were white phase noise, flicker phase noise, white frequency noise, flicker frequency noise, and random walk frequency noise, respectively. Mod $\sigma_y(\tau)$ was computed for 399 data points in each case. One sees the excellent fit to the theory for white phase noise and flicker phase noise, an important new contribution in the ability to characterize oscillators having these noise processes.