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Inertia-Controlled Ambipolar Diffusion

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The general nonlinear ambipolar-flow equations are derived and discussed. The important nonlinearities are caused by the inertia of the ions and by the heat-conduction mechanism. It is shown that all essential effects associated with these nonlinearities can be demonstrated in the plane-parallel case. The influence of the nonlinearity caused by the inertia is discussed in detail for the plane-parallel case and it is shown that this flow is analogous to the flow of a fluid with friction through a contracting nozzle. The limiting velocity—the isothermal or the adiabatic sound velocity—which is found in the exit of a nozzle is also found at the boundary in the case of inertia-controlled diffusion, provided this boundary acts like a perfect sink. Bohm's criterion—the ion drift velocity less than or equal to the sound velocity of the electron–ion gas in front of the boundary or the wall sheath—appears as an integral part of the inertia-controlled-diffusion theory. In the inertia-controlled-diffusion theory are incorporated, as integral parts, all the assumptions that necessarily must be added to the linear-diffusion theory in order to make it realistic. The isothermal and inertia-controlled diffusion leads always to wall-stabilized plasma configurations which directly correspond to the fundamental mode solution in the linear-diffusion theory.

INTRODUCTION

THE concept diffusion applies to the flow of one medium in another under the influence of the density gradients and the thermal motion of the particles in the two media. The analytical description of this phenomenon, the diffusion equation, is obtained by applying the laws of conservation of mass and momentum in their microscopic or macroscopic forms to the diffusing matter. The simple form of the diffusion, the linear diffusion, is obtained when the concentration of the diffusing matter is small in comparison with the concentration of the medium in which it is diffusing and when the nonlinear terms can be neglected. This simple mechanism is modified in practical cases by nonlinear terms. The most important nonlinear terms are the terms caused by the inertia of the diffusing matter and by the heat-conduction mechanism. Both these nonlinearities are included in the basic equations of this paper although only the first kind, the one caused by the inertia of the diffusing matter, is fully analyzed. It will be shown that the inertia-controlled diffusion resolves the dilemma at the boundary which one experiences in the simple linear-diffusion theory when it is treated as an eigenvalue problem. It leads to "wall-stabilized" configurations which change into "free" configurations only when the heat-conduction mechanism is invoked.

The medium treated here is the electron–ion plasma. The simplest practical model of this kind of plasma has three kinds of particles: electrons, ions, and neutral particles. Diffusion in this kind of medium has been discussed in the low-pressure case by Bernstein and Holstein[1] and in the high-pressure case by Allis and Rose.[2] Neither of these authors have considered the influence of the nonlinearities discussed in this paper. Allis and Rose emphasize the influence of the ambipolar space-charge field and the nonlinearity associated with this field. The stand is taken in this paper that the ambipolar space-charge field only gives rise to internal forces which have little influence on the momentum-balance equation for the ionized part of the plasma and that the primary function of this space-charge field is to serve as an energy-exchange mechanism for the transfer of random kinetic energy from the electron gas to the ion gas.

The discussion will be limited to the case of ambipolar flow in the absence of a magnetic field. Then no currents are associated with the ambipolar flow and the plasma can be considered as consisting of two parts, the ionized part of the plasma and the part of the plasma that is formed by the neutral particles. It will be assumed that the concentration of the ionized part of the plasma is small in comparison with the concentration of the neutral particles. The ion-to-neutral-particle collision frequency and the electron-to-neutral-particle collision frequency can then be considered as constants and the drift velocity of the neutral particles neglected

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[1] I. B. Bernstein and T. Holstein, Phys. Rev. **94**, 1475 (1954).
[2] W. P. Allis and D. J. Rose, Phys. Rev. **93**, 84 (1954).

in comparison with the ambipolar drift velocity of the electron–ion gas. In this limit, the electron–ion gas can be considered as diffusing in a random lattice of neutral particles.

AMBIPOLAR-FLOW EQUATIONS

The conservation of mass is expressed through the continuity equation for the ionized part of the plasma

$$d\rho/dt + \rho\nabla\cdot\mathbf{u} = \rho\nu_i, \tag{1}$$

where $\mathbf{u}$ is the ambipolar drift velocity, ν_i the ionization frequency, and ρ the mass density of the ionized part of the plasma defined as

$$\rho = mn_m + Mn_M, \tag{2}$$

with m the electron mass, M the ion mass, n_m the number density of the electrons, and n_M the number density of the ions.

When the viscosity effects are neglected, the momentum balance equation for the ionized part of the plasma can be written as

$$\rho\, d\mathbf{u}/dt + (\nu_i + \nu_M)\rho\mathbf{u} + \nabla(p_m + p_M) = q(\mathbf{E}_q + \mathbf{E}_a), \tag{3}$$

where ν_M is the momentum-transfer collision frequency for the interaction between the ions and the neutral particles referred to an average ion, p_m is the electron pressure, p_M the ion pressure, $\mathbf{E}_q$ the space-charge field that necessarily is associated with the ambipolar diffusion, and q the corresponding free-space charge density, while $\mathbf{E}_a$ is the applied electric field which is necessary for the maintenance of the plasma. Only the pure ambipolar flow will be discussed in this paper, that is, cases where the applied electric field has no direct influence on the ambipolar flow. Electrode phenomena are, therefore, excluded, which means that the applied electric field $\mathbf{E}_a$ generally is perpendicular to the ambipolar space-charge field $\mathbf{E}_q$. The applied electric field $\mathbf{E}_a$ may be parallel with the ambipolar space-charge field $\mathbf{E}_q$ and still not influence the ambipolar flow provided $\mathbf{E}_a$ is an ac field with a frequency that is large in comparison with the inverted significant time constant associated with the ambipolar flow. Therefore, the applied electric field is neglected in the following discussion of the ambipolar flow.

The momentum-balance equation for the ionized part of the plasma, Eq. (3), is obtained by first setting up the momentum-balance equations for the electrons and ions separately and by summing these two equations. By taking the difference between the same two equations appropriately weighted one obtains the so-called generalized Ohm's law. It is easily shown from the latter equation that under ambipolar-flow conditions, the ambipolar space-charge field $\mathbf{E}_q$ can be related to a very good approximation to the electron temperature T_m and the electron pressure p_m as follows:

$$\mathbf{E}_q = -(kT_m/e)\nabla \ln p_m. \tag{4}$$

This equation is an expression for the fact that the electrons have so small mass and so high mobility in comparison with the ions that, for all practical purposes, the electron gas is in static equilibrium with the space-charge field $\mathbf{E}_q$. This statement is more or less obvious when the free-space-charge density q is made up of surplus electrons. It is, however, equally applicable when the space-charge density q is made up of surplus ions, provided one thinks of this space-charge density as made up of deficiency electrons, that is, in an essentially neutral plasma the free-positive-space-charge density q consists of a gas of "holes" with the negative-electron charge, the positive-electron mass, and the electron temperature.

Pure ambipolar flow is found in the limit where the Debye length is small in comparison with the plasma dimensions. The space-charge field $\mathbf{E}_q$ gives only rise to internal forces in this limit. It has practically no influence on the mass flow density of the ionized part of the plasma but acts as a very efficient mechanism, whereby the electrons transfer their kinetic energy to the ions. Therefore, in the ambipolar limit, the ionized part of the plasma can be viewed as a gas with the particle density equal to that of the ion gas, the particle mass equal to the sum of the ion and electron masses, and with a temperature T^* equal to the sum of the electron and ion temperatures. The explicit influence of the space-charge field $\mathbf{E}_q$ on the momentum-balance equation for the ionized part of the plasma can be neglected in this limit as will be done in the following discussion. The solutions to Eqs. (1) and (3) then give spatial distributions of the mass density of the ionized part of the plasma and the ambipolar drift velocity. The mass-density distribution of the plasma is, for all practical purposes, equal to the ion-density distribution. The spatial distribution of the electrons is then obtained by applying Eq. (4) together with Poisson's equation and by assuming that the electron gas in is static equilibrium with the space-charge field and the random-ion lattice as described by Eqs. (1) and (3) corresponding directly to the calculation of the spatial electron distribution in a potential well. The spatial distribution of the ions or the mass density of the ionized part of the plasma is by far more important than the spatial distribution of the electron density in the determination of the possible plasma configurations.

The explicit influence of the space-charge field $\mathbf{E}_q$ and the applied electric field $\mathbf{E}_a$ are, in view of the discussion above, neglected in the following analysis of the ambipolar flow. It is now convenient to write the continuity and momentum-balance equations in their complete Eulerian forms. The discussion will be limited to the three simple configurations: the plane-parallel case, the infinitely long cylindrical, and the spherical cases both with rotational symmetry. Furthermore, it is assumed that the ambipolar flow in all these cases is circulation-free and one-dimensional. The continuity equation can then be written

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^\beta}\frac{\partial}{\partial r}(r^\beta \rho u) = \rho \nu_i, \tag{5}$$

while the momentum-balance equation becomes

$$\frac{\partial \rho u}{\partial t} + \nu_M \rho u + \frac{\partial}{\partial r}\left(\rho \frac{kT^*}{M}\right) + \frac{1}{r^\beta}\frac{\partial}{\partial r}(r^\beta \rho u^2) = 0, \tag{6}$$

with

$$T^* = T_m + T_M. \tag{7}$$

The two equations (5) and (6) describe the ambipolar diffusion in the plane-parallel case with $\beta = 0$, the infinitely long cylindrical case when $\beta = 1$, and the spherical case when $\beta = 2$. The time variations are now accounted for by introducing the time constants

$$\frac{1}{\rho}\frac{\partial \rho}{\partial t} = -\frac{1}{\tau_\rho}, \qquad \frac{1}{u}\frac{\partial u}{\partial t} = -\frac{1}{\tau_u}. \tag{8}$$

The continuity equation then becomes

$$\frac{1}{\rho}\frac{\partial \rho}{\partial r} = \frac{\nu_i + (1/\tau_\rho)}{u} - \frac{\beta}{r} - \frac{1}{u}\frac{\partial u}{\partial r}. \tag{9}$$

The spatial derivative of the density ρ in Eq. (6) can now be eliminated with the help of the equation above and the spatial derivative of u written as

$$\frac{\partial u}{\partial r} = \frac{\left(\nu_M + \nu_i - \frac{1}{\tau_u}\right)u^2 + \left(\nu_i + \frac{1}{\tau_\rho}\right)\frac{kT^*}{M} - \frac{\beta u}{r}\frac{kT^*}{M} + u\frac{\partial}{\partial r}\left(\frac{kT^*}{M}\right)}{[(kT^*/M) - u^2]}. \tag{10}$$

The transformation from Eqs. (5) and (6) into Eqs. (9) and (10) is purely algebraic, allowing the collision frequencies as well as the time constants to be functions of both the coordinate and time. The transformation into Eq. (10) is particularly important as this equation in the limit

$$\nu_M \gg 1/\tau_u, \qquad \nu_i \gg 1/\tau_\rho \tag{11}$$

gives a differential equation for u with respect to r which is explicitly independent of the density ρ.

Equations (9) and (10), together with the equations which determine the energy balance and thereby the thermal gradients and Maxwell's equations, form a complete set of equations which determine the ambipolar diffusion. The two equations (9) and (10) will be written in terms of dimensionless variables and proper parameters before they are analyzed. The inverted time constants will be assumed negligible in comparison with the collision frequencies in this process. A convenient set of parameters and dimensionless variables is then

$$v^2 = \frac{kT^*}{M}, \quad \Lambda_p^2 = \frac{kT^*}{M(\nu_M + \nu_i)\nu_i}, \quad \alpha^2 = \frac{\nu_M + \nu_i}{\nu_i}, \tag{12}$$

$$x = \int_0^r \frac{\alpha}{\Lambda_p}\,dr, \quad x^* = \frac{1}{v}\int_0^x v\,dx, \quad \phi = \frac{\rho}{\rho_0}, \quad \kappa = \frac{u}{v}, \tag{13}$$

$$r = \int_0^x \frac{\Lambda_p}{\alpha}\,dx = \int_0^x \frac{v}{\nu_M}\,dx = \frac{v}{\nu_M}x^* \cong \int_0^x l\,dx, \tag{14}$$

where ρ_0 is the mass density of the ionized part of the plasma in the center of the plasma and l is a mean free path determined by the interaction between the ions and the neutral particles. The origin of the coordinate system will be located at the center of the plasma and it is then necessary that

$$\phi \to 1, \quad \kappa \to 0, \quad \text{when} \quad x \to 0, \tag{15}$$

at least for the solution that applies to the center of the plasma and its neighborhood. The final forms of the two equations (9) and (10) are then

$$\frac{\partial}{\partial x}\ln[\phi\kappa(T^*)^{\frac{1}{2}}] = \frac{1}{\alpha^2\kappa} - \frac{\beta}{x^*} \tag{16}$$

and

$$\frac{\partial \kappa}{\partial x} = \frac{\kappa^2 + \frac{1}{\alpha^2} - \frac{\beta\kappa}{x^*} + \frac{\kappa}{2}(1+\kappa^2)\frac{\partial}{\partial x}\ln T^*}{(1-\kappa^2)}. \tag{17}$$

The variable ϕ in this equation system is the mass density of the ionized part of the plasma normalized with respect to its value at the center of the plasma, κ is the ambipolar drift velcoity normalized with respect to the local value of the isothermal sound velocity of the electron–ion gas, x is the coordinate measured in terms of the local mean free path l, and $T^* = T_m + T_M$ is the temperature of the electron–ion gas. The inertia of the ions give rise to the second term in the denominator of Eq. (17).

The rather elaborate normalization system described by Eqs. (12) and (14) is necessary when the temperature T^* is a strong function of the coordinate since the plasma diffusion length Λ_p and the production parameter α are strong functions of the temperature. The normalization system could be simplified in the isothermal case when Λ_p and α are constant. Very little is gained by this simplification and the more elaborate normalization system has been used here anticipating a future analysis of the influence of the thermal gradient on the ambipolar flow.

The two equations (16) and (17) describe the ambipolar diffusion mechanism as influenced by the production mechanism, the inertia of the ions and the energy dissipated in the plasma through the associated thermal gradient. To these equations must be added the energy-balance equations and Maxwell's equations. All the essential features of the equation system above can be demonstrated in the plane-parallel case. A discussion of the influence of the thermal gradient on the ambipolar flow requires a lengthy analysis of the energy balance in the plasma, too lengthy to be included in the present paper. This limits the discussion in this paper to the plane-parallel and isothermal case and its purpose is to demonstrate the influence of the inertia of the ions on the ambipolar diffusion.

PLANE-PARALLEL AND ISOTHERMAL CASE

The limitation to the plane-parallel case brings about a simplification as β then is zero and the function x^* no longer is present in the differential equations. The isothermal assumption which is applicable in the late afterglow plasma and in some wall-stabilized steady-state discharges brings about an additional simplification. Only the two equations (16) and (17) are then necessary for the description of the ambipolar flow or diffusion. They can, in the isothermal and plane-parallel case, be written as

$$(\partial/\partial x) \ln (\phi\kappa) = 1/\alpha^2\kappa, \tag{18}$$

$$\partial\kappa/\partial x = (\kappa^2 + 1/\alpha^2)/(1 - \kappa^2). \tag{19}$$

One obtains, by eliminating the differential dx between these two equations, a differential equation for ϕ as function of κ

$$\partial \ln \phi/\partial\kappa = -(1 + \alpha^2)\kappa/(1 + \alpha^2\kappa^2), \tag{20}$$

which replaces the differential equation (16). The solution to the differential equations (19) and (20) are, respectively,

$$\frac{x}{\alpha} = \frac{x_1}{\alpha} + \left(1 + \frac{1}{\alpha^2}\right) \text{arc tan } (\alpha\kappa) - \frac{\kappa}{\alpha} \tag{21}$$

and

$$\phi = \phi_1(1 + \alpha^2\kappa^2)^{-\frac{1}{2}(1+1/\alpha^2)}, \tag{22}$$

where x_1 and ϕ_1 are integration constants which are equal to unity and zero, respectively, for the principal solution that applies to the center of the plasma and its neighborhood. The two equations above constitute the parametric general solution to the inertia-controlled (ambipolar) diffusion in the isothermal plane-parallel case. The fourth term in the momentum-balance equation (6)—the inertia term—gives rise to the singularity that appears in Eq. (19). The differential equation (18) is not changed when the inertia term is neglected, while the second term in the denominator of Eq. (19) disappears. The parametric general solution for the case when the inertia term is neglected can be written as

$$x/\alpha = x_1/\alpha + \text{arc tan } (\alpha\kappa) \tag{23}$$

and

$$\phi = \phi_1(1 + \alpha^2\kappa^2)^{-\frac{1}{2}}, \tag{24}$$

giving

$$\phi = \phi_1 \cos [(x - x_1)/\alpha], \tag{25}$$

the standard solution to the plane-parallel isothermal diffusion, when the normalized drift velocity κ is eliminated. The two parametric solutions (21), (22) and (23), (24) are illustrated schematically in Fig. 1 for $\theta_1 = 1$ and $\kappa_1 = 0$. The normalized drift velocity κ and the normalized plasma density ϕ are plotted linearly as function of x/α. The curves labeled $(1/\alpha)$ tan (x/α) and cos (x/α) illustrate the linear diffusion which is obtained when the inertia of the ions is neglected, while the curves B represent the solution for the inertia-controlled diffusion. The effect of the inertia of the ions is easily seen by comparing these two sets of curves. The function $\kappa(x)$ given by formula (23) and which applies to the linear diffusion goes to infinity when x/a approaches $\frac{1}{2}\pi$, giving rise to ambipolar drift velocities much

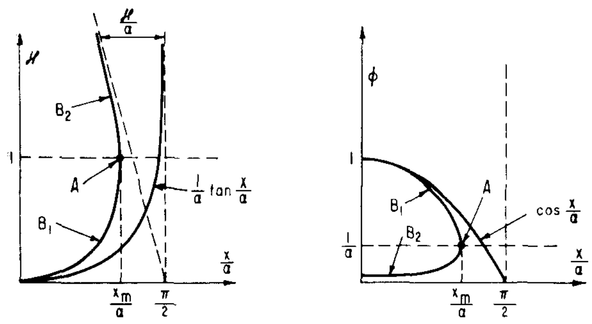

FIG. 1.

higher than the sound velocity of the electron–ion gas. This is a physical absurdity which, in the linear diffusion, is resolved by "assuming" that the ambipolar drift velocity does not exceed the sound velocity and by cutting off the solution at an appropriate point close to $\kappa = 1$. This artifact is not necessary when the effect of the inertia of the ions is included. The solution for the inertia-controlled diffusion, which is illustrated in terms of the curves B in Fig. 1, shows that the coordinate x has a maximum x_m independent of whether it is viewed as a function of κ or ϕ. This maximum is

$$x_m/\alpha = (1 + 1/\alpha^2) \text{ arc tan } \alpha - 1/\alpha < \tfrac{1}{2}\pi \qquad (26)$$

and is obtained when

$$\kappa = 1 \quad \text{and} \quad \phi = (1 + \alpha^2)^{-\frac{1}{2}(1+1/\alpha^2)} \underset{\alpha \gg 1}{\longrightarrow} 1/\alpha. \qquad (27)$$

It is particularly significant that the maximum x_m is obtained for the value $\kappa = 1$ independent of the value of the parameter α. The sound velocity of the electron–ion gas appears naturally as a critical velocity in the inertia-controlled diffusion. Both κ and ϕ are double-valued as function of the coordinate in the inertia-controlled diffusion, while they are single-valued functions in the linear diffusion. The curves B_1 represent the only acceptable solution for the ambipolar diffusion from the center of the plasma while the curves B_2 represent the behavior of a beam of the electron–ion gas shot into the parent gas. The points A are terminal points for the continuous analytical solutions in either one of these two cases, that is, a neutral electron–ion beam shot into a neutral gas will slow down until it reaches the sound velocity which occurs at the coordinate x_m while the ambipolar drift velocity from the center of the plasma will increase with the coordinate until it reaches the sound velocity at the coordinate x_m. Figure 1 illustrates only the principal branches ($\phi_1 = 1$ and $x_1 = 0$) of the solutions for the linear and inertia-controlled diffusion. The nonprincipal branches of the general solutions are generated by translating the principal branches along the x axis. It is quite obvious from the appearance of the principal branches that an analytical and continuous solution for the (ambipolar) diffusion—starting at the center of the plasma—cannot be continued beyond the coordinate x_m as long as the diffusion is isothermal and therefore based on the differential equations (18) and (19). In order to continue the analytical description of the (ambipolar) diffusion beyond the coordinate x_m it is necessary to invoke the thermal gradients as they appear in Eqs. (16) and (17).

It is impossible to conceive of high thermal gradients which could influence the diffusive flow in the late afterglow when the electrons and the ions have assumed the temperature of the parent gas. One comes inescapably to the conclusion that the principal solution [Eqs. (21) and (22) with $\phi_1 = 1$ and $x_1 = 0$] *cannot* be accepted as a (quasi-) steady-state solution unless a wall with the appropriate "boundary conditions" is located somewhere between the center of the plasma and the coordinate x_m. If the wall is assumed to be a perfect sink for the electron–ion gas, then it is necessary that the flow of this gas into the wall be equal to the density of the gas in front of the wall times some average of the random thermal velocity. The wall, which acts like a perfect sink, must be located at the point where the flow is maximum and according to Eq. (21) this occurs at the coordinate x_m. The wall, which cannot be considered as a perfect sink for the electron–ion gas, must be located in the range between the origin end the coordinate x_m. Any such wall can be characterized by a number κ_w, less than or equal to unity, which determines the wall coordinate x_w

$$x_W/\alpha = (1 + 1/\alpha^2) \text{ arc tan } (\alpha\kappa_w) - \kappa_w/\alpha. \qquad (28)$$

The number κ_w is a function of the probability f for the ions and electrons to stick on the wall at the first impact. The number κ_w is zero when f is zero and unity when f is equal to unity. How κ_w depends on f between these two points can only be found through an investigation of the velocity distribution function of the electron–ion gas just in front of the wall. In most gas discharges, it is generally assumed that κ_w is equal to unity, that is, that the wall acts like a perfect sink for the electron–ion gas.

Equations (21) and (22), with $\phi_1 = 1$ and $x_1 = 0$, together with the "boundary condition" Eq. (28) constitute the steady-state solution for the wall-stabilized plasma when it is governed by the inertia-controlled ambipolar diffusion. This solution corresponds directly to eigenvalue solution for the ambipolar diffusion as it was introduced by Schottky,[3] provided one adds to the Schottky solution the auxiliary requirement that the mass drift velocity of the electron–ion gas in front of the wall or the wall sheath is equal to the isothermal sound velocity of the electron–ion gas at this point. This auxiliary requirement agrees identically with Bohm's statement or criterion[4] with regard to the ion current density entering the wall sheath provided the ion temperature is negligible in comparison with the electron temperature. The steady-state solution to the inertia-controlled ambipolar diffusion between two parallel walls—the distance d apart—phrased as Allis[2] does, is obtained from Eq. (28) when the values of the parameters are introduced

$$\frac{d}{2}=\frac{1}{\nu_M}\left(\frac{kT^*}{M}\right)^{\frac{1}{2}} \cdot\left\{\left[\left(\frac{\nu_M}{\nu_i}\right)^{\frac{1}{2}}\left(\frac{\nu_i}{\nu_M}\right)^{\frac{1}{2}}\right]\text{arc tan}\left[\kappa_w\left(\frac{\nu_M}{\nu_i}\right)^{\frac{1}{2}}\right]-\kappa_w\right\}. \quad (29)$$

This equation for the wall-stabilized plasma or discharge does, in the limit $\kappa_w = 1$ and $\nu_M \gg \nu_i$, become identical with the corresponding solution

$$\frac{d}{\pi}=\left(\frac{kT^*}{M\nu_M\nu_i}\right)^{\frac{1}{2}}=\left(\frac{D_a}{\nu_i}\right)^{\frac{1}{2}}=\Lambda_p \quad (30)$$

for the linear ambipolar diffusion.

The solutions above apply, in principle, only to the steady-state plasma. The steady-state solutions are generally also applied in the interpretation of the decay measurements in the afterglow plasma which is looked upon as being in quasi-equilibrium. It is generally assumed that the shape of the plasma remains unchanged during the decay. The original differential equations (5) and (6) are linear in terms of the density ρ but nonlinear in terms of the ambipolar drift velocity u. This means that these equations are separable in terms of the time dependence of the density but not in terms of the time dependence of the ambipolar drift velocity u. A definite requirement for the quasi-equilibrium is then that the electron–ion gas has assumed the temperature of the parent gas and that this temperature is uniform. Equation (29) gives then the time constant τ for the diffusion-controlled, wall-stabilized plasma, provided ν_i is replaced by τ^{-1}. A deviation from this quasi-equilibrium in terms of the shape or in terms of a nonuniform temperature of the parent gas will bring in a time dependence in the ambipolar drift velocity u. The time constants for the processes which bring the disturbed afterglow plasma back to the quasi-equilibrium are, of necessity, of the order of the diffusion times. Small deviations from the exponential diffusion decay in the afterglow can, therefore, not be interpreted as deviations from the diffusion process unless one has other strong evidence to that effect.

The most striking feature of the differential equation (16) is that it has a singularity when the ambipolar drift velocity approaches the random thermal velocity of the electron–ion gas. The behavior of the electron–ion gas in the range of the singularity is, for all practical purposes, independent of the collisions. It is, therefore, reasonable to assume that the electron–ion gas flows adiabatically as an ideal gas in this range. The large spatial variations in the ambipolar drift velocity are then accompanied by large spatial variations in the temperature of the electron–ion gas. The conservation of mass, momentum, and energy gives, then, since the effects of the collisions are neglected, the following relation:

$$(5/2)(kT^*/M)+u^2/2=(5/2)(kT_s^*/M), \quad (31)$$

between the adiabatic temperature T^* and the flow velocity u of the electron–ion gas. The temperature T_s^* is a stagnation temperature which can be considered as constant in the range of the singularity. One finds, when formula (16) is corrected for the adiabatic temperature variations that the singularity occurs when ambipolar drift velocity becomes equal to the adiabatic sound velocity of the electron–ion gas. The "adiabatic correction" has been neglected in the previous discussion. It is assumed that the electron–ion gas in one case, the steady-state case, is in good thermal contact with the electric field that maintains the plasma, and in the other case, applying to the afterglow plasma, that the thermal contact between the electron–ion gas and the parent gas is sufficiently good to prevent the adiabatic mechanism to be effective except just at the singularity. The adiabatic mechanism at the singularity is important in the analysis of the sheath mechanism which, however, is not the subject of discussion in this paper.

IV. SUMMARY

The outstanding feature of the inertia controlled (ambipolar) diffusion is that it introduces in a natural way the isothermal sound velocity $[(k/M)(T_m + T_M)]^{\frac{1}{2}}$—or when the adiabatic cor-

[3] W. Schottky, Physik. Z. **25**, 635 (1924).

[4] *Characteristics of Discharges in Magnetic Fields,* edited by A. Guthrie and R. K. Wakerling (McGraw-Hill Book Company, Inc., New York, 1949), p. 95.

rection is introduced, the adiabatic sound velocity $[(5/3)(k/M)(T_m + T_M)]^{\frac{1}{2}}$—as the critical velocity in the diffusive flow. A result of the naturally appearing critical velocity is the fact that the density in front of a wall or a wall sheath is finite even if the wall is a perfect sink for the electron–ion gas.

The isothermal and the inertia-controlled diffusion leads to a "wall-stabilized" configuration in a much more definite way than does the simple linear diffusion and without any need of patching up the solution close to the boundary. This is caused by the presence of the singularity in the differential equations (10) and (17). This singularity is rendered ineffective if the numerator in any one of these two equations becomes zero before the ambipolar drift velocity u reaches the critical velocity. This is how the heat-conduction mechanism through the thermal gradient, which in most practical cases is negative, causes the plasma to change from a "wall-stabilized" configuration at low powers to a "free" configuration at high powers. This mechanism, which is capable of explaining some of the so-called plasma "striations," will be the subject of discussion in a future paper. The term in the numerator of Eqs. (10) and (17) which contains the configuration factor β is also negative and could possibly in some cases cause the numerator to become zero before the ambipolar drift velocity u reaches the critical velocity and without invoking the heat-conduction mechanism. It is easily seen from Eqs. (13) that the function x^* is identical with x in the isothermal case. Equation (17) can therefore be written as

$$\frac{\partial \kappa}{\partial x} = \frac{\kappa^2 + (1/\alpha^2) - (\beta\kappa/x)}{1 - \kappa^2} \tag{32}$$

in the isothermal case. The fact that the term containing the configuration factor β cannot cause the numerator of this equation to become zero before κ reaches unity is best demonstrated with the help of Fig. 2. This figure illustrates in the x–κ plane the curves along which the derivative $\partial\kappa/\partial x$ either is infinite or zero. An investigation of Eq. (32) shows that $|\kappa| \leqq 1$ is negative in the range a and positive in the range b. The only two possible solutions to the differential equation (32) in the range $|\kappa| \leqq 1$ are shown in terms of the dashed curves A and B. It is rather obvious from these curves that in no case can the effect of the singularity be avoided due to the presence of the term containing the configuration factor β. This is the reason why all the essential mechanisms—linear or nonlinear—that influence the

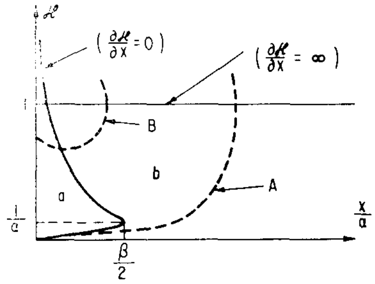

FIG. 2.

ambipolar flow can be demonstrated and discussed in the plane-parallel case.

The inertia-controlled diffusion is directly analogous to the flow of a fluid with friction through a contracting nozzle. The two cases become identical if one assumes that the cross section of the nozzle varies in such a fashion that

$$\frac{1}{A}\frac{\partial A}{\partial r} = -\frac{\nu_i}{u}, \tag{33}$$

and provided the ionization frequency ν_i is small in comparison with the ion–neutral collision frequency ν_M. It is a well-known fact that the limiting flow velocity at the exit of a contracting nozzle is the sound velocity just as has been shown to be the case for the inertia-controlled diffusion.

The significance of the theory of the inertia-controlled diffusion is primarily conceptual. The linear-diffusion problem is solved as an eigenvalue problem leading to a series of solutions. Of these solutions the fundamental mode is picked out as the physical solution since the density of the diffusing medium cannot be negative. The drift velocity of the diffusing fluid is infinite at the boundary of this fundamental mode when the boundary acts like a perfect sink. This is physically absurd and is corrected with the help of Bohm's criterion, which essentially says that the drift velocity of the diffusing medium at the boundary cannot exceed the local sound velocity of the electron–ion gas. These additional assumptions, that must be added to the linear diffusion theory in order to get the physically correct description, appear as integral parts of the inertia-controlled diffusion theory. The description of the diffusion—based on the linear-diffusion theory—is acceptable when the fundamental mode is used and provided the boundaries are far from being perfect sinks. The difference between the two theories is caused by the nonlinear terms. These terms are most important close to the boundary and determine the boundary condition when the boundary acts like a perfect sink. All cold wall and boundaries—excluding the electrodes in the dc and low-frequency cases—are generally considered as perfect sinks for the electron–ion gas; hence, the importance of the inertia-controlled ambipolar diffusion.

The production parameter $\alpha = \nu_M/\nu_i$ is more than two orders of magnitude larger than unity in most practical cases. The numerical results obtained with the inertia-controlled-diffusion theory are then practically indistinguishable from the results obtained with the linear-diffusion theory provided the fundamental mode is used and Bohm's criterion is added. It is important to use the inertia-controlled-diffusion theory when the production parameter is relatively small and when the heat-conduction mechanism is included. The play between the inertia mechanism and the heat-conduction mechanism leads to phenomena which cannot be accounted for in the absence of one or both of these mechanisms.

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