

Relativistic theory of the falling retroreflector gravimeter*

Neil Ashby 

National Institute of Standards and Technology, Boulder, CO, United States of America
University of Colorado, Boulder, CO, United States of America

E-mail: ashby@boulder.nist.gov

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Abstract

We develop a relativistic treatment of interference between light reflected from a falling cube retroreflector in the vertical arm of an interferometer, and light in a reference beam in the horizontal arm. Coordinates that are nearly Minkowskian, attached to the falling cube, are used to describe the propagation of light within the cube. Relativistic effects such as the dependence of the coordinate speed of light on gravitational potential, propagation of light along null geodesics, relativity of simultaneity, and Lorentz contraction of the moving cube, are accounted for. The calculation is carried to first order in the gradient of the acceleration of gravity. Analysis of data from a falling cube gravimeter shows that the propagation time of light within the cube itself causes a significant reduction in the value of the acceleration of gravity obtained from measurements, compared to assuming reflection occurs at the face. An expression for the correction to g is derived and found to agree with experiment. Depending on the instrument, the correction can be several microgals, comparable to commonly applied corrections such as those due to polar motion and earth tides. The controversial ‘speed of light’ correction is discussed.

Keywords: gravimeters, relativity, retroreflectors, acceleration of gravity

(Some figures may appear in colour only in the online journal)

1. Introduction

In a typical falling cube gravimeter, whose purpose is to measure the acceleration of gravity g with precision, a reference beam passes through a beam splitter where part of the beam is sent vertically to a falling corner cube reflector in an evacuated chamber. The cube is accelerated by gravity so the reflected beam suffers first- and second-order Doppler shifts as well as other effects; upon mixing with the reference beam back at the beamsplitter a beat frequency is generated having a rapidly increasing frequency chirp. In the gravimeter analyzed in this paper, a time stamp is recorded repeatedly after some fixed increment in the number of zero crossings of the beat signal. The time series depends on the strength g of the acceleration of gravity and on its gradient, on the structure of

the retroreflector, and the position and velocity of the cube at the initial instant of release; the initial position and velocity and g are then extracted from the data. Typically a great many drops are averaged to obtain the value of g ; fractional uncertainties of the order of 10^{-6} cm s⁻² are currently obtained after applying several corrections that are of microgal order; (1 Gal = 1 cm s⁻²).

A freely falling, locally inertial system can provide an application of the Principle of Equivalence: after expanding the gravitational potential in a Taylor series about the origin of local freely-falling coordinates and transforming the metric to local coordinates, the linear contribution cancels out and the local gravitational field consists primarily of tidal terms, or gravity gradients. The corner cube is an extended body with more mass near the flat face, so the net gravitational force on the body acts at a point above the face. For an ideal solid cube, the center of mass is at a point $d = D/4$ from the face, where

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D is the cube depth from face to corner. Choosing the retroreflector's center of mass as the origin of local coordinates simplifies the metric tensor in the freely falling frame, but complicates the analysis of light propagation within the cube.

Light propagation within the cube can be described as though a reference frame fixed in the falling cube were inertial; this entails a small time delay, during which the cube continues to accelerate in the laboratory. A wave front with a particular phase that enters the cube then leaves the cube at a lesser height and the fitted value of the initial center of mass position Z_0 is diminished. The extra path can amount to several thousand wavelengths. A smaller measured value of g results when this is accounted for; appendix B gives a detailed discussion of this effect in the 'non-relativistic' limit.

We construct a reference frame fixed in the cube using normal Fermi coordinates, which are very close to the Minkowski coordinates of special relativity. We derive a simple expression for the correction to the measured value of g that accounts for the dimensions and refractive index of the cube; the correction can amount to several microgals, comparable to many other commonly applied corrections such as those due to polar motion, and earth tides.

The calculation is carried to first order in the gravity gradient parameter $\gamma \approx 3 \times 10^{-6} \text{ s}^{-2}$ near earth's surface; terms of order γ^2 have been investigated but found to have negligible influence on the phase of electromagnetic waves in the interferometer beams. Also, the reciprocal of the speed of light, c^{-1} provides a parameter that is sufficiently small that Taylor expansions in powers of c^{-1} converge rapidly. Contributions of order c^{-3} have been investigated and terms of order c^{-2} in the phase difference of the test and reference beams have been retained.

The purpose of the present article is to provide a relativistic theory of a falling cube gravimeter, which fully respects relativistic principles such as the equivalence principle, light propagation along null geodesics, relativity of simultaneity and Lorentz contraction between moving frames. Tan *et al* [9] have used normal Fermi coordinates to treat relativistic effects in an absolute gravimeter; they have studied earth rotation effects, and motion of the falling mass perpendicular to the laser beam; neither of these phenomena are investigated here. They have not, however, treated the optical path of laser light in the cube, which is the most important topic discussed here. In this paper we account for the phase change of the reflected beam due to its propagation in the glass. We process data from 5000 drops in a gravimeter of this type, both with and without the assumption that reflection occurs at the face, and find a difference of several microgals in the value of g . An additional issue treated here in full is dependence of the coordinate speed of light on gravitational potential and the time delays of the test beam upward from the beamsplitter, through the glass, and down to the beamsplitter.

In section 2 we discuss the action of a gravity field gradient on a cube of pyramidal shape. Section 3 applies the result to the construction of a cube-fixed locally inertial frame; this work is supported by detailed derivations in appendix A. In section 4 the coordinate speed of light is shown to depend on the gravitational potential, and the phase of the upward-propagating test

beam reflected from the beam splitter is calculated. Section 5 completes the derivation of the test beam phase and its interference with the reference beam. Results of data analysis of 5000 drops are discussed in section 7. The 'speed of light' correction is briefly discussed in section 8.

2. Free fall of an extended body

For a freely-falling point mass, the gravitational potential gradient at the position of the mass is equal to (except for a negative sign) the mass's acceleration. Construction of local normal Fermi coordinates with origin at the mass's position, results in the elimination of linear terms in the Taylor expansion of the gravitational potential, so that in the local coordinate system the mass is not accelerated; only quadratic (and possibly higher order) terms in the gravitational potential remain. This is a consequence of the Equivalence Principle: the acceleration \mathbf{a} of the local coordinate system results in an additional local gravitational potential $\mathbf{a} \cdot \mathbf{r}$; the negative gradient of this additional potential cancels the negative gradient of the original potential so that the mass is 'weightless' in the local coordinate system. For an extended body, however, the center of mass may not fall along a geodesic. Below we show the center of mass follows a geodesic if terms no higher than quadratic order in the Taylor expansion of the gravitational potential are retained, as is the case in the present work. If the total gravitational force on the body equals the (negative of the) gradient of the gravitational potential evaluated at the center of mass, then the center of mass will follow a geodesic [7].

The linear mass density of a corner cube of depth D , along the symmetry axis, increases quadratically with distance along the axis from the corner. We use capital letters to denote coordinates in the laboratory frame. We assume that the gravitational potential is approximated by

$$\Phi(Z) = gZ - \gamma Z^2/2, \quad (1)$$

with the gravity gradient parameter $\gamma \approx 3 \times 10^{-6} \text{ s}^{-2}$ near earth's surface. The gradient parameter γ represents the rate of decrease of g , per meter of vertical distance upwards. Let dM be the mass in a slice of retroreflector material between Z and $Z + dZ$. The gravitational force on dM will be $(-g + \gamma Z)dM$. Let the cube be placed at rest with its face at Z and its corner at $Z + D$. The center of mass \bar{Z} is defined by

$$M\bar{Z} = \int Z dM, \quad (2)$$

so the net force per unit mass on the retroreflector is

$$-g + \gamma\bar{Z} = -\left. \frac{\partial\Phi}{\partial Z} \right|_{Z=\bar{Z}}. \quad (3)$$

Thus the net gravitational force on the retroreflector is identical to the (negative) gradient of the potential evaluated at the center of mass, so the center of mass will follow the same trajectory as that of a point mass at the center of mass position. Then if the center of mass, at distance d above the face, should be chosen as the origin of locally inertial coordinates, quadratic terms will survive in the Taylor expansion of the

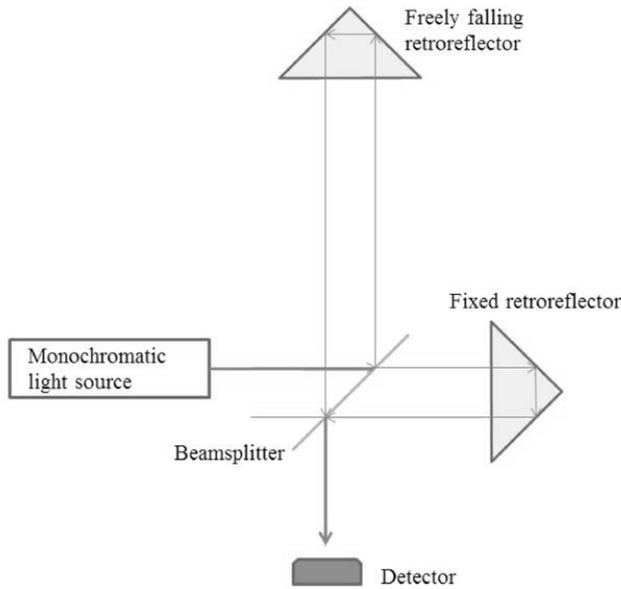


Figure 1. Simplified diagram of a freely falling cube gravimeter.

local metric. This is proved explicitly in appendix A. For a perfect solid cube with face ground normally to the $(1, 1, 1)$ axis, $d = D/4$.

3. The falling cube

Figure 1 illustrates the operation of a gravimeter that is analyzed in this paper. A solid retroreflector is diagrammed schematically in figure 2, showing an incoming ray, followed by three total internal reflections, and an exiting ray. We take as the reference point and origin of laboratory coordinates the point where the beams recombine. We assume the beamsplitter is perfectly aligned so that the optical path differences in vacuo in the two arms, from the point where the beam is split, to the recombination point, are compensated and do not have to be considered explicitly. The cube is assumed to be perfectly aligned and to fall without rotation. In the laboratory the gravitational potential is given by equation (1). We use lower case letters to denote coordinates fixed in the frame of the falling cube, and take the center of mass of the falling cube to be the origin of coordinates $z = 0$ in the falling frame. Both Z and z are measured positive upwards. The equation of motion of the center of mass of the retroreflector is

$$\ddot{Z} = -g + \gamma Z. \quad (4)$$

The solution corresponding to release from position Z_0 of the center of mass, with initial velocity V_0 at $T = 0$ is

$$Z(T) = Z_0 + V_0 T - \frac{gT^2}{2} + \gamma \left(\frac{Z_0 T^2}{2} + \frac{V_0 T^3}{6} - \frac{gT^4}{24} \right). \quad (5)$$

The velocity of the center of mass is

$$V(T) = V_0 - gT + \gamma \left(Z_0 T + \frac{V_0 T^2}{2} - \frac{gT^3}{6} \right). \quad (6)$$

Given the gravitational potential in equation (1), the geodesic equations of free fall vertical motion give equation (4), but with relativistic corrections of order c^{-2} that can be neglected. Also, in computing quantities such as $V(T)^2$ terms that are of order γ^2 or higher are neglected throughout this paper.

To fully describe the physics within the falling cube we need to introduce a transformation relating coordinates and time between the laboratory frame (T, Z) , and the accelerating falling frame (t, z) . We begin by computing the proper time τ of a reference clock at the center of mass, elapsed from the drop instant $T = 0$.

Let the metric tensor in the laboratory frame be:

$$G_{00} = -\left(1 + \frac{2\Phi(Z)}{c^2}\right); \quad (7)$$

$$G_{ZZ} = \left(1 - \frac{2\Phi(Z)}{c^2}\right), \quad (8)$$

where c is the speed of light at $Z = 0$ determined with the aid of an atomic reference clock. The potential is assumed to be static in the laboratory frame during the time required for one drop. The fundamental scalar invariant is

$$ds^2 = (cd\tau)^2 = -G_{\mu\nu}dX^\mu dX^\nu. \quad (9)$$

We have adopted the metric signature $-1, 1, 1, 1$ so $d\tau^2 > 0$; repeated Greek indices are summed from 0 to 3, except that only two components of the metric tensor need to be considered. Transverse motion is neglected in this paper; we do not consider Coriolis forces or rotation of the retroreflector.

The proper time of a falling clock at the origin of cube-fixed coordinates is then

$$\begin{aligned} \tau &= \frac{1}{c} \int_0^T \sqrt{-(G_{00}(c dT)^2 + G_{ZZ}dZ^2)} \\ &\approx \int_0^T \left(1 + \frac{\Phi(Z(T))}{c^2} - \frac{V(T)^2}{2c^2} \right) dT \\ &= T \left(1 + \frac{1}{c^2} \left(Z_0 g - \frac{V_0^2}{2} \right) \right) + \frac{V_0 g}{c^2} T^2 - \frac{g^2 T^3}{3c^2} \\ &\quad + \frac{\gamma}{c^2} \left(-\frac{Z_0^2 T}{2} - Z_0 V_0 T^2 \right. \\ &\quad \left. + \left(-\frac{V_0^2}{3} + \frac{2gZ_0}{3} \right) T^3 + \frac{gV_0 T^4}{3} - \frac{g^2 T^5}{15} \right), \end{aligned} \quad (10)$$

where here and throughout the paper we keep the leading relativistic correction terms of order c^{-2} , but keep only the contributions linear in γ .

This has an application to atomic fountain clocks, since equation (10) applies to an atom launched upwards. For atomic fountains, we neglect γ . Suppose an atom is projected upwards from $Z_0 = 0$ with velocity V_0 sufficient to reach height H , with $V_0^2 = 2gH$. A total time $2V_0/g$ is required for the atom to fall back down to the starting point. During this interval the proper time elapsed on the atom from equation (10), is

$$\tau = \frac{2V_0}{g} \left(1 + \frac{V_0^2}{6c^2} \right). \quad (11)$$

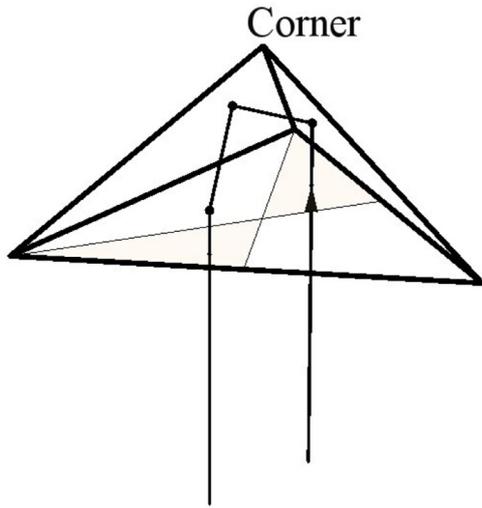


Figure 2. An ideal solid retroreflector showing an entering ray, three reflections, and an exiting ray. Most such retroreflectors will have the points near the entry/exit face ground away, so the center of mass is closer to the corner than is the case for the cube shown here.

The ratio of the relativistic part of this, to the non-relativistic part, is just the fractional frequency shift of the hyperfine splitting of the atom, due to relativistic effects, and is

$$\frac{\Delta\tau}{\tau} = \frac{\Delta f}{f} = \frac{V_0^3/(3gc^2)}{2V_0/g} = \frac{V_0^2}{6c^2} = \frac{1}{3} \frac{gH}{c^2}. \quad (12)$$

Including γ in this calculation yields only tiny additional corrections.

Equation (10) for the proper time can be solved for T in terms of τ at the center of mass by iteration:

$$T = \tau \left(1 + \frac{V_0^2}{2c^2} - \frac{gZ_0}{c^2} - \frac{gV_0\tau^2}{c^2} + \frac{g^2\tau^3}{3c^2} + \frac{\gamma}{c^2} \left(\frac{Z_0^2\tau}{2} + Z_0V_0\tau^2 + \frac{(V_0^2 - 2gZ_0)\tau^3}{3} - \frac{gV_0\tau^4}{3} + \frac{g^2\tau^5}{15} \right) \right). \quad (13)$$

As can be seen, the derivations are straightforward but the expressions are algebraically lengthy. Therefore the complete calculation of the transformation equations, including the transformation function $Z(t, z)$, is provided in appendix A and we proceed mostly by quoting the results of those calculations.

In addition to the terms in equation (13), there is a correction $V(T)z/c^2$ arising from the relativity of simultaneity. Adding this term (see appendix A) gives the net time transformation

$$T(t, z) = t \left(1 + \frac{V_0^2}{2c^2} - \frac{gZ_0}{c^2} - \frac{gz}{c^2} \right) + \frac{V_0z}{c^2} - \frac{gV_0t^2}{c^2} + \frac{g^2t^3}{3c^2} + \frac{\gamma}{c^2} \left(\left(\frac{Z_0^2}{2} + Z_0z \right) t + \left(V_0Z_0 + \frac{V_0z}{2} \right) t^2 + \left(\frac{V_0^2}{3} - \frac{gz}{6} - \frac{2gZ_0}{3} \right) t^3 - \frac{gV_0t^4}{3} + \frac{g^2t^5}{15} \right). \quad (14)$$

The symbol τ is reserved for the proper time on an ideal clock at the origin of falling coordinates, while t represents the time variable extended to the region including the interior of the

retroreflector, the vertical arm of the interferometer, and the beamsplitter. At the center of mass, $z = 0$ and $t = \tau$.

The transformation for coordinate $Z(t, z)$ is derived in appendix A and is

$$Z(t, z) = Z_0 + z \left(1 + \frac{V_0^2}{2c^2} + \frac{gZ_0}{c^2} \right) + \frac{gz^2}{2c^2} + t \left(V_0 + \frac{V_0^3}{2c^2} - \frac{gV_0Z_0}{c^2} \right) + t^2 \left(-\frac{g}{2} - \frac{3gV_0^2}{2c^2} + \frac{g^2Z_0}{c^2} \right) + \frac{4g^2V_0t^3}{3c^2} - \frac{g^3t^4}{3c^2} + \gamma \left(-\frac{Z_0^2z}{2c^2} - \frac{Z_0z^2}{2c^2} + t \left(\frac{Z_0^2V_0}{2c^2} - \frac{V_0z^2}{2c^2} \right) + t^2 \left(\frac{Z_0}{2} + \frac{3V_0^2Z_0}{2c^2} - \frac{3gZ_0^2}{2c^2} + \frac{gz}{4c^2} \right) + t^3 \left(\frac{V_0}{6} + \frac{7V_0^3}{12c^2} - \frac{19gZ_0V_0}{6c^2} \right) + t^4 \left(-\frac{g}{24} - \frac{5gV_0^2}{4c^2} + \frac{7g^2Z_0}{6c^2} \right) + \frac{11g^2V_0t^5}{15c^2} - \frac{11g^3t^6}{90c^2} \right). \quad (15)$$

It is shown in the appendix A that these transformations eliminate the term in g in the gravitational potential in the falling frame, with a small contribution remaining from the gravity gradient; in the falling frame the gradient contribution to the gravitational potential is $-\gamma z^2$ where z is measured upwards from the center of mass. The term γz^2 is precisely the contribution needed for the cube to remain unaccelerated in the falling frame. The inverses of these transformations are given in appendix A.

4. Signal phase and propagation speed

The output signal of the gravimeter can be analyzed in several ways. One way is to follow the frequency from source through the beamsplitter, through the cube and back down to the beamsplitter where it is combined with the reference beam. Another way is to follow the phase of a monochromatic signal through the apparatus. Another way is to imagine sharp pulses emitted from the source, and to calculate the propagation time to the recombination point. One useful fact in such analyses is, because the gravitational field is static, coordinate frequency during propagation up or down through the gravitational field is conserved. Another useful fact is: the wavevector of a monochromatic wave is a null fourvector in vacuum. The approach we choose here is to follow the phase of the test signal from the beamsplitter, through the cube, and back down to the beamsplitter. This is simpler in some respects since phase is a relativistic invariant. We shall neglect dispersion and make no distinction between group and phase speeds of signals. Also, we are justified in treating propagation of signals in the cube neglecting gravitational potentials, provided that analysis is done in the freely falling frame.

In the lab, a signal propagating in a vacuum in the upwards direction is null and at each point along its path satisfies:

$$0 = G_{00}c^2 dT^2 + G_{ZZ}dZ^2. \quad (16)$$

The coordinate speed of the signal will therefore be:

$$V_s = \frac{dZ}{dT} = \sqrt{-G_{00}/G_{ZZ}} \approx c \left(1 + \frac{2\Phi(Z)}{c^2} \right). \quad (17)$$

We choose the reference point to be at the beam splitter and set $\Phi(0) = 0$ there; the speed of light at this point in the lab

is assumed to be the defined speed, 299 792 458 m s⁻¹. The coordinate speed of light above the reference point will be greater than c .

Suppose the angular frequency of the reference signal is Ω , and that the signal from the reference beam that strikes the splitter and is reflected up has phase $\phi(T_0) = -\Omega T_0$ at the origin $Z = 0$. The wavelength of the light will be $\Lambda = 2\pi c/\Omega$. The phase propagates upwards with coordinate speed given by equation (17), and will reach height Z at time T given by

$$T = T_0 + \int_0^Z \frac{dZ}{V_s} = T_0 + \frac{1}{c} \int_0^Z \left(1 + \frac{2gZ}{c^2} - \frac{\gamma Z^2}{c^2} \right) dZ$$

$$= T_0 + \frac{1}{c} \left(Z + \frac{gZ^2}{c^2} - \frac{\gamma Z^3}{3c^2} \right). \quad (18)$$

The phase of the signal at (T, Z) is therefore

$$\phi(Z, T) = \phi(0, T_0) = -\Omega \left(T - \frac{1}{c} \left(Z - \frac{gZ^2}{c^2} + \frac{\gamma Z^3}{3c^2} \right) \right). \quad (19)$$

The signal four-vector K_μ can be obtained from this phase by differentiation:

$$K_\mu = \frac{\partial \phi(Z, T)}{\partial X^\mu}, \quad (20)$$

where $X^0 = cT$, $X^3 = Z$. This is a null fourvector satisfying

$$G^{00}(K_0)^2 + G^{ZZ}(K_Z)^2 = \frac{(K_0)^2}{G_{00}} + \frac{(K_Z)^2}{G_{ZZ}} = 0. \quad (21)$$

The wavevector components obtained from equation (20) are consistent with equation (21) to the order c^{-2} of the present calculation. In particular, the coordinate frequency $K_0 = -\Omega/c$ in the lab is conserved in the static gravitational field.

5. Phase in the falling cube

We now follow the phase of the signal through the cube until it exits going in the $-z$ direction. The phase of the signal impinging on the front face of the falling cube is denoted by ϕ_{in} and will be

$$\phi_{in} = \phi(Z(t, -d), T(t, -d)) \quad (22)$$

where $Z(t, -d)$ is the position of the face at time $T(t, -d)$ given by the transformation equations (15) and (14). This substitution naturally gives the phase entering the cube in terms of the time t in the local frame. Substituting and expanding to order c^{-2} , the signal phase at the retroreflector face is

$$\phi_{in}(t) = \Omega \left(\frac{Z_0 - d}{c} \right) + \Omega t \left(-1 + \frac{V_0}{c} - \frac{dg}{c^2} - \frac{V_0^2}{2c^2} + \frac{gZ_0}{c^2} \right)$$

$$+ \Omega t^2 \left(-\frac{g}{2c} + \frac{gV_0}{c^2} \right) + \frac{\Omega d V_0}{c^2} - \frac{\Omega^3 g^2}{3c^2} + \gamma \Omega \left(t \left(-\frac{Z_0^2}{2c^2} + \frac{dZ_0}{c^2} \right) + \frac{Z_0}{2c} \right)$$

$$+ t^2 \left(+\frac{dV_0}{2c^2} - \frac{V_0 Z_0}{c^2} \right) + t^3 \left(+\frac{V_0}{6c} - \frac{dg}{6c^2} - \frac{V_0^2}{3c^2} + \frac{2gZ_0}{3c^2} \right)$$

$$+ t^4 \left(-\frac{g}{24c} + \frac{gV_0}{3c^2} \right) - \frac{t^5 g^2}{15c^2}. \quad (23)$$

The value of ϕ in equation (23) is labeled with a subscript ‘in’ since it corresponds to the phase that strikes the cube face. The phase is a relativistic invariant, so this is the phase entering the retroreflector in the local freely falling frame. In this frame (but in vacuum outside the glass) at this point, the coordinate speed of light differs slightly from c because $z = -d$ and the metric tensor still has gravity gradient terms. However, the difference is negligible, see (A.9) and (A.10). We shall neglect dispersion in the cube and assume that the coordinate phase speed of light in the glass is reduced by a factor $1/n$. The phase speed is therefore

$$v_z = \frac{dz}{dt} = \frac{c}{n} \sqrt{-g_{00}/g_{zz}} = \frac{c}{n} \left(1 - \frac{\gamma z^2}{c^2} \right). \quad (24)$$

In the falling frame, the local time interval needed to reach height z going upwards is then

$$\delta t = \int_{-d}^z \frac{dz}{v_z} \approx \frac{n}{c} \int_{-d}^z dz \left(1 + \frac{\gamma z^2}{c^2} \right) = \frac{n}{c} \left(z + d + \frac{\gamma}{3c^2} (z^3 + d^3) \right). \quad (25)$$

We denote the distance from the face at $z = -d$ to the corner by D . The total local time required for the phase front to propagate to the corner and back to the face is then double the amount calculated just above:

$$\delta t_+ = \frac{2n}{c} \left(D + \frac{\gamma}{c^2} (d^2 D - d D^2 + \frac{1}{3} D^3) \right). \quad (26)$$

This time interval is the same for every ray entering the face normally. The phase ϕ_{out} exiting the face is numerically equal to ϕ_{in} , but the local time at which the phase front exits is at a later local time. Therefore the replacement

$$\phi_{out} = \phi_{in}|_{t \rightarrow t - \delta t_+} \quad (27)$$

will give an outgoing phase that represents the phase at a later t . In making this replacement, the last terms in γ in equation (26) do not contribute, to the order of the present calculation, so

$$\phi_{out} = -\frac{\Omega d}{c} + \frac{\Omega Z_0}{c} + \frac{2Dn\Omega}{c} - \frac{2Dn\Omega V_0}{c^2} + \frac{\Omega d V_0}{c^2}$$

$$+ t\Omega \left(-1 + \frac{V_0}{c} - \frac{V_0^2}{2c^2} + \frac{2Dng}{c^2} + \frac{gZ_0}{c^2} - \frac{dg}{c^2} \right)$$

$$+ t^2\Omega \left(-\frac{g}{2c} + \frac{gV_0}{c^2} \right) - \frac{\Omega g^2 t^3}{3c^2} + \gamma\Omega \left(t \left(-\frac{Z_0^2}{2c^2} - \frac{2DnZ_0}{c^2} + \frac{dZ_0}{c^2} \right) \right)$$

$$+ t^2 \left(\frac{Z_0}{2c} + \frac{dV_0}{2c^2} - \frac{V_0 Z_0}{c^2} - \frac{DnV_0}{c^2} \right) + t^3 \left(\frac{V_0}{6c} - \frac{dg}{6c^2} + \frac{V_0^2}{3c^2} + \frac{2gZ_0}{3c^2} + \frac{Dng}{3c^2} \right)$$

$$+ t^4 \left(-\frac{g}{24c} + \frac{gV_0}{3c^2} \right) - t^5 \frac{g^2}{15c^2}. \quad (28)$$

Viewed from the falling frame, the phase front propagates in vacuum downward to negative values of z with a phase velocity

$$v_z = -c \left(1 - \frac{\gamma z^2}{c^2} \right). \quad (29)$$

The local time required to propagate to $z < -d$ is

$$\delta t_- = -\frac{1}{c} \int_{-d}^z dz \left(1 + \frac{\gamma z^2}{c^2} \right) = -\frac{1}{c} \left(z + d + \frac{\gamma (z^3 + d^3)}{3c^2} \right). \quad (30)$$

The phase field in the region below the cube is then

$$\phi(t, z) = \phi_{\text{out}}|_{t \rightarrow t - \delta t_-} \quad (31)$$

Substituting and expanding to order c^{-2} gives

$$\begin{aligned} \phi(t, z) = & \frac{\Omega Z_0}{c} + \frac{2Dn\Omega}{c} + z\left(-\frac{\Omega}{c} + \frac{\Omega V_0}{c^2}\right) - \frac{2Dn\Omega V_0}{c^2} \\ & + t\left(-\Omega + \frac{\Omega V_0}{c} + \frac{2Dgn\Omega}{c^2} - \frac{\Omega V_0^2}{2c^2} + \frac{g\Omega Z_0}{c^2} - \frac{g\Omega z}{c^2}\right) + t^2\left(-\frac{2dDn}{c^2} - \frac{g\Omega}{2c} + \frac{g\Omega V_0}{c^2}\right) \\ & - t^3\frac{g^2\Omega}{3c^2} + \gamma\Omega\left(t\left(-\frac{2dDn}{c^2} - \frac{2DnZ_0}{c^2} - \frac{Z_0^2}{2c^2} + \frac{dz}{c^2} + \frac{Z_0z}{c^2}\right)\right. \\ & \left. + t^2\left(\frac{d}{2c} - \frac{dV_0}{2c^2} + \frac{Z_0}{2c} - \frac{DnV_0}{c^2} - \frac{V_0Z_0}{c^2} + \frac{V_0z}{2c^2}\right)\right) \\ & + t^3\left(\frac{dg}{2c^2} + \frac{Dgn}{3c^2} + \frac{V_0}{6c} - \frac{V_0^2}{3c^2} - \frac{gz}{6c^2} + \frac{2gZ_0}{3c^2}c^2\right) \\ & + t^4\left(-\frac{g}{24c} + \frac{gV_0}{3c^2}\right) - t^5\frac{g^2}{15c^2} - \frac{\gamma\Omega n D^2 t}{2c^2}. \end{aligned} \quad (32)$$

This phase needs to be evaluated at the reference point $Z = 0$ and expressed as a function of the time T . Substituting from the transformation equations equation (A.5) and (A.6) and then setting $Z = 0$ yields the phase to be combined with the reference phase at the splitter:

$$\begin{aligned} \phi(T, 0) = & \frac{2(Dn-d)\Omega}{c} + \frac{2\Omega Z_0}{c} - \frac{2(Dn-d)\Omega V_0}{c^2} - \frac{2\Omega Z_0 V_0}{c^2} \\ & + T\left(-\Omega + \frac{2\Omega V_0}{c} - \frac{2\Omega V_0^2}{c^2} + \frac{2\Omega g(Dn-d)}{c^2} + \frac{2g\Omega Z_0}{c^2}\right) \\ & + T^2\left(-\frac{g\Omega}{c} + \frac{3g\Omega V_0}{c^2} - \frac{g^2\Omega T^3}{c^2}\right) \\ & + \gamma\Omega\left(T\left(-\frac{2(Dn-d)Z_0}{c^2} - \frac{2Z_0^2}{c^2}\right) + T^2\left(\frac{Z_0}{c} - \frac{(Dn-d)V_0}{c^2} - \frac{4V_0Z_0}{c^2}\right)\right. \\ & \left. + T^3\left(\frac{V_0}{3c} + \frac{g(Dn-d)}{3c^2} - \frac{4V_0^2}{3c^2} + \frac{7gZ_0}{3c^2}\right) + T^4\left(-\frac{g}{12c} + \frac{5gV_0}{4c^2}\right) - \frac{g^2T^5}{4c^2}\right). \end{aligned} \quad (33)$$

To the order of this calculation, solving the relativistic equations for the hyperbolic center of mass motion of the cube gives additional terms of order c^{-2} in equations (5) and (6). These terms contribute to the transformation equations but because of their high order have no effect on the final phase, equation (33).

Not all of the relativistic correction terms are important; for example many of the terms multiplying γ are very small. Within the falling cube, z is usually no more than a few centimeters; with a nominal value of the gravity gradient at Earth's surface $\gamma \approx 3 \times 10^{-6}$, then $\gamma z^2/(2c^2) \approx 10^{-26}$ and could be neglected. Nevertheless these terms are included since they may be important in other applications.

The signal is mixed with the reference signal, which has phase $-\Omega T$, and fringes are counted. The phase difference is obtained from equation (33) by omitting the term $-\Omega T$ in the second line. It is possible to absorb some of the terms in the cube depth D in the difference by redefining the initial position Z_0 , however it is not possible to eliminate all such terms and a significant effect remains, which will be discussed in Sect. 6.

The transformations between laboratory coordinates and freely falling coordinates provide alternative descriptions of phenomena, which must agree when results are expressed in terms of observables such as numbers of interference fringe counts. For example, in the above calculation of the test beam phase at the detector the calculation treated the source at $z = -d$ as at rest and the detector as moving. Alternatively,

one may calculate the test beam phase at the detector in laboratory coordinates considering the source (the cube face) to be moving while the detector is at rest. In the latter case, however, one must take care to compute the time delay between the source and arrival at the detector accounting for the finite speed of light. This is analagous to computing the propagation time of a signal sent from a moving transmitter to a receiver at rest.

Let the face of the cube at transmission time T_T be $Z_{\text{face}}(T_T)$ and let the time required for the signal to reach the detector at $Z = 0$ at time T be denoted by $\Delta(Z_{\text{face}}(T_T))$. Then the time of transmission is determined by a retarded time:

$$T_T = T - \Delta(Z_{\text{face}}(T_T)). \quad (34)$$

This can be solved by iteration:

$$T_T = T - \Delta(Z_{\text{face}}(T - \Delta(Z_{\text{face}}(T - \Delta(Z_{\text{face}}(T - \dots)))))). \quad (35)$$

Each iteration introduces terms having one more factor of c in the denominator; convergence is rapid.

We denote the phase out of the cube as a function of T_T in laboratory coordinates as $\Phi_{\text{out}}(T_T)$. This may be obtained by solving the equation $T_T = T(t, -d)$ from the time transformation, for t in terms of T_T and substituting for t in favor of T_T into the scalar, equation (28). The phase at the detector is then

$$\begin{aligned} \Phi(T) = & \Phi_{\text{out}}(T_T) \\ = & \Phi_{\text{out}}(T - \Delta(Z_{\text{face}}(T - \Delta(Z_{\text{face}}(T - \Delta(Z_{\text{face}}(T - \dots)))))). \end{aligned} \quad (36)$$

The result agrees with the test beam phase at the splitter, equation (33).

5.1. Origin of time

The choice of zero of time is arbitrary; in the present work we have chosen to make $T = 0$ at the instant the cube is dropped, introducing initial position Z_0 and velocity V_0 to account for imperfections in release. Suppose instead the cube were dropped at a different instant T_0 . The center of mass position and velocity at this instant will be:

$$\begin{aligned} Z(T_0) = & Z_0 + V_0 T_0 - \frac{1}{2}gT_0^2 + \gamma\left(\frac{Z_0 T_0^2}{2} + \frac{V_0 T_0^3}{6} - \frac{gT_0^4}{24}\right); \\ V(T_0) = & V_0 - gT_0 + \gamma\left(Z_0 T_0 + \frac{V_0 T_0^2}{2} - \frac{gT_0^3}{6}\right). \end{aligned} \quad (37)$$

Solving for the original position and velocity, keeping linear terms in γ , gives

$$\begin{aligned} Z_0 = & Z(T_0) - V(T_0)T_0 - \frac{gT_0^2}{2} + \gamma\left(\frac{Z(T_0)T_0^2}{2} - \frac{V(T_0)T_0^3}{6} - \frac{gT_0^4}{24}\right); \\ V_0 = & V(T_0) + gT_0 + \gamma\left(-Z(T_0)T_0 + \frac{V(T_0)T_0^2}{2} + \frac{gT_0^3}{6}\right). \end{aligned} \quad (38)$$

If one substitutes the replacements indicated in equations (38) into (33) or one lets $T \rightarrow (T - T_0)$, and makes the replacements $Z_0 \rightarrow Z(T_0)$, $V_0 \rightarrow V(T_0)$, then upon neglecting powers of γ higher than 1 it is found that the interference phase difference is form-invariant with respect to this change of time origin. For example, consider only the terms proportional to c^{-1} in equation (33). These are

$$\frac{\Omega}{c} \left(-2d + 2Dn - gT^2 + 2Z_0 + 2V_0T + \gamma \left(-\frac{1}{12}gT^4 + \frac{1}{3}V_0T^3 + Z_0T^2 \right) \right). \quad (39)$$

Making the replacements given in equation (38) and keeping terms of order γ , this becomes

$$\frac{\Omega}{c} \left(-2d + 2Dn - gT^2 + 2gTT_0 + gT_0^2 + 2TV(T_0) - 2T_0V(T_0) + 2Z(T_0) + \frac{1}{12}\gamma(g(T-T_0)^4 - 4(T-T_0)^2V(T_0) + 3Z(T_0)(T-T_0)^2) \right), \quad (40)$$

which is of the same form as equation (39) upon making the replacements $Z_0 \rightarrow Z(T_0)$, $V_0 \rightarrow V(T_0)$, $T \rightarrow T - T_0$. The remaining contributions in equation (33) behave similarly, as do all the terms proportional to D . This is a useful self-consistency check of the theory.

6. Dispersion and modulation

6.1. Dispersion

As the cube falls, the wavelength of laser radiation within the cube decreases slightly. This results in a change of phase velocity that is negligibly small, as can be seen from the following argument. The acceleration g acts for a time T that is only a few tenths of a second, causing the velocity to build up to $gT \approx 5 \text{ m s}^{-2}$. The first-order Doppler shift of the laser wavelength within the cube is then no more than $\Delta\lambda = -V\lambda/c = -gT\lambda/c \approx -10^{-14} \text{ m}$. Chromatic dispersion for typical corner-cube glass is $dn/d\lambda \approx 5 \times 10^4 \text{ m}^{-1}$, [10] so $\Delta n \approx 5 \times 10^{-10}$. The change in time delay within the cube due to dispersion is negligible.

6.2. Modulation

Low-frequency modulation may be applied to the laser signal to aid in locking the laser to a frequency reference—e.g. a Rubidium oscillator. This modulation produces sidebands whose strength depends on the amplitude of the modulation. Every term in the beat frequency is proportional to the original laser frequency of the source, Ω , so both reference beam and retroreflected beam will have sidebands that undergo a chirp proportional to the frequency chirp of the main signal. We shall write the unmodulated phase of the signal $\Phi(T, 0)$ in (33) in the form

$$\Phi(T, 0) = \Omega F(Z_0, V_0, g, \gamma, T) = \Omega F. \quad (41)$$

Assuming the signal is frequency modulated with relatively low frequency ω_m , and has a small modulation index β , the modulated reference beam signal at distance x from the origin can be represented by

$$E_{\text{ref}} = E_0 e^{i(\Omega x/c - i\Omega T)} \left(1 - \frac{\beta}{2} e^{(-ix\omega_m + i\omega_m T)} + \frac{\beta}{2} e^{(ix\omega_m - i\omega_m T)} \right). \quad (42)$$

The electric field of the retroreflected beam may have a slightly attenuated amplitude, but will have sidebands such that

$$E_{\text{test}} = E_1 e^{i\Omega F} \left(1 - \frac{\beta}{2} e^{i\omega_m F} + \frac{\beta}{2} e^{-i\omega_m F} \right). \quad (43)$$

These signals are superimposed and sent into a photodiode. The measurements consist of time stamps of zero-crossings of the time varying quantity

$$E_{\text{test}} E_{\text{ref}}^* + \text{c.c.} \quad (44)$$

The sideband frequencies also suffer from the frequency chirp that occurs as the cube falls and can be described by the phase function equation (33) with an appropriate frequency. However, unless the frequency modulation index is large we find there is almost no effect on the interference fringe counts.

7. Data analysis

Three data sets produced by a falling cube gravimeter were analyzed. These were characterized by differing numbers of drops, and differing numbers of zero-crossings of the interference signal between time stamps. These details are summarized in table 1.

The data were processed using the result in equation (33), with the term $-\Omega T$ in the second line omitted, and refractive index $n = 1.52$. An estimated value of gravity gradient $\gamma = 3.0724615 \times 10^{-6} \text{ s}^{-2}$ was used for all drops; the data is not sufficiently robust to determine γ itself as the covariance matrix becomes singular if γ is included as a variable to be determined by the drop data. For each drop in each ‘project’, the data were first processed with $D = 0$, as though the reflection from the falling cube occurred at the cube face. The variables Z_0 , V_0 , and g were determined by the fit with a fixed value of γ . Then the data were processed with $D = 0.0175 \text{ m}$ corresponding to the distance from cube face to corner of a typical ‘1 inch’ cube, and $d = D/4$. (The values of D and d for the particular gravimeter from which this data were taken were not available.) None of the usual corrections such as those arising from polar motion or tides were applied since such corrections would be the same whether D was or was not included in the calculation. The value of g obtained when $D \neq 0$ was invariably smaller than the value obtained when $D = 0.0175 \text{ m}$. In fact the difference for all drops in all projects, including short drops obtained by selecting fewer time stamps, was found to be

$$\delta g = g(D = 0.0175 \text{ m}) - g(D = 0 \text{ m}) = -6.82854 \mu \text{ Gal.} \quad (45)$$

with negligible variation. The same number arises if all terms of order c^{-2} in equation (33) are neglected.

This is a significant correction, comparable to the magnitudes of other corrections such as arise from polar motion, or earth tides.

The fact that the difference δg was found to be the same for all drops implies this constant difference should be derivable from equation (33). Suppose that fitting the drop data to equation (33) with $D = 0$ results in values Z_0, V_0, g for these three parameters. Then suppose that fitting the drop data with $D \neq 0$ results in the values $Z_0 + \delta Z_0, V_0 + \delta V_0, g + \delta g$. All these quantities are constants independent of T , so the two phase functions at an arbitrary value of T must match. Let us denote the phase given in equation (33) by $\phi(D, Z_0, V_0, g, T)$. Linearizing the phase with respect to the small increments $\delta Z_0, \delta V_0, \delta g$

Table 1. Description of data processed using equation (33).

Project #	Number of drops	Number of time stamps	Number of zero-crossings between time stamps
#1	200	1200	1000
#2	2400	1400	800
#3	2400	1000	1200

then upon subtracting the phase without D the difference should be identically zero. Thus we expect

$$\phi(D, Z_0 + \delta Z_0, V_0 + \delta V_0, g + \delta g, T) - \phi(0, Z_0, V_0, g, T) = 0. \quad (46)$$

Carrying out the calculation, we find neglecting terms of order γ^2 but without further approximation, that

$$\begin{aligned} \delta Z_0 &= -Dn + d; \\ \delta V_0 &= 0; \\ \delta g &= -\gamma(Dn - d). \end{aligned} \quad (47)$$

It is remarkable that these replacements reduce both non-relativistic terms and terms proportional to c^{-1} in equation (46) to negligible levels. This value of the correction to g is indeed equal to that found by fitting the data:

$$\delta g = -6.828\,54 \mu \text{ Gal}. \quad (48)$$

The changes in Z_0 and V_0 as given in equation (47) are similarly verified by fitting. Although equation (46) is a fourth-order polynomial involving five coefficients, the adjustment of only three quantities as in equation (47) is sufficient because $\delta V_0 = 0$ eliminates all odd terms in T . The residuals after applying the remaining two adjustments in equation (47) are proportional to γ^2 and are negligible.

8. ‘Speed of light’ corrections

One of the more controversial issues surrounding measurements with falling cube gravimeters is the so-called ‘speed of light’ correction. Many measurement models of gravimeters of this type attempt to account for propagation delays between the instant the waves are reflected from the cube and the instant they are delivered to the detector, due to the finite value of the speed of light [5, 6]. A review of such speed of light corrections has been given in [8]. It should be emphasized that the result given in equation (33) *does not require any such correction* as light propagation has been fully described in that expression using relativistic principles.

Suppose one nevertheless attempts to derive a ‘speed of light correction’ by analogy with equation (46) by first fitting only with the leading c^{-1} terms, then linearizing. To compare with earlier treatments we take $D = 0$ and separate the phase contributions into c^{-1} and c^{-2} contributions:

$$\phi(0, Z_0, V_0, g) = \frac{1}{c} \phi_1(Z_0, V_0, g) + \frac{1}{c^2} \phi_2(Z_0, V_0, g). \quad (49)$$

Let Z_0, V_0, g be determined by data fitting using only ϕ_1 , the non-relativistic part of the phase. Then one might expect corrections to satisfy

$$\frac{1}{c} \phi_1(Z_0 + \delta Z_0, V_0 + \delta V_0, g + \delta g) - \frac{1}{c^2} \phi_2(Z_0, V_0, g) = 0. \quad (50)$$

However, this is a fourth-order polynomial in the time having five constant coefficients and cannot be satisfied by adjusting three fitting parameters. Even if terms in γ/c were neglected, there are still four constant coefficients. Thus it appears to be infeasible to derive speed-of-light corrections in this way. In case of a three-level schema in which only three fringe shifts are recorded at three times, one can imagine obtaining relativistic corrections by adjusting only three parameters.

In any case, such corrections are unnecessary; the full relativistic phase difference including propagation delays of light through the apparatus is available in equation (33).

9. Conclusions

The large value of the correction in equation (47), obtained from data analysis as well as from theory, supports the contention that in falling cube gravimeters, the time delay entailed by penetration of the light into the cube cannot be ignored. It also points up the need for a better value of the gravity gradient—not just a ‘standard’ value-at the position of the apparatus, as well as a precise value for the refractive index of the glass and an accurate position for the center of mass. No additional ‘speed of light’ correction is needed in this picture, as relativistic corrections including Lorentz contraction of the falling cube, dependence of the coordinate speed of light on gravitational potential, relativity of simultaneity, propagation along null geodesics, and first- and second-order Doppler shifts have been accounted for. The correction for time delay within the falling cube has been shown to be several microgals and to depend on properties of the cube; this correction can simply be added to other relevant corrections that are commonly applied.

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Appendix A. Coordinate transformations

The prescription for constructing accelerated cube-fixed normal Fermi coordinates that cover the falling cube, extending to a space-time patch containing the cube and beamsplitter, is developed in [1]. Normal Fermi coordinates[2–4], similar to Cartesian coordinates falling with the cube, are constructed

by parallel propagation of a tetrad of orthonormal four-vectors along the trajectory of the freely falling test object. In the present case the trajectory is that of the center of mass of the falling cube. The 0th member of the tetrad is just the four-velocity, tangent to the trajectory of the center of mass. The time in the falling frame is the proper time on an imaginary ideal atomic clock carried along at the center of mass. The other members of the tetrad are obtained by solving equations for spacelike geodesics that intersect the trajectory of the center of mass orthogonally. In the present case the only additional coordinate of interest is the Z -coordinate so only two members of the tetrad are relevant: these are the 0-member of the tetrad tangent to the center of mass trajectory, and the spacelike 3-vector directed along the vertical Z -axis. The construction yields the following expression for the time transformation (equation (A.12) in [1]):

$$T(t, z) = \int_0^t \frac{dt}{\sqrt{-G_{\mu\nu}dX^\mu dX^\nu}} + \frac{\mathbf{V} \cdot \mathbf{r}}{c^2}. \quad (\text{A.1})$$

The integral has already been calculated in equation (13). The last term in equation (A.1) representing relativity of simultaneity is

$$\begin{aligned} \frac{V(T)z}{c^2} &= \frac{1}{c^2} (V_0 z - gzT + \gamma z(Z_0 T + \frac{V_0 T^2}{2} - \frac{gT^3}{6})) \\ &= \frac{1}{c^2} (V_0 z - gtz + \gamma(Z_0 zt + \frac{V_0 zt^2}{2} - \frac{gt^3 z}{6})). \end{aligned} \quad (\text{A.2})$$

where to the order of the calculation T can be replaced by t in equation (A.2). Combining equations (13) and (A.2) gives the transformation from time in the lab to the falling coordinates in the cube, equation (14).

We also need the transformation of vertical coordinates from the lab coordinate Z to the cube-fixed coordinate z , which is given by (equation (A.10) in [1]):

$$\begin{aligned} Z = Z(t, z) &= \left(Z_{\text{cm}}(T) + z \left(1 - \frac{\Phi(Z(T))}{c^2} - \frac{A(T)z}{c^2} \right) \right. \\ &\quad \left. + \frac{V(T)^2 z}{2c^2} + \frac{z^2 A(T)}{2c^2} \right) \Big|_{T=T(t,0)}. \end{aligned} \quad (\text{A.3})$$

Here quantities such as $Z(T)$ and $\Phi(T)$ are evaluated at the cube's center of mass, which is the origin of locally inertial, freely falling coordinates. Then T is replaced by its value at the center of mass given by equation (14). The acceleration $A(T)$ of the cube is obtained by differentiating equation (6). The first term in equation (A.3) is the accelerating value of the Z -coordinate at the center of mass. The coefficients of z include a change of scale arising from the earth's gravitational potential, a Lorentz contraction term, and additional acceleration contributions. These contributions arise during construction of normal Fermi coordinates, while solving for spacelike geodesics that intersect the trajectory orthogonally [1]. In General Relativity, arbitrary coordinate transformations are allowed; of course then the physics in the resulting coordinate system must be interpreted in terms of coordinate-independent observables.

After expanding equation (A.3) to order c^{-2} , the transformation for the Z coordinate is

$$\begin{aligned} Z(t, z) &= Z_0 + z \left(1 + \frac{V_0^2}{2c^2} + \frac{gZ_0}{c^2} \right) + \frac{gz^2}{2c^2} \\ &\quad + t \left(V_0 + \frac{V_0^3}{2c^2} - \frac{gV_0 Z_0}{c^2} \right) + t^2 \left(-\frac{g}{2} - \frac{3gV_0^2}{2c^2} + \frac{g^2 Z_0}{c^2} \right) + \frac{4g^2 V_0 t^3}{3c^2} - \frac{g^3 t^4}{3c^2} \\ &\quad + \gamma \left(-\frac{Z_0^2 z}{2c^2} - \frac{Z_0 z^2}{2c^2} + t \left(\frac{Z_0^2 V_0}{2c^2} - \frac{V_0 z^2}{2c^2} \right) + t^2 \left(\frac{Z_0}{2} + \frac{3V_0^2 Z_0}{2c^2} - \frac{3gZ_0^2}{2c^2} + \frac{gz^2}{4c^2} \right) \right. \\ &\quad \left. + t^3 \left(\frac{V_0}{6} + \frac{7V_0^3}{12c^2} - \frac{19gZ_0 V_0}{6c^2} \right) + t^4 \left(-\frac{g}{24} - \frac{5gV_0^2}{4c^2} + \frac{7g^2 Z_0}{6c^2} \right) \right. \\ &\quad \left. + \frac{11g^2 V_0 t^5}{15c^2} - \frac{11g^3 t^6}{90c^2} \right). \end{aligned} \quad (\text{A.4})$$

The relativistic correction terms are easily identified by the factors of c in the denominators. The inverses of these transformations, obtained by an iterative process, are

$$\begin{aligned} t(T, Z) &= \frac{V_0 Z_0}{c^2} - \frac{V_0 Z}{c^2} + T \left(1 + \frac{V_0^2}{2c^2} + \frac{gZ}{c^2} \right) - \frac{gV_0 T^2}{2c^2} + \frac{g^2 T^3}{6c^2} \\ &\quad + \gamma \left(T \left(\frac{Z_0^2}{2c^2} - \frac{Z_0 Z}{c^2} \right) + T^2 \left(\frac{V_0 Z_0}{c^2} - \frac{V_0 Z}{2c^2} \right) \right. \\ &\quad \left. + T^3 \left(-\frac{gZ_0}{2c^2} + \frac{V_0^2}{3c^2} + \frac{gZ}{6c^2} \right) - \frac{7gT^4 V_0}{24c^2} + \frac{7g^2 T^5}{120c^2} \right); \end{aligned} \quad (\text{A.5})$$

$$\begin{aligned} z(T, Z) &= -Z_0 - \frac{V_0^2 Z_0}{2c^2} + \frac{gZ_0^2}{2c^2} + \left(1 + \frac{V_0^2}{2c^2} \right) Z - \frac{gZ^2}{2c^2} - \frac{3g^2 T^3 V_0}{2c^2} + \frac{3g^3 T^4}{8c^2} \\ &\quad + T \left(-V_0 - \frac{V_0^3}{2c^2} + \frac{2gV_0 Z_0}{c^2} - \frac{gV_0 Z}{c^2} \right) + T^2 \left(\frac{g}{2} + \frac{7gV_0^2}{4c^2} - \frac{g^2 Z_0}{c^2} + \frac{g^2 Z}{2c^2} \right) \\ &\quad + \gamma \left(-\frac{Z_0^2 Z}{2c^2} + \frac{Z_0 Z^2}{2c^2} + T \left(-\frac{V_0 Z_0^2}{c^2} + \frac{V_0 Z^2}{2c^2} \right) \right. \\ &\quad \left. + T^2 \left(-\frac{Z_0}{2} - \frac{7V_0^2 Z_0}{4c^2} + \frac{3gZ_0^2}{2c^2} - \frac{gZZ_0}{2c^2} - \frac{gZ^2}{4c^2} \right) \right. \\ &\quad \left. + T^3 \left(-\frac{V_0}{6} + \frac{10gV_0 Z_0}{3c^2} - \frac{7V_0^3}{12c^2} - \frac{gV_0 Z}{6c^2} \right) \right. \\ &\quad \left. + T^4 \left(\frac{g}{24} + \frac{61gV_0^2}{48c^2} - \frac{29g^2 Z_0}{24c^2} + \frac{g^2 Z}{24c^2} \right) \right. \\ &\quad \left. - \frac{3g^2 V_0 T^5}{4c^2} + \frac{g^3 T^6}{8c^2} \right). \end{aligned} \quad (\text{A.6})$$

These transformations can be used to investigate transformations of laser light wavevectors from the laboratory frame to the cube frame. Partial derivatives such as

$$\frac{\partial X^i}{\partial x^i}, \quad (\text{A.7})$$

where $X^i = \{cT, Z\}$ and $x^i = \{ct, z\}$ can be easily evaluated from the polynomials given in equations (14) and (A.4). For example, the time-time-component of the metric tensor in the falling frame will be

$$g_{00} = \left(\frac{\partial cT}{\partial ct} \right)^2 G_{00} + \left(\frac{\partial Z}{\partial ct} \right)^2 G_{ZZ}. \quad (\text{A.8})$$

Carrying out the partial differentiations and evaluating equation (A.8), expanding to order c^{-2} , keeping linear contributions in γ , and quadratic contributions in z , we obtain

$$g_{00} = -(1 - \gamma \frac{z^2}{2}). \quad (\text{A.9})$$

A similar calculation gives:

$$g_{zz} = \left(\frac{\partial cT}{\partial z} \right)^2 G_{00} + \left(\frac{\partial Z}{\partial z} \right)^2 G_{ZZ} = 1 + \gamma \frac{z^2}{c^2}. \quad (\text{A.10})$$

There are no linear terms in g_{00} or g_{zz} in the local z -coordinate because the origin was chosen to be at the point where the net force acts. In the cube-fixed frame the center of mass is at

$$z_{\text{cm}} = \frac{1}{M} \int_{-d}^{D-d} z \, dm = 0, \quad (\text{A.11})$$

while the force per unit mass at z is proportional to $-\gamma z$. The total force is then proportional to

$$-\gamma \int_{-d}^{D-d} z \, dm = 0, \quad (\text{A.12})$$

so the retroreflector is unaccelerated in the local freely falling frame. If one should choose the flat face of the falling object as origin of the local frame, the local metric tensor would have linear terms in z .

Such features express the equivalence principle: in a freely falling reference frame, the linear term in the expansion of the gravitational potential is cancelled by terms arising from the acceleration. This is primarily an algebraic consequence of the simultaneity term equation (A.2) that was added into the time transformation equation (14). The time derivative of this term is proportional to the acceleration of the origin of falling coordinates, times a ; in carrying out the standard tensor transformations of the components of the metric tensor, this term cancels the linear term in the Taylor expansion of the gravitational potential.

Although one may have questions about the transformations quoted in equations (14) and (A.4), the result clearly describes the physics in a freely-falling, locally inertial system with a static gravity gradient, with coordinates that are locally very similar to those used in Special Relativity. The linear term in the potential involving g has been transformed away in the local frame.

Appendix B. Non-relativistic derivation of corrections

The corrections given in equation (47) do not involve the constant c explicitly, and can be derived by taking the non-relativistic limit of the equations presented above. It is instructive to summarize the argument. For example, the difference between the time coordinates in the lab and in the falling frame can be neglected. From the transformation equations, the position of the cube face is then approximately

$$Z_{\text{face}}(T) = Z_{\text{cm}}(T) - d. \quad (\text{B.1})$$

The phase of the signal penetrating the cube is then

$$\phi_{\text{in}}(T) = -\Omega \left(T - \frac{Z_{\text{face}}(T)}{c} \right). \quad (\text{B.2})$$

The time delay during signal propagation within the cube is $2Dn/c$ so the phase of the signal leaving the cube is

$$\begin{aligned} \phi_{\text{out}}(T) &= \phi_{\text{in}}(T - 2Dn/c) \\ &= -\Omega \left(T - \frac{2Dn}{c} - \frac{Z_{\text{face}}(T - 2Dn/c)}{c} \right) \\ &\approx -\Omega \left(T - \frac{2Dn}{c} - \frac{Z_{\text{face}}(T)}{c} \right), \end{aligned} \quad (\text{B.3})$$

where in the last term the higher-order relativistic corrections can be neglected. The time required for the signal to reach the origin $Z = 0$, to leading order in c^{-1} , is $Z_{\text{face}}(T)/c$, so the phase of the signal at the detector is

$$\begin{aligned} &-\Omega \left(T - \frac{Z_{\text{face}}(T)}{c} - \frac{2Dn}{c} - \frac{Z_{\text{face}}(T - Z_{\text{face}}(T)/c)}{c} \right) \\ &= -\Omega \left(T - \frac{2Dn}{c} - \frac{2Z_{\text{face}}(T)}{c} \right). \end{aligned} \quad (\text{B.4})$$

Replacing Ω by $2\pi c/\lambda$, the ‘nonrelativistic’ contributions to the phase are

$$-\frac{2\pi c}{\lambda} \left(T - \frac{2Dn}{c} - \frac{2Z_{\text{face}}(T)}{c} \right). \quad (\text{B.5})$$

The first term is removed by interference with the reference beam so the net nonrelativistic phase difference is

$$2\pi \left(\frac{2Dn}{\lambda} + \frac{2Z_{\text{face}}(T)}{\lambda} \right). \quad (\text{B.6})$$

Combining this with equations (B.1) and (5) and rearranging terms gives the phase difference

$$2\pi \left(\frac{2(Dn - d + Z_0)}{\lambda} + \frac{2V_0(T + \gamma \frac{T^3}{6})}{\lambda} + \frac{T^2}{2\lambda} (\gamma Z_0 - g) - \frac{g\gamma T^4}{12\lambda} \right). \quad (\text{B.7})$$

In particular, the constant term gives rise to the correction in the first line of equation (47). This correction inserted into the T^2 term in equation (B.7) then leads to the last line of equation (47). The remaining terms have no effect on the conclusions stated in equation (47).

ORCID iDs

Neil Ashby  <https://orcid.org/0000-0003-0719-3979>

References

- [1] Ashby N and Bertotti B 1986 Relativistic effects in local inertial frames *Phys. Rev. D* **34** 2246–59
- [2] Fermi E 1922 *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Nat. Rend.* **31** 21
Fermi E 1922 *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Nat. Rend.* **31** 51
- [3] Manasse F K and Misner C W 1963 *J. Math. Phys.* **4** 735
- [4] Ni W-T and Zimmermann M 1978 *Phys. Rev. D* **17** 1473–6
- [5] Hammond J A and Faller J E 1971 *A Laser-Interferometer System for the Absolute Determination of the Acceleration due to Gravity* NBS Special Publication 343, Precision Measurement and Fundamental Constants (Washington, DC: US Government Printing Office) Catalog C 13.10:343 pp 457–64
- [6] Hammond J A and Faller J E 1971 Results of absolute gravity determinations at a number of different sites *J. Geophys. Res.* **76** 7850–4
- [7] Kopejkin S M 1988 Celestial coordinate reference systems in curved space-time *Celest. Mech.* **44** 87–115
- [8] Nagorni V D, Zanimonskiy Y M and Zanimonskiy Y Y 2011 Correction due to the finite speed of light in absolute gravimeters *Metrologia* **48** 101–13
- [9] Tan Y, Shao C, Li J and Hu Z 2016 A general relativistic model for free-fall absolute gravimeters *Metrologia* **53** 846–52
- [10] The following link has extensive information about properties of BK 7 glass: <https://refractiveindex.info/>

Reply

Reply to ‘Comment on Relativistic theory of the falling cube gravimeter’*

Neil Ashby¹ 

National Institute of Standards and Technology, Boulder, CO 80305, United States of America
University of Colorado, Boulder, CO 80309, United States of America

E-mail: ashby@boulder.nist.gov

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Abstract

The comment (Křen and Pálinkás 2017 *Metrologia* **55** 314–5) claims that the paper *Relativistic theory of the falling cube gravimeter* (Ashby 2017 *Metrologia* **55** 1–10) is incorrect. The authors of this comment assert that optical paths in the two interferometer arms of an absolute gravimeter shift only the absolute phase difference between interferometer arms and therefore cannot affect the measured value of g , and that the only needed relativistic correction is the commonly applied ‘speed of light correction’. Neither claim stands up to scrutiny.

Keywords: gravimeters, relativity, acceleration, retroreflectors

The value of δg quoted in [1] results from assuming reasonable values of D , d , n , and γ , since actual values were not known. Such numerical values are needed as inputs to fitting routines, should depend on the specific situation and will affect results.

The falling cube receives a signal at its face, shifted downwards in wavelength due to motion. The wavelength in the glass is diminished due to the refractive index. In the cube’s rest frame the signal frequency chirps upwards and the wavelength is further slightly diminished, thus many wavefronts become entrained in the glass and unavailable for interference until they emerge later from a lower position. Contrary to what the authors of [2] assert, this time-dependent dynamical effect on the phase of the emerging signal is not compensated for elsewhere in the apparatus. Whereas one might think at first that the constant term in the non-relativistic phase difference (equation (69) in appendix B of [1]) could be cancelled by contributions from the reference beam, this term arises from analysis of the signal that passes through the falling cube *only*. The reference beam could be produced by a reflector of different properties (e.g. a hollow cube with $n = 1$ and $D \approx d$) and its phase may be modified by a constant depending on distance from the beamsplitter, passage through windows, etc,

but such a constant cancels in the differences computed in equation (46) of [1]. Further, the relativistic terms separately satisfy the same conditions, (46) in [1].

In the cube rest frame, a wavefront emerges from the face after spending proper time $2Dn/c$ in the glass. Coordinate transformations between the laboratory and cube rest frame show the delay in the laboratory frame is also equal to $2Dn/c$ to order c^{-2} , but the frequency that emerges later from the face differs from the entering frequency by additional time-dependent contributions, than if reflection occurred at the face. The coordinate transformations provide a means for analyzing wavefront propagation in two ways—either in the cube rest frame or in the laboratory frame. The two approaches give the same results; the simplest parts are described in the publication [1].

In a gravitational field g with a gradient, the net force acts at the center of mass (CM) so the CM falls in the same way that a point mass at the CM would fall. The comment dismisses the details of the actual physical processes taking place in the gravimeter, and denies the relevance of parameters d , the distance of the cube face from the CM, and n , the refractive index of the glass. If this were true, a given software data fitting package would give the same result for g independent of n , whether or not the prism is large and solid or small and hollow, whereas software based on the model in [1] would account for such properties and give differing results.

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¹ Author to whom any correspondence should be addressed.

The comment states ‘The counting interferometer ... does not measure differences of measuring arm with respect to reference arm’. The meaning of this comment is unclear since the measured quantities of interest are related to phase differences between the two arms.

Also, the comment states ‘the second term γd is irrelevant too, because the position of the testing body is standardly related to its CM (i.e. it is not shifted to the face by d)’. Actually the paper [1] always relates the position of the cube to its CM and never states its position is shifted to the face. It should be standard to relate the cube’s position to the CM, but it is untenable to compute the delay with speed c from cube to detector from the CM since the CM is inside the glass.

The comment states that ‘only the finite “speed of light”, that is already commonly applied by accounting for the variable time delay in the measuring arm of the interferometer, is the necessary “relativistic” correction for the absolute gravimeters with macroscopic objects’. This statement misses one of the main points explained in [1], namely that the expression for retarded time must be iterated twice, to obtain relativistic corrections to order c^{-2} , rather than once as appears to be the case normally [4, 5]. Suppose a plane wave source at moving position $Z_{\text{face}}(T_T)$ in the lab emits a wave that propagates downward at speed c in vacuum, the observable being the time T at which the wave arrives at a detector at a fixed position $Z = 0 < Z(T_T)$. The emission time T_T is a retarded time and is given by an implicit equation [3]: $T_T = T - Z_{\text{face}}(T_T)/c$. This equation can be solved by iteration, as discussed in [1]. Alternatively, a Taylor expansion to order c^{-2} gives the approximation

$$T_T = T - \frac{Z_{\text{face}}(T)}{c} + \frac{Z_{\text{face}}(T)V_{\text{face}}(T)}{c^2}, \quad (1)$$

which has non-negligible time-dependent contributions to the phase difference between the arms, of order c^{-2} . These were incorporated into derivations in [1], where they play a role in arriving at the stated conclusions. The ‘speed of light correction as commonly stated’ [4, 5] assumes light propagates from the CM with speed c , whereas part of the optical path is certainly through the glass. Further, the commonly used expression is incapable of providing all terms of order c^{-2} correctly. In summary, no credible grounds are given for denial of the simplified explanation in appendix B of [1], and the comment [2] does not provide any helpful arguments that would show that the derivations in [1] are flawed.

ORCID iDs

Neil Ashby  <https://orcid.org/0000-0003-0719-3979>

References

- [1] Ashby N 2017 Relativistic theory of the falling retroreflector gravimeter *Metrologia* **55** 1–10
- [2] Křen P and Pálinkás V 2017 Comment on ‘Relativistic theory of the falling retroreflector gravimeter’ *Metrologia* **55** 314–5
- [3] Griffiths D J 1989 *Introduction to Electrodynamics* 2nd edn (Englewood Cliffs, NJ: Prentice-Hall) p 416
- [4] Niebauer T M, Sasagawa G S, Faller J E, Hilt R and Klotting F 1995 A new generation of absolute gravimeters *Metrologia* **32** 159–80
- [5] Nagorny V D, Zanimonskiy Y M and Zanimonskiy Y Y 2011 Correction due to the finite speed of light in absolute gravimeters *Metrologia* **48** 101–13 (equations (2) and (3))

Reply

Reply to ‘No correction for the light propagation within the cube: Comment on *Relativistic theory of the falling cube gravimeter*’*

Neil Ashby 

National Institute of Standards and Technology, Boulder, CO 80305, United States of America
University of Colorado, Boulder, CO, United States of America

E-mail: ashby@boulder.nist.gov

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Abstract

The comment (Nagorny 2018 *Metrologia*) claims that, notwithstanding the conclusions stated in the paper *Relativistic theory of the falling cube gravimeter* (Ashby 2008 *Metrologia* 55 1–10), there is no need to consider the dimensions or refractive index of the cube in fitting data from falling cube absolute gravimeters; additional questions are raised about matching quartic polynomials while determining only three quantities. The comment also suggests errors were made in Ashby (2008 *Metrologia* 55 1–10) while implementing the fitting routines on which the conclusions were based. The main contention of the comment is shown to be invalid because retarded time was not properly used in constructing a fictitious cube position. Such a fictitious position, fixed relative to the falling cube, is derived and shown to be dependent on cube dimensions and refractive index. An example is given showing how in the present context, polynomials of fourth order can be effectively matched by determining only three quantities, and a new compact characterization of the interference signal arriving at the detector is given.

Keywords: gravimeters, relativity, acceleration, retroreflectors

The phase of the reflected signal derived in [1] comes from a detailed relativistic analysis of signal phase arriving at the point of recombination of the test and reference beams in an absolute gravimeter, and provides an alternative to the currently used or ‘standard’ approach [3]. The ‘corrections’ calculated in [1] represent differences between two models of the retroreflector: one model characterized by parameters D , n , and d , which are respectively the cube depth, refractive index, and distance of the face from the center of mass (COM), and a model with $D = d = 0$ corresponding to reflection from the face. The difference arises when these two models are used to fit the same data—other conditions remaining

unchanged—and the results quoted are large enough to show that the index of refraction and other retroreflector properties should be accounted for in fitting data from absolute gravimeters.

The comment states that the correction translates the gravity value to the point above the cube’s COM by the distance $Dn - d$. However, such a point would be in motion, would not give a unique value for g , and is inconsistent with the definition of g used in [1]. The gravity value referred to in the original paper [1] is the value at $Z = 0$, the point of recombination. This can be seen from the form of the gravitational potential as a function of the vertical coordinate Z , in equation (1) of [1].

After quoting the phase of the signal at the beamsplitter from the paper [1], the comment multiplies the interference

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signal by $2c/\Omega$ where Ω is the angular frequency of reference beam light, thus deriving a quantity called ‘the position of the cube at the moment T' ’ (equation (3) of [2]). This is a fictitious position, since the interference signal is generated at the origin $Z = 0$. The calculation is based on the idea that successive interference fringes correspond to changes in position of the point of reflection by a half wavelength. The times T are times of arrival at the reference origin $Z = 0$, whereas the fictitious retroreflector position is at some height $Z(T')$ above the beamsplitter at which there is no reference beam to generate an interference signal. If this cube position had meaning, the position would have to be related to the COM position by no more than an added constant. This is not so, as can be seen by comparing the COM position, used repeatedly in [1]—

$$Z_{cm}(T) = Z_0 + V_0T - \frac{1}{2}gT^2 + \gamma\left(\frac{1}{2}Z_0T^2 + \frac{1}{6}V_0T^3 - \frac{1}{24}gT^4\right) \tag{1}$$

—with equation (2) of [2], which has additional time-dependent terms. The main conclusion of the comment, as expressed in its title, is thus without justification.

However, following this line of argument leads to an interesting result. Let us account for retardation by assuming that at moving position $Z(T')$ a wavefront of phase $\phi(T')$ is emitted and travels with speed c through vacuum to the detector, where it arrives at time T later. The retarded time is determined implicitly by $T' = T - Z(T')/c$. Let the interference phase at the detector be the function $\phi_0(T)$ as provided in equation (33) of [1] and quoted in the comment [2], equation (1) with the reference phase removed. The phase at emission would be

$$\phi(T') = \phi_0(T) = \phi_0(T' + Z(T')/c). \tag{2}$$

Proceeding as in the comment, we identify the position $Z(T')$ as

$$Z(T') = \frac{c\phi(T')}{2\Omega} = \frac{c}{2\Omega} \left(\phi_0(T') + \frac{d\phi_0(T')}{dT'} \frac{Z(T')}{c} \right). \tag{3}$$

This leads to

$$Z(T') = \frac{c}{2\Omega} \frac{\phi_0(T')}{1 - \frac{1}{2\Omega} \frac{d\phi_0(T')}{dT'}}. \tag{4}$$

Carrying out this calculation, we obtain for the fictitious position

$$Z(T') = Z_{cm}(T') + Dn - d + O(c^{-2}). \tag{5}$$

This result would have been obtained in the comment [2], if retardation had been included. The form of equation (5), above, fully supports the conclusion in [1] that properties of the retroreflector are important in absolute gravimeters. Equation (5) suggests there is a hypothetical surface at distance $Z(T')$ above the COM, from which the phase at the detector can be obtained by application of a simple retardation calculation with light speed c . The hypothetical phase at this surface at time T' would be

$$\phi(T') = \frac{2\Omega}{c} \left(Z_{cm}(T') + Dn - d \right) \tag{6}$$

and, if transferred to the origin at $Z = 0$ with speed c using retarded time, yields $\phi_0(T)$. This compact characterization of the interference signal in [1] works very well. The fictitious position is above the COM and in fact, above the cube.

The constant in equation (5) contains all the corrections discussed in [1], and provides a useful example of the problem of matching two polynomials in the present context. We ask, in a model based on equation (5), what corrections $\delta Z_0, \delta V_0, \delta g$ arise compared to simply using a model based on the uncorrected COM position, equation (1)? This leads to the condition

$$Dn - d + \delta Z_0 + \delta V_0T - \frac{1}{2}\delta gT^2 + \frac{\gamma}{24}(12\delta Z_0T^2 + 4\delta V_0T^3 - \delta gT^4) = 0. \tag{7}$$

This is a polynomial of order four involving five coefficients, while only three quantities are being determined. However, $\delta V_0 = 0$ removes the two terms in equation (7) that are odd in T ; the constant term gives $\delta Z_0 = -(Dn - d)$, and the quadratic term gives $\delta g = \gamma\delta Z_0$. The residual quartic term is $-\gamma^2T^4\delta g/24$. This term is of second order in γ and is extremely small; even for $T = 1$ s, it is less than 10^{-14} meters. Such terms have been systematically neglected throughout [1]. This explains how two fourth-order polynomials can be matched, with more than adequate precision, with only three quantities. The issue was already discussed near the end of section 7 in [1]. Thus, there is no basis for the suggestion in the comment, that a mistake was made in the fitting routines; this is extremely unlikely anyway because of the agreement between the fitting of data for thousands of drops and the algebraic calculations carried out independently.

In sum, the suggestion in the comment that it is not necessary to consider retroreflector properties in fitting data from absolute gravimeters has been shown to be invalid. Other statements in the comment have been responded to; in particular, it has been explained how for this theory only three parameters are precisely determined by matching fourth-order polynomials. A compact characterization of the interference signal has been presented. No errors or substantive criticisms of the original paper [1] as claimed in the comment [2] are justified.

ORCID iDs

Neil Ashby  <https://orcid.org/0000-0003-0719-3979>

References

- [1] Ashby N 2008 Relativistic theory of the falling retroreflector gravimeter *Metrologia* **55** 1–10
- [2] Nagorny V D 2018 No correction for the light propagation within the cube: Comment on ‘Relativistic Theory of the falling retroreflector gravimeter’ *Metrologia*
- [3] Nagorny V D, Zanimonskiy Y M and Zanimonskiy Y Y 2011 Correction due to the finite speed of light in absolute gravimeters *Metrologia* **48** 101–13