Nonlinear Ambipolar Diffusion of an Isothermal Plasma across a Magnetic Field

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Solutions are presented of the first two moment equations, including nonlinear terms, for the ambipolar diffusion of an isothermal plasma across a magnetic field. The two geometries considered are the plane parallel case and the infinite cylinder with axial symmetry. The Bohm criterion is automatically satisfied by the solutions. It is shown that the singularity in the space derivative of the ambipolar drift velocity at the plasma boundary cannot be removed by an axial magnetic field of any strength. Thus the plasma drift velocity and the plasma density remain monotonic functions of the position coordinate. It is also shown, under the assumptions of this theory, that the ambipolar space charge field is always directed outward and does not reverse direction in this isothermal approximation even for extremely high magnetic fields. One is forced to conclude that a realistic theory of ambipolar diffusion requires the consideration of thermal gradients within the plasma.

THE traditional approach to a theoretical treatment of the ambipolar diffusion of a plasma to the walls of its container has been to start with the general transport equation, use the first and second moment equations neglecting all nonlinearities, and then impose the boundary condition that the electron and ion densities are zero at the wall. Such an assumption leads to the nonphysical result that, in order to assure mass transport across the boundary, the diffusion velocity must be infinite at the boundary.

In a previous paper,[1] it was pointed out that, from a treatment of the complete first and second moment equations, including all nonlinearities, one obtains a solution, without imposing boundary conditions, which automatically obeys Bohm's criterion.[2] That is, the ambipolar diffusion velocity at the boundary is equal to the ambipolar thermal velocity, $v = (kT^*/M)^{\frac{1}{2}}$, where T^* is the sum of

[1] K.-B. Persson, Phys. Fluids 5, 1625 (1962).

[2] A. Guthrie and R. K. Wakerling, *Characteristics of Electrical Discharges in Magnetic Fields* (McGraw-Hill Book Company, Inc., New York, 1949), p. 77.

electron and ion temperatures and M is the ion mass.

In the present paper, the equations are extended to include the effects of an applied magnetic field, including the shielding of this magnetic field due to the diamagnetism of the plasma. As before, the starting point is the general transport equation.[3] We immediately restrict our consideration to two specific geometries: (1) the plane parallel case with symmetry about the central plane, and (2) the infinite cylinder with axial symmetry. By using a configuration parameter β equal to 0 or 1 respectively for these two cases, one can express both geometries in terms of a single set of differential equations. The above assumption of symmetry requires that all spacial derivatives be zero except for the derivative with respect to the first coordinate which, in both cases, is the coordinate perpendicular to the plasma boundary.

Considering the plasma as a three-component gas consisting of electrons, one species of positive ion and the background neutral gas, we then write the first and second moment equations for each component in the following form:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^\beta} \frac{\partial}{\partial r} (r^\beta \rho w_1) = \rho \nu_s, \tag{1}$$

$$\frac{\partial}{\partial t} \rho_m w_m^i + \left[\delta_\alpha^i \delta_1^i \frac{1}{r^\beta} \frac{\partial}{\partial r} r^\beta - \frac{\beta}{r} \delta_1^i \delta_2^\alpha \delta_2^i + \frac{\beta}{r} \delta_2^i \delta_2^\alpha \delta_1^i \right]$$

$$\cdot [\rho_m w_m^\alpha w_m^i + P_m^{\alpha i}] + \nu_m \rho_m w_m^i + \nu_{mM} \rho_m (w_m^i - w_M^i)$$

$$= -\frac{\rho_m}{m} e[E^i + e^{ijk} w_{mj} B_k], \tag{2}$$

$$\frac{\partial}{\partial t} \rho_M w_M^i \left[\delta_\alpha^i \delta_1^i \frac{1}{r^\beta} \frac{\partial}{\partial r} r^\beta - \frac{\beta}{r} \delta_1^i \delta_2^\alpha \delta_2^i + \frac{\beta}{r} \delta_2^i \delta_2^\alpha \delta_1^i \right]$$

$$\cdot [\rho_M w_M^\alpha w_M^i + P_M^{\alpha i}] + \nu_M \rho_M w_M^i - \nu_{mM} \rho_m (w_m^i - w_M^i)$$

$$= +\frac{\rho_M}{M} e[E^i + e^{ijk} w_{Mj} B_k]. \tag{3}$$

In Eqs. (2) and (3), the first bracket is to be considered as an operator acting on the second bracket. The Kronecker δ and e^{ijk} symbol are defined in Ref. (4). The drift velocity of the neutral gas is here assumed to be negligible relative to the ambipolar drift velocity of the ionized component of the plasma, in which case the moment equation of the neutral gas can be neglected. In the above equations n is the particle density (electron or ion);

ρ is the mass density; ν, the charged particle–neutral collision frequency; T, the temperature: w_i is the ith component of the mean particle velocity averaged over the velocity distribution; and $P^{\alpha i}$ is the electron or ion pressure. The subscripts m and M are used with the above symbols to distinguish between electron and ion quantities. The symbol ν_{mM} refers to the electron–ion collision frequency and ν_s, to the frequency of ionizing electron–neutral collisions.

Equations of this type for both ions and electrons are combined into sum and difference equations, the difference equations being formed by multiplying the ion equation by m/M and from this subtracting the electron equation.

By combining the electron and ion equations in this manner, the resulting equations are in terms of the following macroscopic quantities: the mass density of the ionized component, $\rho = \rho_m + \rho_M$, the net space charge density, $q = e(\rho_M/M - \rho_m/m)$, the mass flow of the ionized component, $\rho u^i \equiv \rho_m w_m^i + \rho_M w_M^i$, and the current density, $J_i = e(\rho_M w_M^i/M - \rho_m w_m^i/m)$. These are the quantities which describe the macroscopic behavior of the plasma and which are, at least in principle, directly measurable. In order to simplify the resulting equations, it is assumed that $q/e \ll \rho/M$, i.e., that the ambipolar condition holds and thus that the Debye length is much smaller than the smallest dimension of the plasma. In addition, we assume $u_2 = u_3 = 0$, $J_1 = 0$, $B_1 = B_2 = 0$, and $E_2 \doteq 0$. That is, mass transport is only allowed perpendicular to the boundary[5]; current flow perpendicular to the boundary is not allowed; and only an axial magnetic field is considered, that is, the induced magnetic field, B_2, due to J_3, is neglected relative to B_3. We are thus considering here a plasma maintained by a relatively small current in the z direction. The condition on E_2 follows directly from Maxwell's equations and the symmetry assumptions. The electron and ion pressures, as well as the collision frequencies, are taken to be scalar quantities.

Under these assumptions the following equations result (where u is written for u_1).
The sum equation for $i = 1$ becomes

$$\frac{\partial}{\partial t} \rho u + \frac{m \nu_m + M \nu_M}{M + m} \rho u + \frac{\partial}{\partial r} \rho v^2 + \frac{1}{r^\beta} \frac{\partial}{\partial r} (r^\beta \rho u^2)$$

$$= J_2 \left(B_3 + \beta \frac{mM}{re^2} \frac{J_2}{\rho} \right). \tag{4}$$

[3] S. Chapman and T. G. Cowling, *Mathematical Theory of Non-Uniform Gases* (Cambridge University Press, London, 1960), p. 48.
[4] I. S. Sokolnikoff, *Tensor Analysis* (John Wiley & Sons, Inc., New York, 1951).

[5] Objection may be made to the excluding of u_2 in the combined presence of nonvanishing u_1 and B_3. However, it can be shown that terms involving u_2 are negligibly small relative to the other terms in the equation.

The difference equation for $i = 1$ becomes

$$\frac{e\rho}{M} E_1 = J_2\left(B_3 + \beta \frac{mM}{re^2} \frac{J_2}{\rho}\right) - \frac{\partial}{\partial r} \rho v^2$$

$$+ \frac{m(\nu_M - \nu_m)}{M + m} \rho u. \quad (5)$$

The difference equation for $i = 2$ becomes

$$\frac{\partial}{\partial t} J_2 + (\nu_m + \nu_{mM})J_2 + \frac{1}{r^\beta} \frac{\partial}{\partial r} (r^\beta u J_2)$$

$$= -\frac{e^2 \rho}{mM} u\left(B_3 + \beta \frac{mM}{re^2} \frac{J_2}{\rho}\right). \quad (6)$$

The difference equation for $i = 3$ becomes

$$\frac{\partial}{\partial t} J_3 + (\nu_m + \nu_{mM})J_3 = \frac{e^2 \rho}{mM} E_3 - \frac{1}{r^\beta} \frac{\partial}{\partial r} (r^\beta u J_3), \quad (7)$$

and the continuity equation is

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^\beta} \frac{\partial}{\partial r} (r^\beta \rho u) = \rho \nu_s. \quad (8)$$

Here v, the ambipolar sound velocity is given by $(kT^*/M)^{\frac{1}{2}}$ in which T^* is the ambipolar temperature, $T_m + T_M$.

Under the further assumption that the plasma be isothermal and that the time derivatives be zero, these equations can be written in the following dimensionless form:

$$\frac{dy}{d\kappa} = \frac{1 - \kappa^2}{\left\{\frac{M}{M + m}\left[\frac{1}{\delta} + \frac{m}{M}\left(\frac{1}{\eta^2} + \frac{1}{\xi}\right)\right] + \Gamma\zeta_0(1 - \epsilon)\left[1 - \beta \frac{\kappa}{y}\zeta_0(1 - \epsilon)\right]\right\}\kappa^2 + \frac{1}{\eta^2} - \beta \frac{\kappa}{y}}, \quad (9)$$

$$\frac{e}{M(\nu_m + \nu_{mM})} \frac{E_1}{\kappa v} = \frac{d\kappa}{dy} + \frac{1}{\delta}, \quad (10)$$

$$\frac{d\epsilon}{d\kappa} = \frac{1 - \epsilon}{\kappa} + \frac{1}{\kappa} \frac{dy}{d\kappa}\left[(1 - \epsilon)\left(1 + \frac{1}{\eta^2} + \beta \frac{\kappa}{y}\right)\right.$$

$$\left. - \frac{1}{\zeta_0} + \zeta_0 C W \kappa (1 - \epsilon)^2\right], \quad (11)$$

$$\frac{dW}{d\kappa} = \left(\frac{1}{\eta^2} - \beta \frac{\kappa}{y}\right) \frac{W}{\kappa} \frac{dy}{d\kappa}, \quad (12)$$

$$\frac{d\Gamma}{d\kappa} = 2C\zeta_0(1 - \epsilon)\Gamma W \frac{dy}{d\kappa}, \quad (13)$$

$$C = \frac{\omega_{p0}^2}{(\nu_m + \nu_{mM})^2}\left(\frac{v}{c}\right)^2,$$

$$\delta \equiv \frac{\nu_m + \nu_{mM}}{\nu_M + \nu_s}, \qquad \eta^2 \equiv \frac{\nu_m + \nu_{mM}}{\nu_s},$$

$$\zeta_0 \equiv \frac{\eta^2}{\eta^2 + 2}, \qquad \xi \equiv \frac{\nu_m + \nu_{mM}}{\nu_m},$$

$\omega_p^2 \equiv e^2\rho/mM\epsilon_0$, with units of sec^{-2}, is the square of the plasma frequency, where ϵ_0 is the dielectric constant of free space; ω_{p0} is the plasma frequency at the center of the plasma; $\Omega_g \equiv eB/(mM)^{\frac{1}{2}}$, with units of sec^{-1}, is the hybrid gyrofrequency; and c is the velocity of light in free space.

No boundary conditions were forced on the equations. Instead, the "initial" conditions at the center of the plasma were used and Eqs. (9) and (11)–(13) were solved simultaneously, with κ as the independent variable, on an IBM 7090 computer using the Runge–Kutta method to calculate out from the center of the plasma. The machine solution of the epsilon equation was unstable unless the calculation was started at some nonzero value of κ. Accurate asymptotic forms were therefore used to obtain the appropriate starting values of y, ϵ, W, and Γ. When the starting value of ϵ was varied from its correct value, the epsilon solution always converged rapidly to the same function, provided only that the incrementation in κ was kept sufficiently small. The ϵ-function is thus apparently an eigenfunction, there being only one solution of the epsilon differential equation for a given set of parameters. This is not true for the other variables which exhibit a whole family of solutions depending on the initial value assumed.

where the various dimensionless quantities are: $\kappa \equiv u/v$, the ambipolar drift velocity normalized to the ambipolar thermal velocity; $y \equiv (\nu_m + \nu_{mM})r/v$, a dimensionless variable proportional to position within the plasma;

$$\epsilon \equiv 1 + \frac{\eta^2 + 2}{\eta^2} \frac{(\nu_m + \nu_{mM})}{\Omega_g^2} \frac{J_2 B_3}{\rho u}$$

a Hall-type quantity depending on the interaction of a crossed drift velocity and magnetic field; $\Gamma \equiv [\Omega_g/(\nu_m + \nu_{mM})]^2$, a magnetic parameter proportional to the square of the magnetic field; $\Phi \equiv \rho/\rho_0$, the mass density of the ionized component of the plasma normalized to its value at the center of the plasma; $W \equiv \Phi\kappa$, a dimensionless variable proportional to the mass flow of the ionized component;
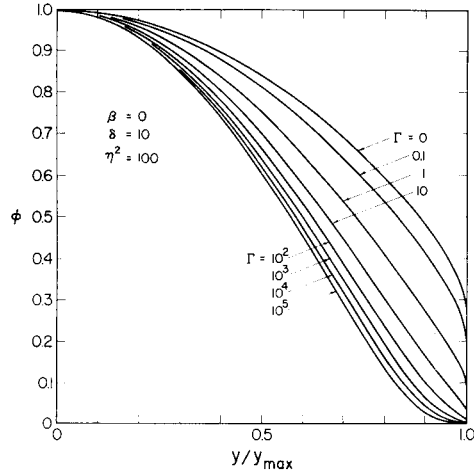
FIG. 1. Normalized mass density of the ionized component of the plasma as a function of normalized position coordinate for the plane parallel case where $C = 0$.

For most laboratory plasmas ($n_e < 10^{11}$ cm^{-3}, $P \geq 1$ Torr, and $T_e \leq 1$ eV, hence $C \leq 10^{-5}$) the diamagnetic effect of the plasma is quite small. For $C \leq 10^{-3}$, the solutions were negligibly different from those where $C = 0$. Most of the calculations were therefore carried out under the assumption $C = 0$. Figures 1–3 show the result of calculations for the case $\delta = 10$, $\eta^2 = 100$ which are parameters very roughly appropriate for helium. These calculations are for the plane parallel case and neglect the diamagnetism of the plasma. The variation in the curves as the magnetic field is changed is somewhat smaller for the case of cylindrical geometry, $\beta = 1$, but qualitatively the effects are the same. In Fig. 4, the form of Φ is shown for $\delta = 0.1$ for various values of η^2. Note that as $\eta^2 \to \infty$ the form
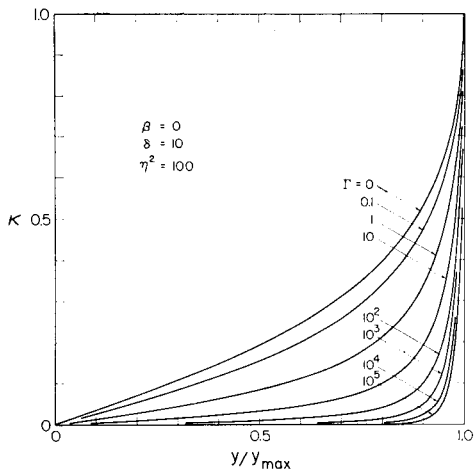

FIG. 2. Normalized ambipolar drift velocity as a function of normalized position coordinate for the plane parallel case where $C = 0$.

of Φ approaches that of the linear theory. We have here assumed, in deciding on appropriate values for δ, that $\nu_{mM} \ll \nu_m$ so that $\xi = 1$. This assumption is not necessary and, in fact, directly influences only Eq. (9).

The form of Eq. (9) indicates that κ is a monotonically increasing function of y until a singularity is reached at some critical value of y for which $\kappa = 1$ (i.e., $u = v$). Beyond this critical value of y, no solution for κ exists. The possibility, however, remains that if $\epsilon > 1$ for some range of y, then the singularity at $\kappa = 1$ might be removed. It was of considerable interest, therefore, to investigate the behavior of $\epsilon(\kappa)$ as a function of applied magnetic field.
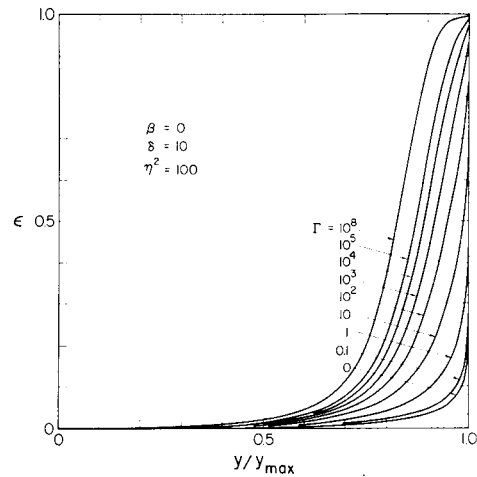

FIG. 3. The function ϵ as a function of normalized position coordinate for the plane parallel case where $C = 0$.

Defining the quantity $Z = \kappa(1 - \epsilon)$ and substituting in Eq. (11), we obtain

$$\frac{dZ}{d\kappa} = \frac{dy}{d\kappa}\left[\frac{1}{\zeta_0} - \frac{Z}{\kappa}\left(1 + \frac{1}{\eta^2} + \beta\frac{\kappa}{y}\right) - \zeta_0 C\Phi Z^2\right]. \quad (14)$$

In order to show that ϵ remains positive but less than unity over the range of κ, it is sufficient to show that Z is always > 0. We therefore investigate the derivative field of Eq. (14) over the open interval $0 < \kappa < 1$ by setting $dZ/d\kappa = 0$ and solving for $Z_0(\kappa)$ which is then the value of Z for which $dZ/d\kappa = 0$. We note that, as $\kappa \to 0$, $0 \leq Z < Z_0 \to 0$ and that also $0 < dZ/d\kappa < dZ_0/d\kappa$. Thus near $\kappa = 0$, $Z(\kappa)$ is positive and is trapped below $Z_0(\kappa)$ in a region where its derivative is positive. The function Z can decrease only if $Z_0(\kappa)$ has a maximum within the range of κ. It can be shown that no such maximum exists and therefore that Z remains positive over the range of κ.

Two features are of particular interest in the behavior of the space charge field as given by Eq. (10), which represents the equilibrium between the electron and ion pressures and the local force fields. First the derivative $d\kappa/dy$ has a minimum value of $1/\eta^2(\beta + 1)$ as $\kappa \to 0$ and thus the space charge field is always positive regardless of the strength of applied magnetic field. Secondly, this derivative has a singularity at $\kappa = 1$ so that the space charge field becomes infinite at the plasma boundary. This nonphysical result implies that the isothermal assumption must be eliminated and temperature gradients considered near the boundary.

It remains to discuss the solution for very large η^2, that is for a very low percentage of ionizing collisions. As η^2 increases, the machine solutions show that $\epsilon \ll 1$ farther and farther out toward the boundary. In order to obtain the asymptotic form of the solutions for $\beta = 0$ as $\eta^2 \to \infty$, we therefore solve the equations under the assumption $\epsilon = 0$. The resulting analytic solutions are those of linear diffusion theory. For $\beta = 1$, the solutions also converge toward the linear solutions as seen in Fig. 4. Only when η^2 is relatively small or the magnetic field high are the deviations from the linear theory very large. Where these conditions are not involved, the value of the theory is mainly conceptual.

CONCLUSIONS

It has been demonstrated that a uniform magnetic field is incapable of removing the singularity in $d\kappa/dy$ at $\kappa = 1$. Thus the plasma variables remain monotonic functions of the position coordinate.

Calculation of the space-charge field shows that reversal of the field (field directed inward) is not possible in the ambipolar isothermal case with nonconducting boundaries, under the symmetry assumptions used here.

Note that in Eq. (10), the space-charge field approaches infinity as $\kappa \to 1$. This singularity in the space-charge field (which is then also a singularity in energy density), occurring in the isothermal approximation at the maximum velocity, $u = v$, is analogous to the singularity in the velocity or momentum which occurs in the linear theory at the maximum position, namely the plasma boundary.

Just as one removes the singularity in the linear theory by allowing a nonzero plasma density at the boundary, so one would expect to remove the
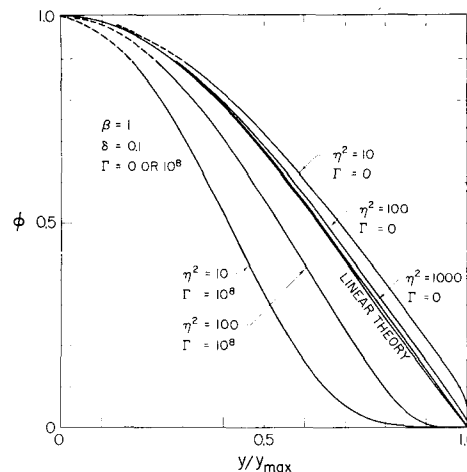

FIG. 4. Normalized mass density of the ionized component of the plasma as a function of normalized position coordinate for cylindrical geometry with axial symmetry showing the approach to $J_0(2.4048\ y/y_{max})$, the solution of the linear theory, as $\eta^2 \to \infty$.

singularity in energy density (i.e., in space-charge field) by allowing a non-zero thermal conductivity at the boundary. That is, one must consider temperature gradients within the plasma.

That the isothermal equations of the present paper are still a nonrealistic description of a plasma is also illustrated by the following. For given δ, η^2, and Γ, the value of y_{max} is uniquely determined by the machine solution. But $y_{max} = (\nu_m + \nu_{mM})R/v$. Thus for a plasma of given dimension R and given temperature, with a given applied magnetic field, the electron collision frequency is fixed. Thus the neutral gas density is not separately variable. In other words, the isothermal condition is not realizable for all combinations of plasma temperature, size, and neutral gas density. We conclude, therefore, that the isothermal requirement on the plasma must be dropped and energy transport through the plasma must be considered in order to obtain a physically realistic theory of the ambipolar diffusion of a plasma to its boundaries.