STATISTICAL MODELING AND FILTERING FOR OPTIMUM ATOMIC TIME SCALE GENERATION

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ABSTRACT

Statistical models for the fractional frequency fluctuations in atomic clocks, clock ensembles, and some of the propagation media are developed. Using these models, near optimum time prediction algorithms are employed to generate time for a clock ensemble or for a set of laboratories' time scales. An example using data from the BIH Circular D bulletin is illustrated and the results compared with IAT.

Accuracy and uniformity problems are considered in light of the CCDS June 1970 recommendations. A model for an evaluable primary frequency standard is developed as well as for a time scale (flywheel frequency standard). It is shown that under certain conditions the accuracy of a time scale can be better than the accuracy of the primary standard for the current calibration, if there is a sufficient number of independent past calibrations. A method of simultaneously achieving accuracy and uniformity is discussed.

Key Words: Clock stability model; Frequency calibration; Frequency stability; International time scale; Time scale accuracy; Time scale stability.

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OUTLINE

I. Introduction: Intriguing concept of time.

   A. Four points to be made

      1. The stability characteristics of clocks can be well-modeled

      2. Optimum algorithm? The choice depends on assumptions

      3. Time scale accuracy can be better than that of primary (evaluated) frequency standard

      4. Uniformity and accuracy are compatible

   B. Methods

      1. Time Scale Algorithm--presented in part at the 24th Annual Symposium on Frequency Control [1]

      2. International example: IAT. Member clocks are the contributing laboratories as seen through transmissions

II. Modeling of Clocks (Define a clock as a frequency standard plus divider or counter)

   A. Define terms: [2], [3]

      1. Accuracy: The degree of conformity of a measured and/or calculated value to some specified value or definition

      2. Stability: Frequency Domain or Time Domain behavior of a process (a description of the degree of uniformity in time)

      3. Reproducibility: The degree of agreement across a set of independent devices of the same design after adjustment of appropriate specified parameters

      4. Precision: The uncertainty associated with measurement process

   B. Time and/or Frequency Dispersion Catagorization

      1. Deterministic (drift, offset, biases)

         a. IAT: Rate and origin arbitrarily chosen. Independent rate for each contributing laboratory's time scale

         b. In principle at least gh/c^2 frequency difference between laboratories

         c. Accuracy and reproducibility concepts important to describe deterministic effects.

2. Random (non-deterministic)

a. Described statistically (may yield information on deterministic influences)

b. $\langle \sigma_y^2(N, T, \tau, f_B) \rangle$ where $y \equiv \frac{\delta\nu}{\nu}$, $N \equiv$ number of sequential data points used in variance, $T \equiv$ period of data sampling, $\tau \equiv$ nominal sampling time for each data point, $f_B \equiv$ system bandwidth, and angular brackets "$\langle \rangle$" denote the time average

c. $S_y(f) \equiv$ Fourier frequency noise intensity of the fractional frequency fluctuations

d. Define $\sigma_y^2(\tau) \equiv \langle \sigma_y^2(N = 2, T = \tau, \tau, f_B) \rangle$
If $\sigma_y^2(\tau) = a\,\tau^{\mu}$ and $S_y(f) = hf^{\alpha}$ then straight-forward relationships exist between a and h and between μ and α [4], [5]

e. Experimentally verified for $\mu = +2, +1, 0, -1, -2$. Discuss noise processes involved for each μ and the advantage of a log σ versus log τ plot

C. Examples of Statistical Modeling

1. Frequency standards and clocks

a. $\sigma_y(\tau)$ for Cs, Rb, CH_4, H, Quartz (actual data measured at NBS) (Fig. 1)

b. Barger and Hall have data for $\sigma_y(\tau)$ for CH_4 at $\sim 3 \times 10^{-14}$ from 10^2 s to 10^4 s

c. JPL have data for $\sigma_y(\tau)$ on H at $\sim 1 \times 10^{-14}$ from 10^2 s to 10^6 s

2. Time dissemination noise

a. WWVL and WWVB measured at Palo Alto (Fig. 2)

b. Loran-C and TV; NBS $\leftrightarrow$ USNO (Fig. 3)

3. Seven laboratories each treated as independent clocks (from BIH Circular D)

a. $\sigma_i^2 = \frac{1}{2}[\sigma_{ij}^2 + \sigma_{ik}^2 - \sigma_{jk}^2]$. Stability of the ith laboratory's clock and transmission perturbation as calculated from stability comparisons with other independent time scales (j, k)

b. Laboratories and observatories involved: Physikalisch Technische Bundesanstalt (PTB), U. S. Naval Observatory (USNO), Paris Observatory† (OP), National Bureau of Standards (NBS), Royal Greenwich Observatory (RGO), Neuchatel Observatory (ON) (Fig. 4 to Fig. 10)

c. Discuss Propagation, Processing, Time Scale composition

D. Infer Spectral Density

1. Apply weighting and filtering to each laboratory's time scale (See Section III. C.)

III. Optimum Algorithm

A. Large number of algorithms--depends upon assumptions

1. Present NBS approach--minimize squared error and deterministic effects

2. Work being done at NBS on least biased estimate approach and on a systems analysis approach

B. Construction and Output of NBS Time Scale System

1. Evaluated primary standard and n-clock ensemble

2. AT(NBS) is independent; SAT(NBS) and UTC(NBS) were combined on 1 January 1972 and continued to be coordinated in rate with UTC(USNO). UTC(NBS) now coordinated with UTC(BIH) also.

C. Basic Algorithm Concepts

1. Algorithm rate equation: $\hat{T}_{ie}(t+\tau) = \tau \cdot \hat{R}_{ie}(t) + T_{ie}(t)$, where

$$\hat{R}_{ie}(t) = \sum_{m=o}^{-\infty} W_i(m) \cdot R_{ie}(t + m\tau).$$

$\hat{T}_{ie}(t+\tau)$ is the minimum squared error prediction time for i^{th} clock with respect to ensemble at the time $t+\tau$; τ is the prediction interval; $\hat{R}_{ie}(t)$ is optimum rate of i^{th} clock with respect to ensemble, i. e. the rate that will give the minimum squared error of prediction; $W_i(m)$ is the optimum weight to multiply the rate, $R_{ie}(t+m\tau)$, for the i^{th} clock. Note, m takes on only negative values.

† The atomic time scale for France AT(F) which contributes to the International Atomic Time Scale IAT(BIH) is composed of seven commercial cesium standards located at various laboratories and observatories in France. The Circular D bulletin does not list the apparent time of the AT(F) scale, but does list it for the OP scale which has three of the seven standards.

2. Algorithm time equation:

$$\hat{T}_e(t+\tau) = \sum_{i=1}^{n} w_i(t+\tau) \cdot \hat{T}_e^i(t+\tau),$$

where $\hat{T}_e(t+\tau)$ is the optimum estimate of ensemble time at $(t+\tau)$, and $w_i(t+\tau)$ is the optimum weight to multiply the optimum estimate of ensemble time as given by the i^{th} clock, $\hat{T}_e^i(t+\tau)$, as inferred from $\hat{T}_{ie}(t+\tau)$

3. For the example, W_i depends primarily on Lab T-Scale stability and w_i depends primarily on propagation stability

4. Need to measure deterministic effects, i.e., time jumps, rate changes or jumps, drift, internal and external performance comparisons

IV. Accuracy Concepts

Assume that the random aspects of each of the past N frequency calibrations, $\nu_{c,n}$, can be modeled by white frequency noise; i.e., each calibration is independent of any other. Map each calibration forward to a time t via an independent time scale (flywheel). As illustrated in Fig. 11, an optimum frequency can be obtained utilizing all of the calibrations.

A. Frequency and Accuracy of International Atomic Time Scale (IAT)

1. Listed in Fig. 12 are the fractional frequencies of time scales contributing to IAT as of June 1971 and after some filtering of propagation noise.

2. The accuracies listed for the three laboratories' time scales with evaluated primary frequency standards (PTB, NRC,† and NBS) are as published by them. Accuracies for other time scales were those stated for the commercial frequency standards employed divided by the square root of the number of standards contributing to each of their time scales.* See Fig. 13 for plot of the tabulated values. Accuracies are not shown for time scales with non-evaluable primary frequency standards as the limits extend off the figure.

3. Applying equations in Fig. 11 gives $(+11.3 \pm 0.3) \times 10^{-13}$ for the relative frequency of IAT with respect to a near optimum estimate of absolute frequency as given by contributing laboratories.

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† In private conversation with Al Mungall, I learned that NRC's published value may be in part a 2σ accuracy.

* The procedure of dividing by the square root of the number of standards is statistically valid if the standards are independent and of the same quality.

B. Frequency and Accuracy* of an independent time scale with an evaluable primary frequency standard

1. Assuming multiple past calibrations we cannot map each calibration forward to a time t since usually there is not available an independent time scale for each calibration. A reasonable assumption, however, is that the uncertainties due to the time scale (flywheel frequency standard) are independent for each interval between calibrations. Therefore, after a particular calibration giving a frequency, $\nu_{c,1}$, we have as a near optimum frequency (compare Fig. 11):

$$\nu_{AT,1} = w_2\, \nu_{AT,2} + w_1\, \nu_{c,1}\,.$$

2. The weights w_1 and w_2 are as given in Fig. 11. However, for a single time scale the accuracies* are: $\sigma_1 (= \sigma_{c,1})$ is as determined for the current calibration and σ_2 is equal to the square root of the sum of the squares of the accuracy* of the time scale at the previous calibration, $\sigma_{AT,2}$, and of the stability of the time scale $\langle \sigma_y^2(2, T, \tau)\rangle^{1/2}$, where T is the time from the beginning of one calibration to the next, and τ is the sample time (duration of a calibration). The current time scale accuracy* then becomes:

$$\sigma_{AT,1} = \left[\frac{1}{\sigma_{c,1}^2} + \frac{1}{\langle \sigma_y^2(2, T, \tau)\rangle + \sigma_{AT,2}^2}\right]^{-1/2}.$$

Hence, under the above assumptions the accuracy* of the time scale is better than the accuracy* of the last frequency calibration, and will continue to improve as the number of calibrations increases--asymptoting toward the time scale stability.

3. Example using NRC*

   a. NRC adjusts the rate of their time scale to conform with the frequency of their primary (evaluated) cesium beam frequency standard nominally once a week.

   b. Figure 14 illustrates application of equation in IV. B. 2. above to the AT(NRC) time scale. The stability used is that inferred from Fig. 9. The duration of calibration (τ = 2 weeks) and the time from the beginning of one calibration to the next (T = 4 weeks) were arbitrarily chosen.

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*The accuracy considered herein assumes that each calibration is independent of any other. Frequency biases may persist in a given primary (evaluated) frequency standard and thus limit the accuracy. Therefore, such biases should be carefully considered at each laboratory.

V. Uniformity versus Accuracy

A. Review Consultative Committee for the Definition of the Second (CCDS) document of June 1970 [6]

1. p. S13. Mr. Guinot desires to receive on the part of CCDS an indication of how to have accuracy and uniformity simultaneously

2. Recommendation S 2: IAT to conform with the definition of the second

3. Item 3 of Mise en Pratique, p. S22. IAT to be stepped in rate only when it differs significantly from SI definition of the second. Rate steps are inconsistent with V. A. 1. and V. A. 2. above

B. From above results a model of the stability of IAT is achievable

1. Determine frequency, ν_{opt}, in an optimum way from all available calibrations

2. Determine the accuracy of this optimum frequency

3. Design a filter based on the above so that the frequency of IAT always gravitates toward the optimum estimate of frequency. Thus achieving uniformity and accuracy simultaneously consistent with V. A. 1. and V. A. 2. stated above.

VI. Conclusion

A. Modeling of general clocks over large range of sample time, τ, is practical

B. Near optimum algorithms can and should be implemented

C. Accuracy of a time scale can be better than an evaluated standard

D. Accuracy and stability are compatible through proper data processing

E. Recommend

1. More emphasis on evaluated primary frequency standards

2. Each lab should have independent scale

3. Correct by $\delta = \frac{gh}{c^2}$ plus deterministic effects as determined by BIH

4. All members of CCDS study further the accuracy and stability problems

References

[1] David W. Allan, James E. Gray, and Howard E. Machlan, "The NBS Atomic Time Scale System: AT(NBS), SAT(NBS), and UTC(NBS)" (Summary), Proc. of the 24th Annual Symposium on Frequency Control, Fort Monmouth, N. J., p. 361, 27-29 April 1970.

[2] A. O. McCoubrey, "A Survey of Atomic Frequency Standards," Proc. IEEE, Vol. 54, No. 2, p. 116, February 1966.

[3] R. E. Beehler, R. C. Mockler, and J. M. Richardson, "Cesium Beam Atomic Time and Frequency Standards," Metrologia, Vol. 1, No. 3, pp. 114-131, July 1965.

[4] David W. Allan, "Statistics of Atomic Frequency Standards," Proc. IEEE, Vol. 54, No. 2, pp. 221-230, February 1966.

[5] J. A. Barnes et al., "Characterization of Frequency Stability," IEEE Trans. on Instrumentation and Measurement, Vol. IM-20, No. 2, pp. 105-120, May 1971.

[6] Comité International des Poids et Mesures; Comité Consultatif pour la Definition de la Seconde; 5^{e} Session (18-19 juin) 1970.

Figure Captions

Fig. 1 Fractional frequency stability characteristics of some state-of-the-art frequency standards (actual data measured at NBS).

Fig. 2 Some fractional frequency stability characteristics of propagation media at LF and VLF.

Fig. 3 Some fractional frequency stability characteristics of Loran-C and TV line-10 time transfer system between Washington, D. C., and Boulder, Colorado, as well as of a cesium beam portable clock.

Fig. 4 through Fig. 10 The inferred fractional frequency stability from the BIH Circular D data of each of the contributing time scales plus propagation instabilities and assuming each time scale is independent of any other time scale. The contributing time scales are PTB, USNO, OP, NBS, RGO, NRC, and ON, and the stabilities are plotted in Figs. 4 - 10, respectively.

Fig. 11 An illustration of how multiple independent frequency calibrations can be utilized to improve the accuracy such that a time scale(s) accuracy can be better than the accuracy of the current frequency calibration.

Fig. 12 A tabulation, after some filtering of the propagation noise, of the fractional frequency deviations with respect to IAT [AT(BIH)] of contributors to the IAT [AT(BIH)] scale as of 5 June 1971. Also listed are estimates of the accuracy (1σ) for each scale's entry. The bottom row is a calculation of an optimum fractional frequency (in the sense of minimum squared error) of a combination of the seven scales along with an estimate of its accuracy (1σ).

Fig. 13 A plot of the fractional frequencies of the time scales of laboratories contributing in the IAT [AT(BIH)] scale as of June 1971. Assuming independence, a combined optimally weighted (in the sense of a minimum squared error) estimate of the fractional frequency from the contributors is plotted along with its accuracy (1σ).

Fig. 14 Using the NRC time scale as an example, illustrated is a plot of how the accuracy of their time scale would improve with an increasing number of calibrations spaced four weeks apart-- assuming each calibration has a duration of two weeks and is independent of any other calibration.

Some Experimental Frequency Stabilities

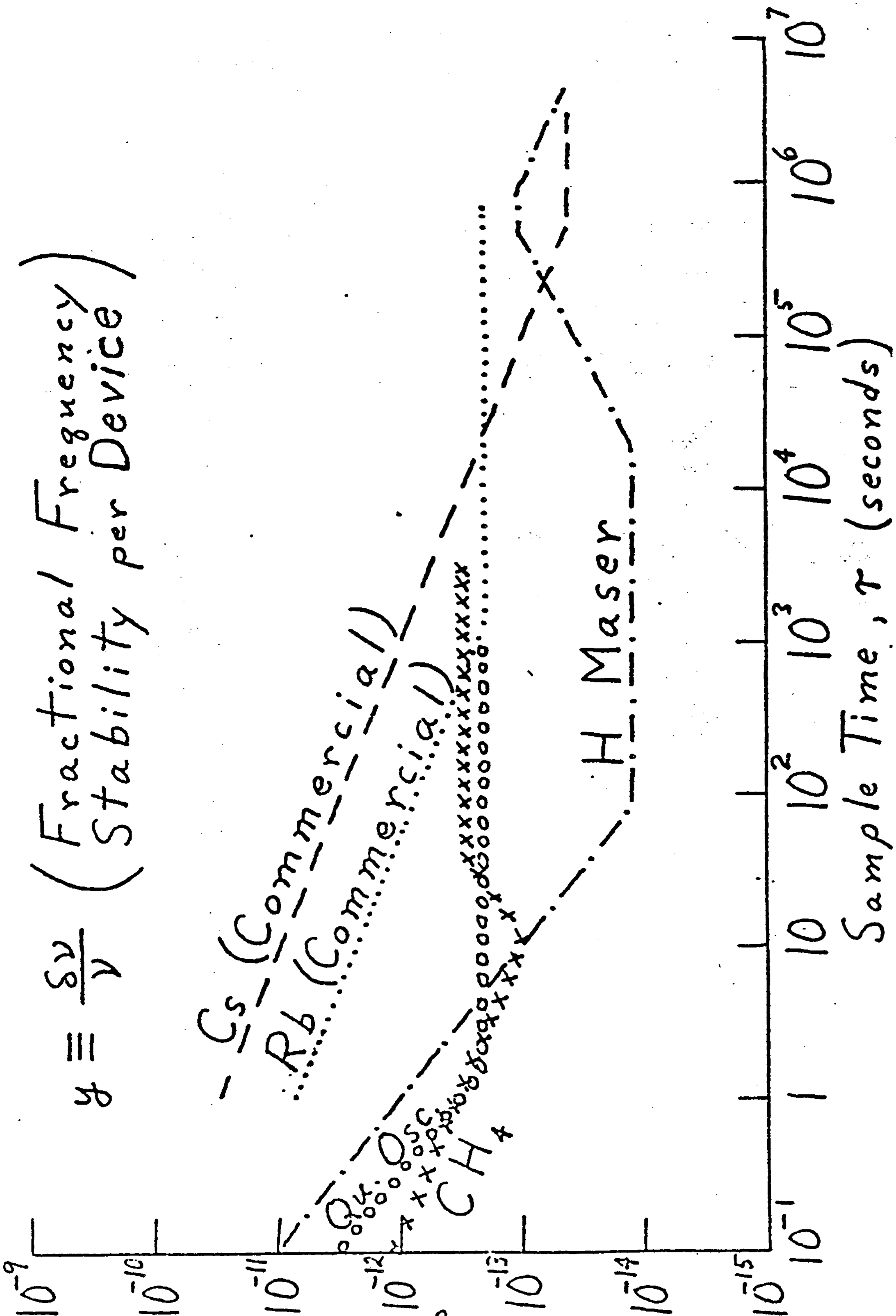


Figure 1

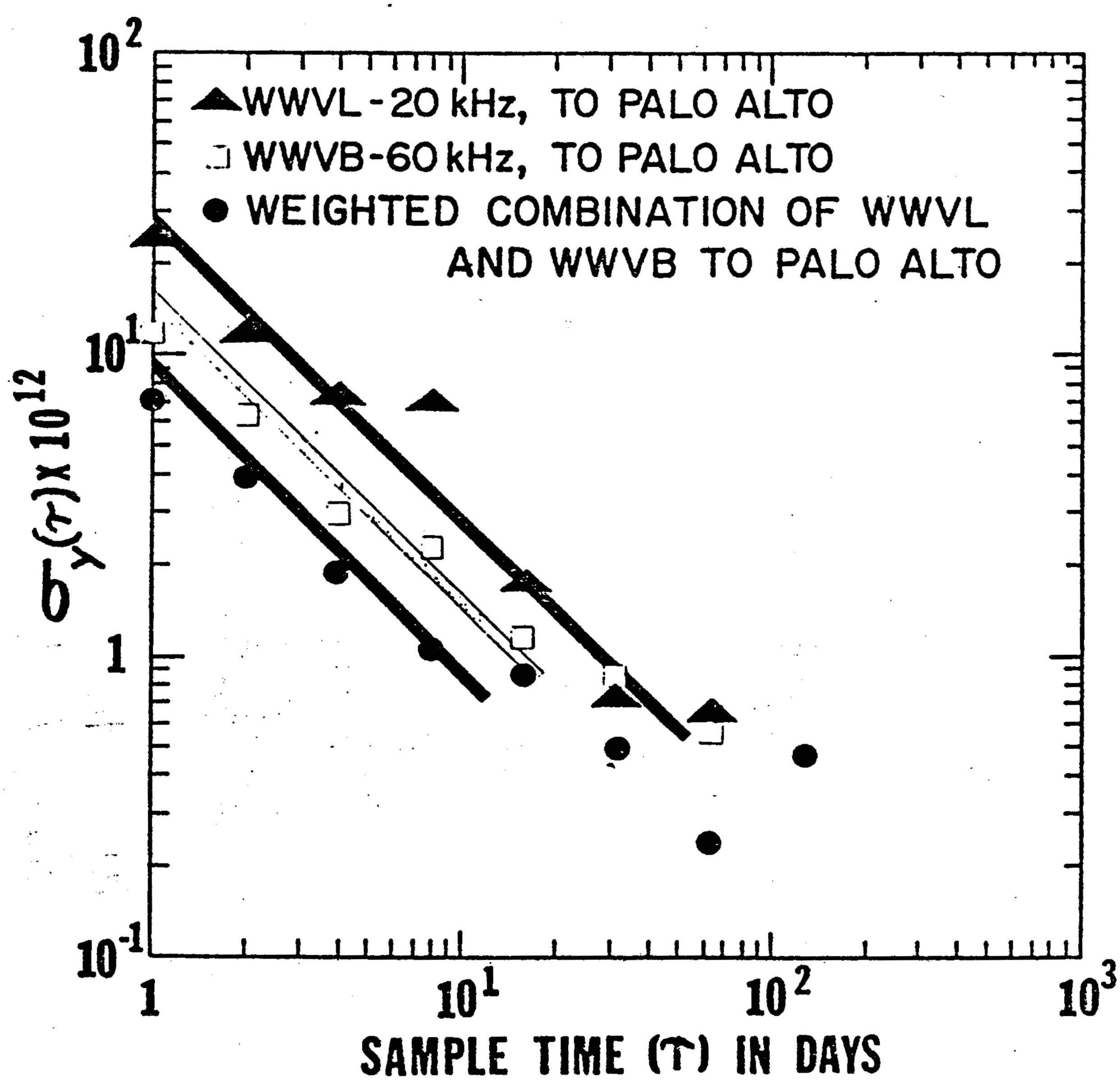

WWVL - 20 kHz, TO PALO ALTO
WWVB - 60 kHz, TO PALO ALTO
WEIGHTED COMBINATION OF WWVL AND WWVB TO PALO ALTO
$\sigma_y(\tau) \times 10^{12}$
10^2
10^1
1
10^{-1}
1
10^1
10^2
10^3
SAMPLE TIME (τ) IN DAYS

Figure 2

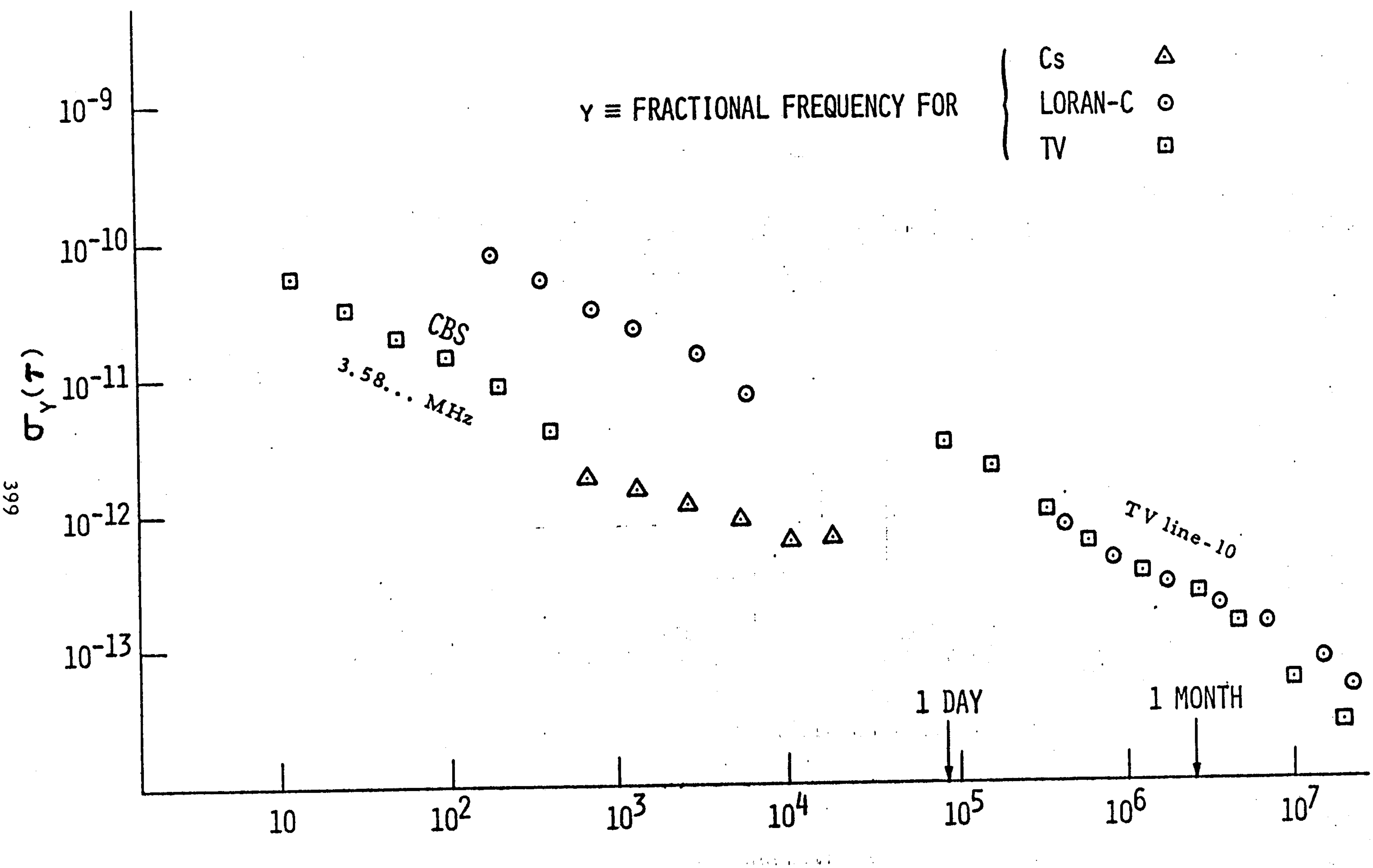

$Y \equiv$ FRACTIONAL FREQUENCY FOR
Cs
LORAN-C
TV
$\sigma_y(\tau)$
10^{-9}
10^{-10}
10^{-11}
10^{-12}
10^{-13}
CBS
3.58... MHz
TV line-10
1 DAY
1 MONTH
10
10^2
10^3
10^4
10^5
10^6
10^7
τ, SAMPLE TIME (SECONDS)

Figure 3

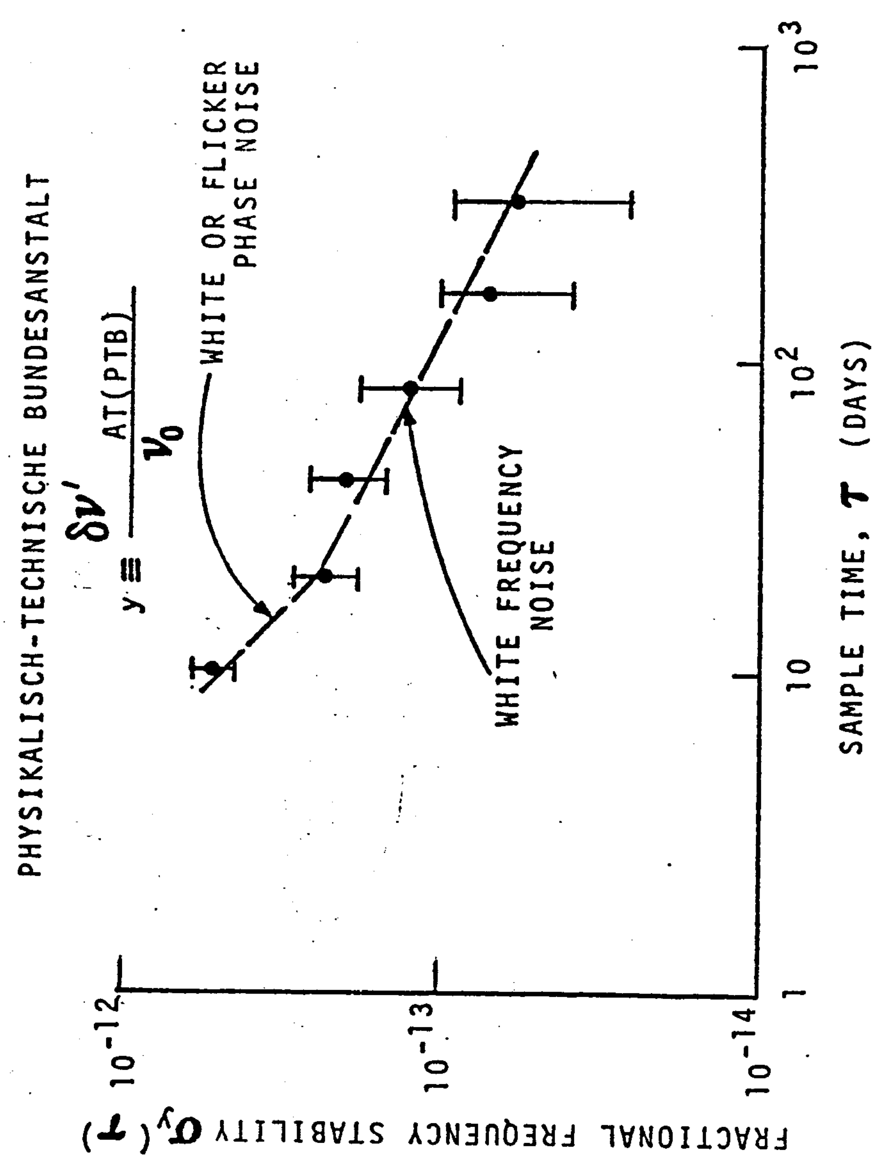

PHYSIKALISCH-TECHNISCHE BUNDESANSTALT
$y \equiv \frac{\delta \nu'}{\nu_0}$ AT(PTB)
WHITE OR FLICKER PHASE NOISE
WHITE FREQUENCY NOISE
FRACTIONAL FREQUENCY STABILITY $\sigma_y(\tau)$
10^{-12}
10^{-13}
10^{-14}
1
10
10^2
10^3
SAMPLE TIME, τ (DAYS)

Figure 4

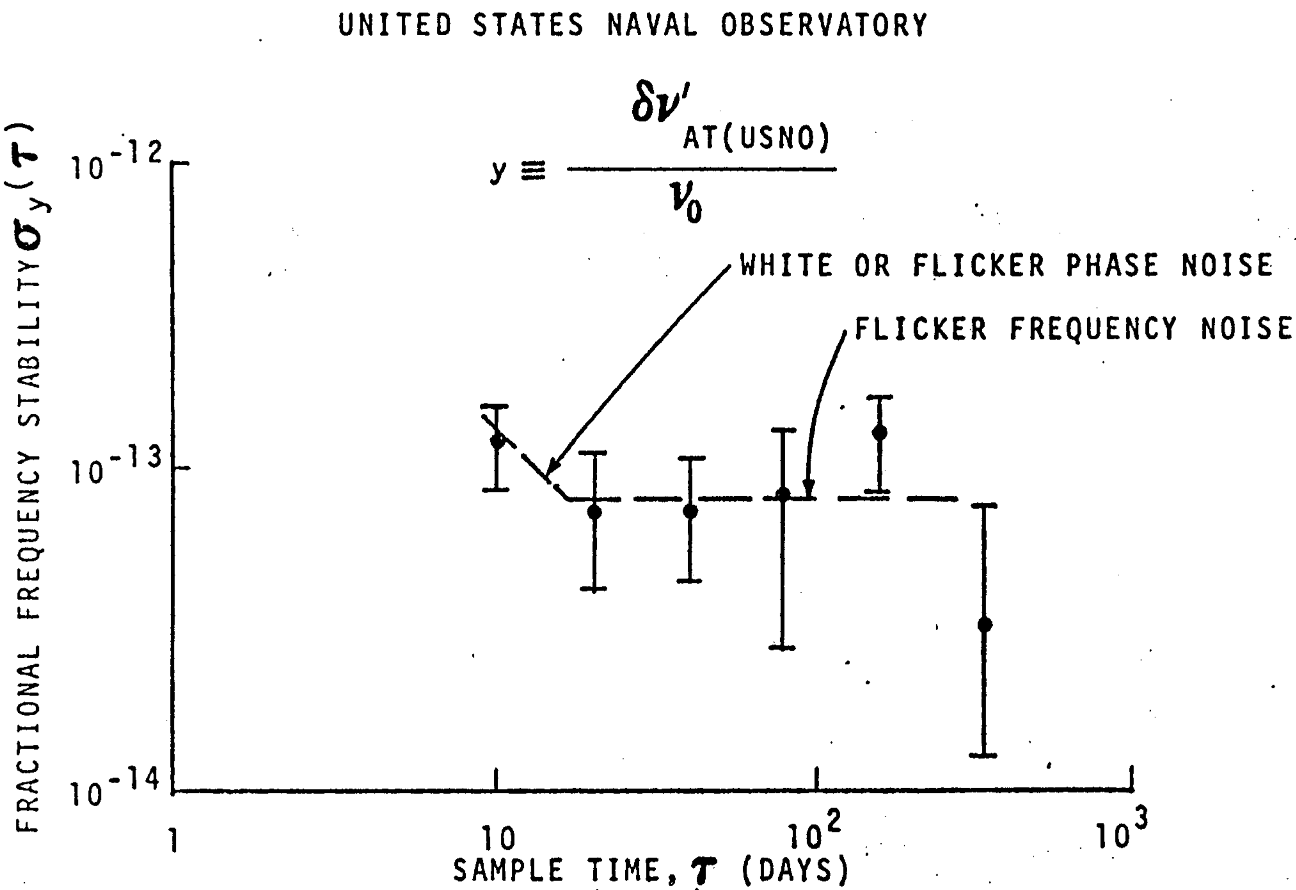

UNITED STATES NAVAL OBSERVATORY
$y \equiv \frac{\delta\nu'_{AT(USNO)}}{\nu_0}$
WHITE OR FLICKER PHASE NOISE
FLICKER FREQUENCY NOISE
FRACTIONAL FREQUENCY STABILITY $\sigma_y(\tau)$
10^{-12}
10^{-13}
10^{-14}
1
10
10^2
10^3
SAMPLE TIME, τ (DAYS)

Figure 5

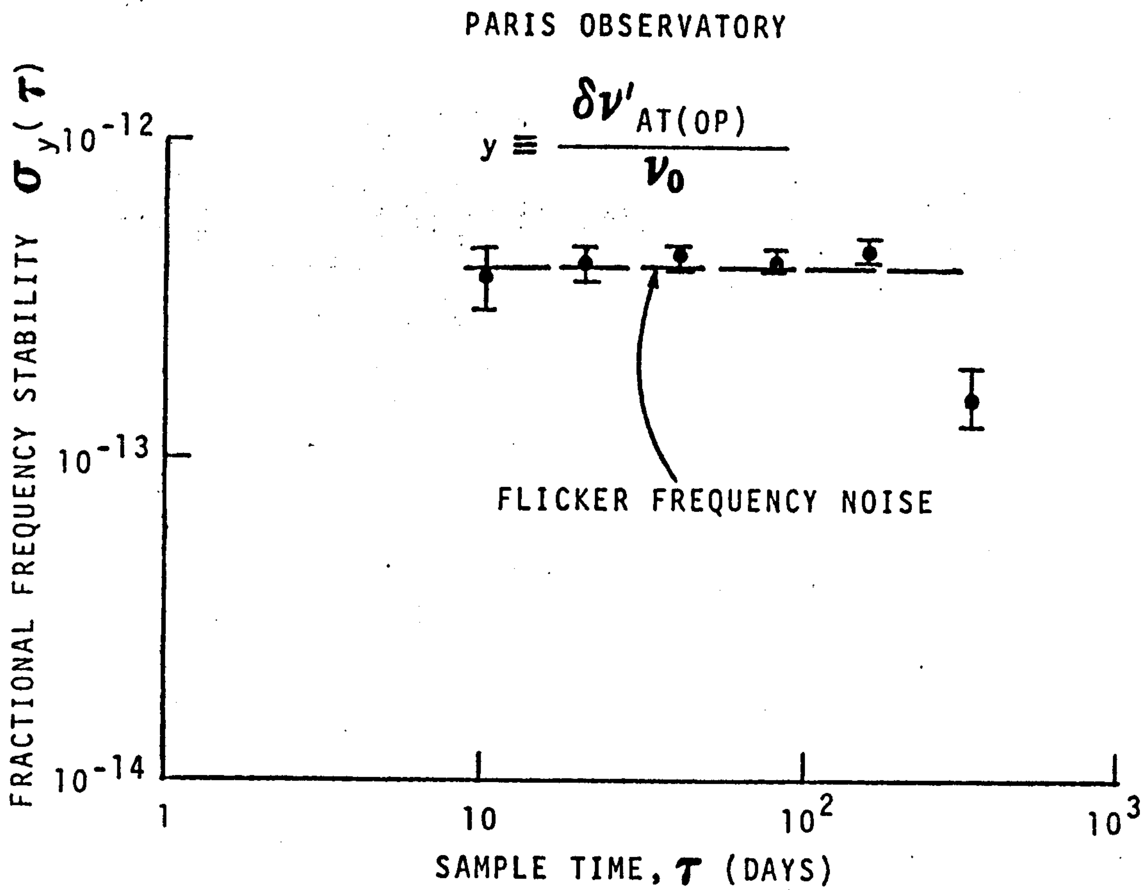
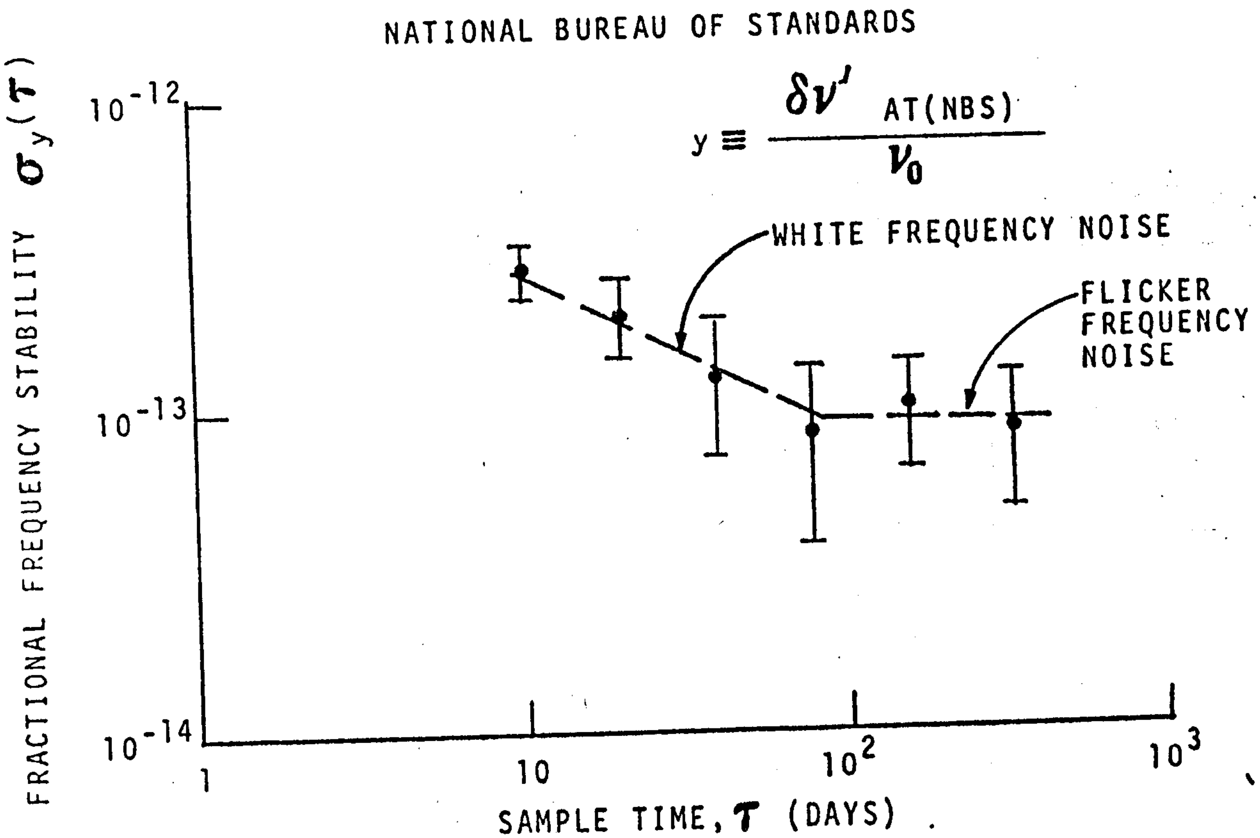

PARIS OBSERVATORY
$y \equiv \frac{\delta\nu'_{AT(OP)}}{\nu_0}$
FLICKER FREQUENCY NOISE
FRACTIONAL FREQUENCY STABILITY $\sigma_y(\tau)$
10^{-12}
10^{-13}
10^{-14}
1
10
10^2
10^3
SAMPLE TIME, τ (DAYS)

Figure 6



NATIONAL BUREAU OF STANDARDS
$y \equiv \frac{\delta \nu'_{AT(NBS)}}{\nu_0}$
WHITE FREQUENCY NOISE
FLICKER FREQUENCY NOISE
FRACTIONAL FREQUENCY STABILITY $\sigma_y(\tau)$
10^{-12}
10^{-13}
10^{-14}
1
10
10^2
10^3
SAMPLE TIME, τ (DAYS)

Figure 7

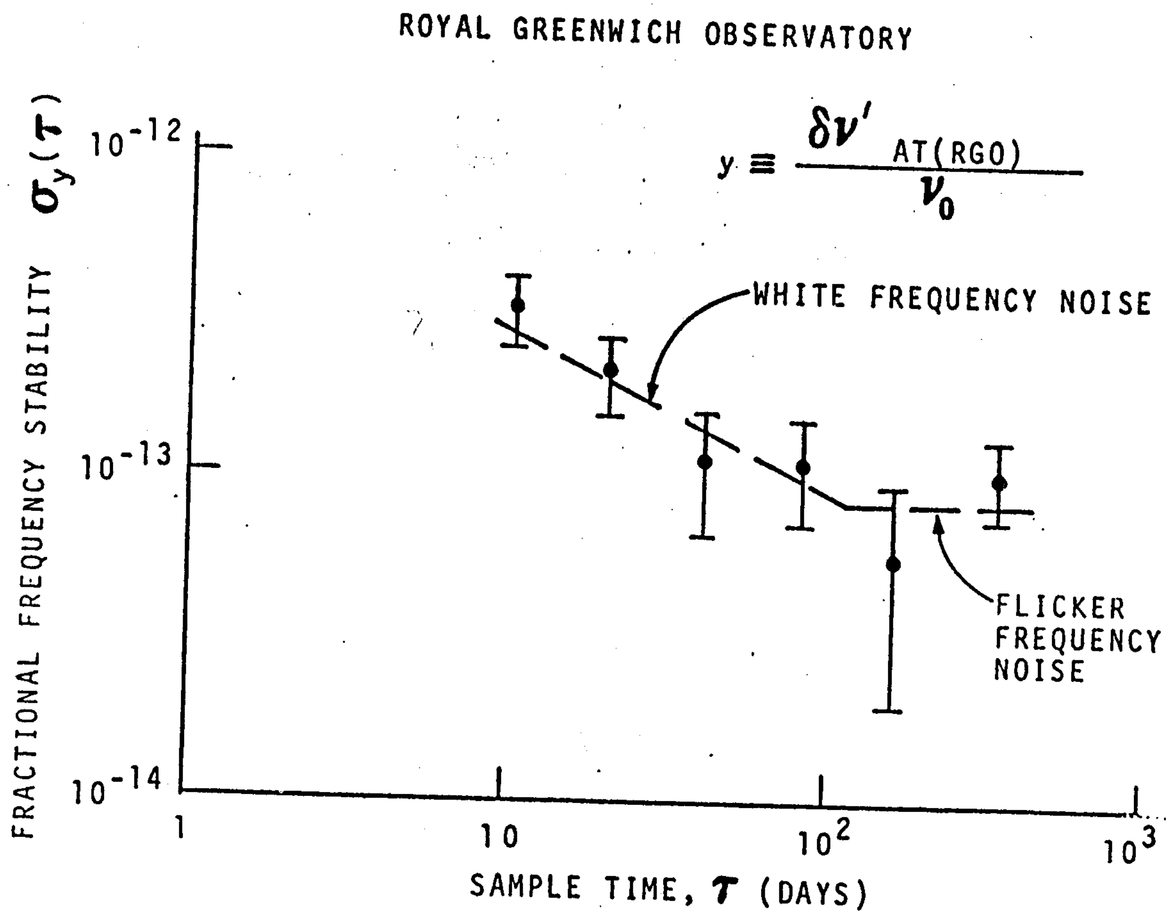

ROYAL GREENWICH OBSERVATORY
$y \equiv \frac{\delta\nu'_{AT(RGO)}}{\nu_0}$
WHITE FREQUENCY NOISE
FLICKER FREQUENCY NOISE
FRACTIONAL FREQUENCY STABILITY $\sigma_y(\tau)$
10^{-12}
10^{-13}
10^{-14}
1
10
10^2
10^3
SAMPLE TIME, τ (DAYS)

Figure 8

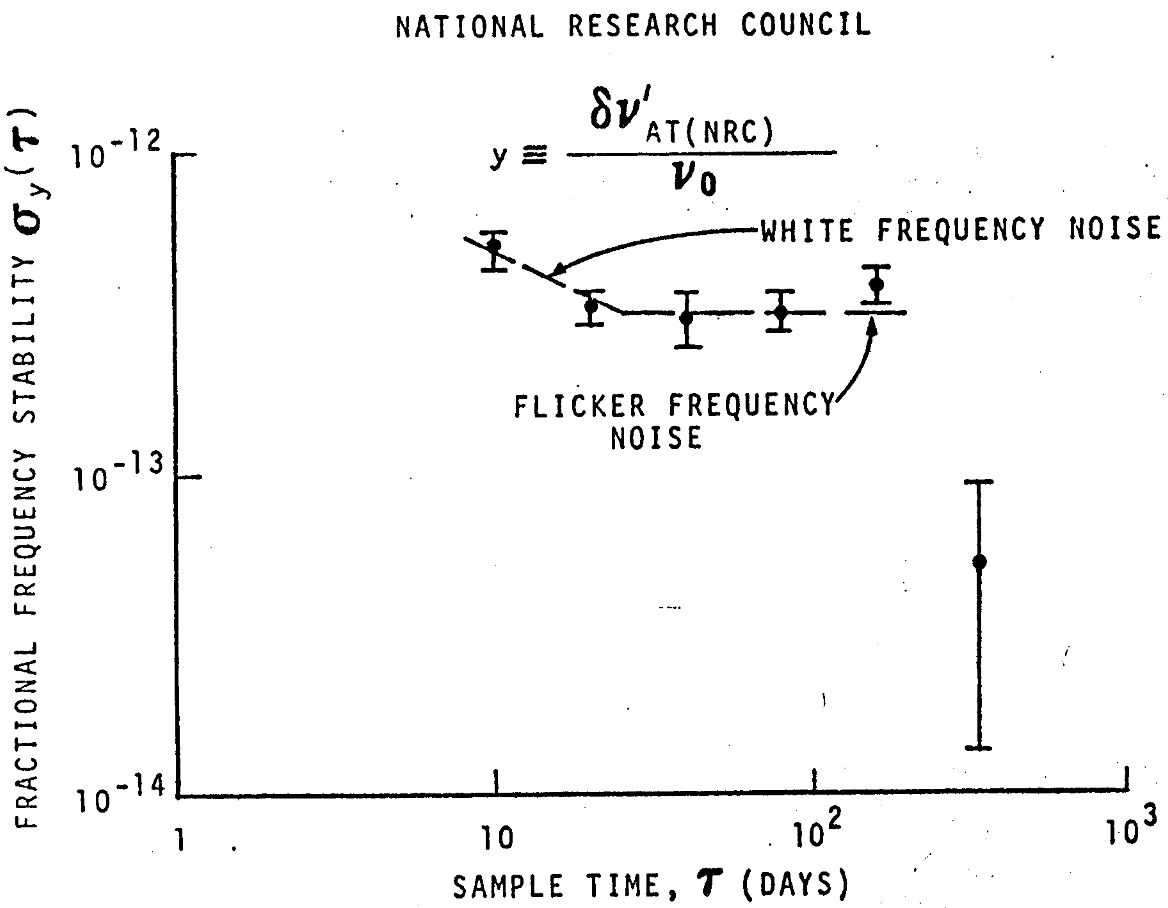

NATIONAL RESEARCH COUNCIL
$y \equiv \frac{\delta\nu'_{AT(NRC)}}{\nu_0}$
WHITE FREQUENCY NOISE
FLICKER FREQUENCY NOISE
FRACTIONAL FREQUENCY STABILITY $\sigma_y(\tau)$
10^{-12}
10^{-13}
10^{-14}
1
10
10^2
10^3
SAMPLE TIME, τ (DAYS)

Figure 9

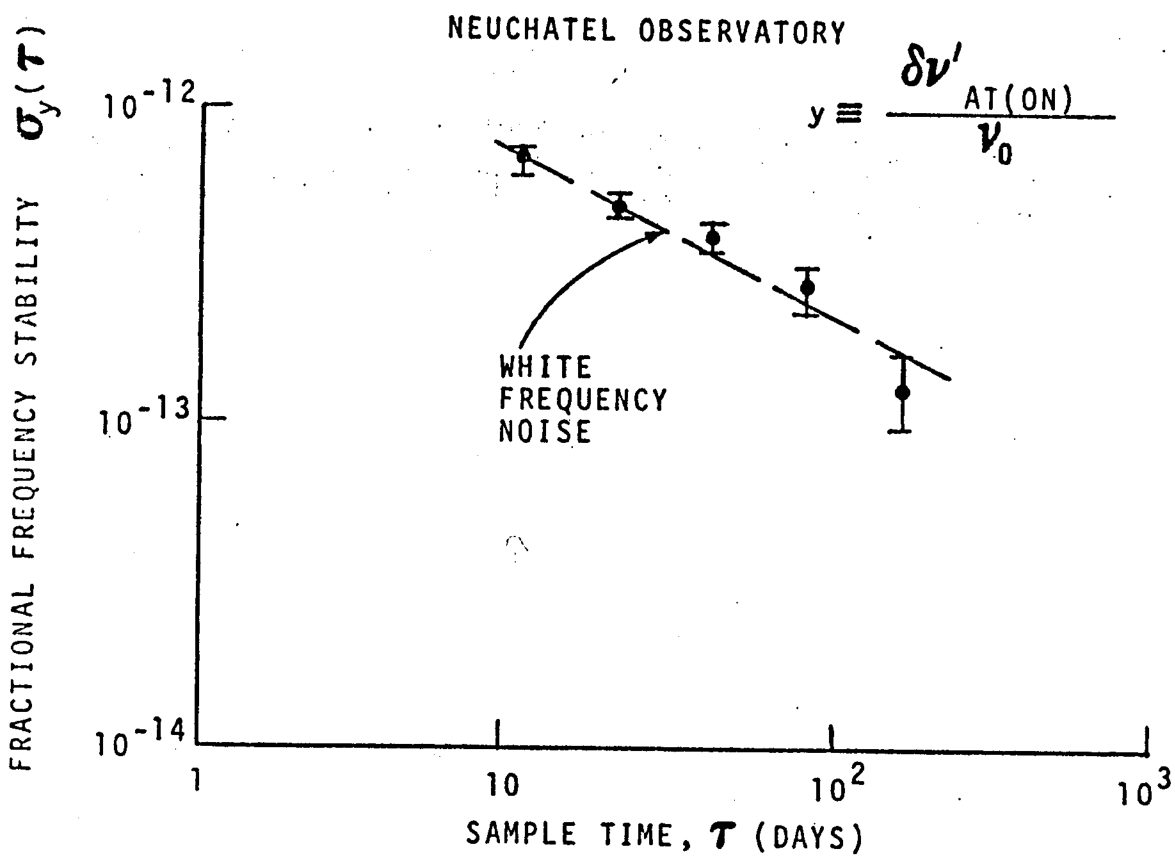
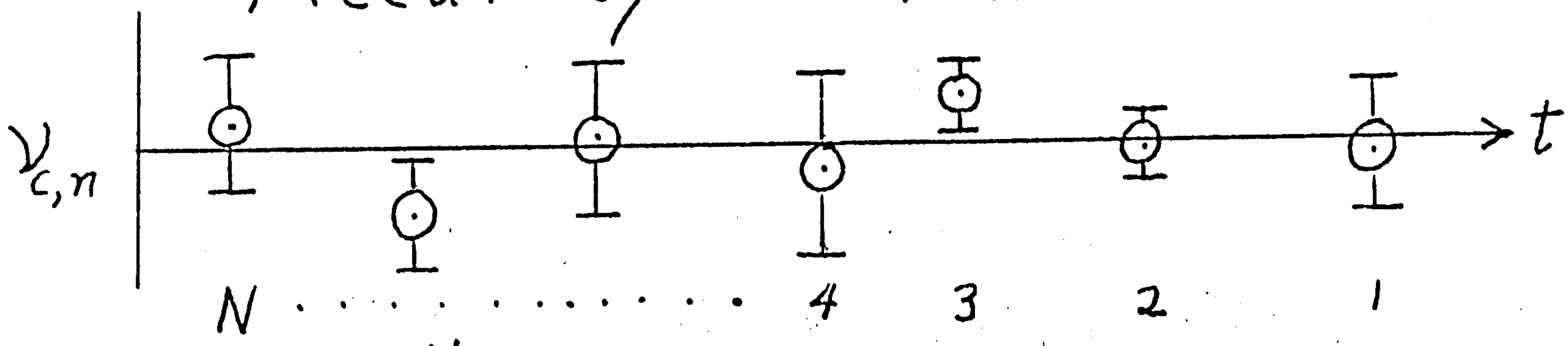

NEUCHATEL OBSERVATORY
$y \equiv \frac{\delta\nu'_{AT(ON)}}{\nu_0}$
WHITE FREQUENCY NOISE
FRACTIONAL FREQUENCY STABILITY $\sigma_y(\tau)$
10^{-12}
10^{-13}
10^{-14}
1
10
10^2
10^3
SAMPLE TIME, τ (DAYS)

Figure 10

Accuracy of Time Scale



$$\nu_{opt.} = \sum_{n=1}^{N} w_n \, \nu_{c,n} , \qquad w_n = \frac{a}{\sigma_n^2}$$

where $\sigma_n^2 = \sigma_{Scale,n}^2 + \sigma_{c,n}^2$

and $\text{opt. Accuracy} \equiv a = \left[\sum_{n=1}^{N} \frac{1}{\sigma_n^2} \right]^{-1}$

Figure 11

Fractional Frequency Deviations of AT(i); $i \equiv$ Lab Contributing in AT(BIH)

LAB:	$\frac{\nu_i - \nu_{AT(BIH)}}{\nu_{AT(BIH)}}$	(one-sigma) ACCURACY
PTB	-12.0×10^{-13}	4.1×10^{-13}
USNO	-2.8 "	25 "
OP	-2.7 "	58 "
NBS	-13.8 "	10 "
RGO	$+1.9$ "	50 "
NRC	-3.0 "	15 "
ON	-2.0 "	71 "
Weighted Standard	-11.3 "	3.6 "

Figure 12

Fractional Frequency Deviations of
AT(i); i ≡ Lab Contributing in AT(BIH)

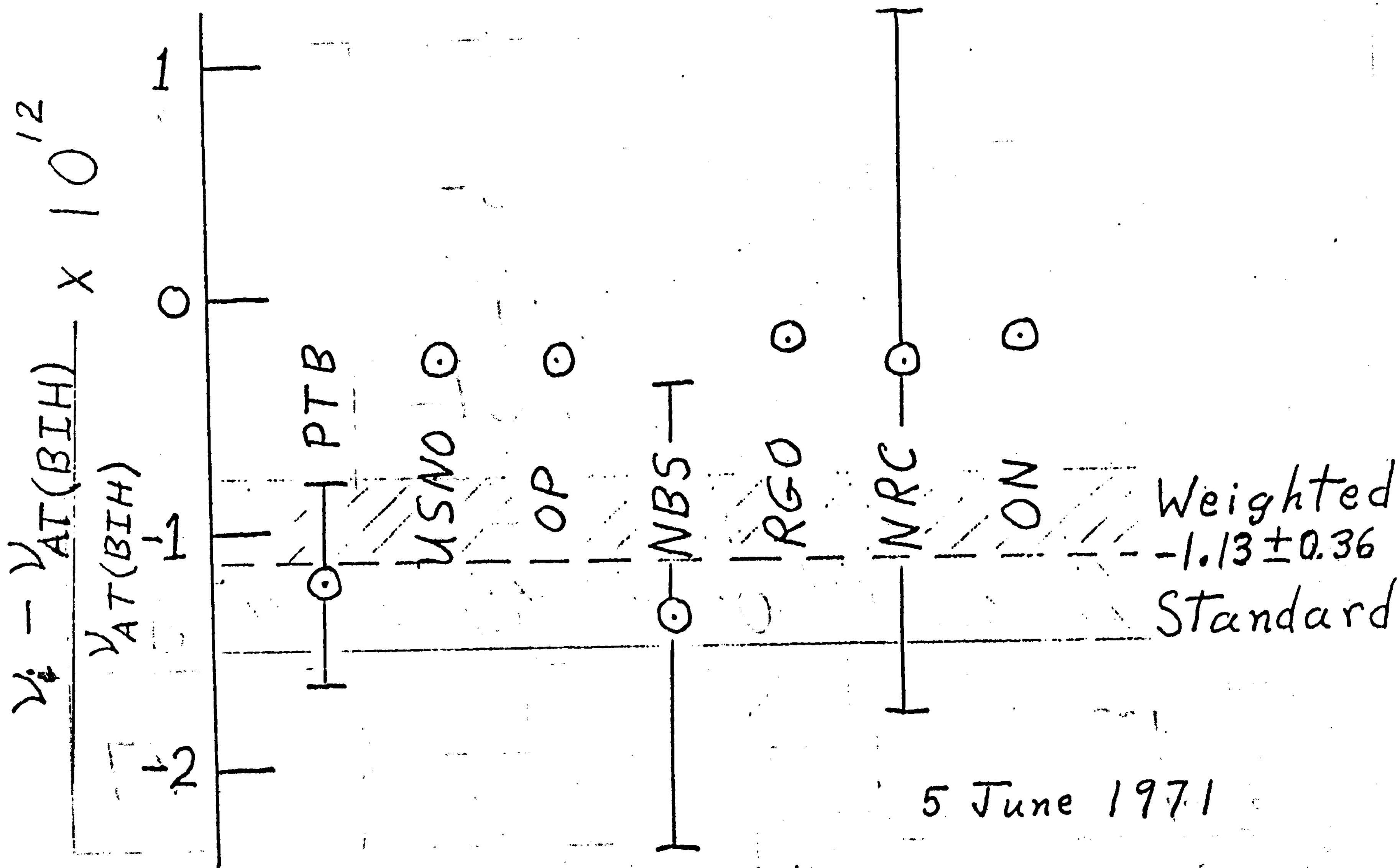


Figure 13

Time Scale Accuracy

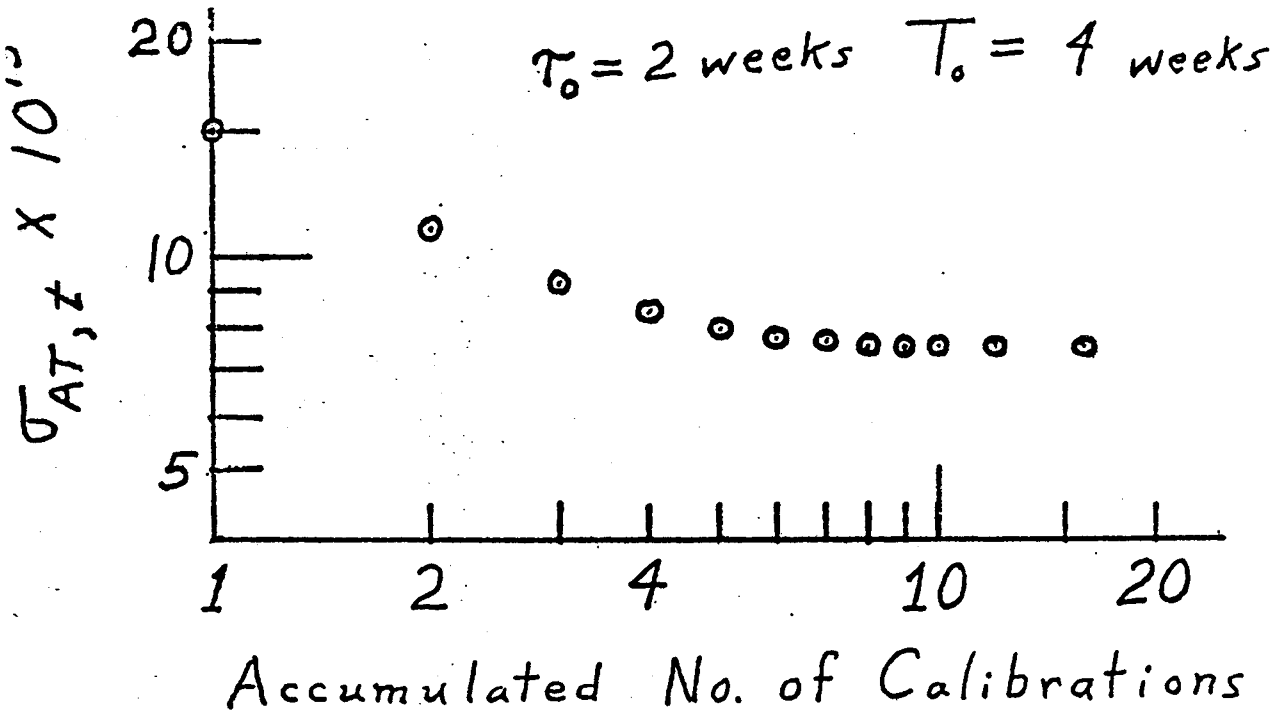


Figure 14