Confidence Estimates in Simulation of Phase Noise or Spectral Density

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***Abstract*—In this paper, we apply the method of discrete simulation of power-law noise, developed by Timmer and König, Ashby, and Ashby and Patla, to the problem of simulating phase noise for a combination of power-law noises. We derive analytic expressions for the probability of observing a value of phase noise $\mathcal{L}(f)$ or of any of the one-sided spectral densities $S_\phi(f)$, $S_y(f)$, or $S_x(f)$, for arbitrary superpositions of power-law noise.**

***Index Terms*—Phase noise, power-law noise, simulation, spectral density.**

I. Introduction

POWER-LAW noise is widely used to describe the performance of many types of clocks, particularly those used in modern standards laboratories. Simulation of such power-law noise can be very useful for testing software algorithms that extract spectral density and stability information from measured time deviations, and particularly in predicting the probability of observing values of some clock stability variances. In a series of investigations [2]–[4], a method of simulating power-law noise has been developed that is based on creating a spectrum for white phase noise, modifying the spectrum in the frequency domain to correspond to some chosen power-law noise, and then transforming to the time domain, producing a time series with the desired statistical properties. This method was first proposed by Timmer and König [1] who applied it to the study of AGN light curves. The method has been successfully applied to the Allan and Hadamard Variances, with and without dead time, in modified, overlapping, and nonoverlapping forms [3], [4]. In this paper, we apply the method to one-sided spectral densities, thus not requiring a transformation to the time domain. In Section II, we summarize the process of discrete simulation of power-law noise and show how it leads directly to simulations of the phase noise $\mathcal{L}(f)$. In Section III, we develop probability distributions for observation of phase noise as a function of offset frequency, and discuss probability distributions for observations of the various forms of spectral densities. Section VIII compares the predictions for a particular type of flicker-like noise with real data from an amplifier.

II. Simulation Method

We imagine that the noise amplitudes at Fourier frequencies f_m are generated by a set of normally distributed random complex numbers w_m having mean zero and variance σ^2 that would by themselves generate a simulated spectrum for white phase noise. These random numbers are then divided by a function of the frequency, $|f|^\lambda$, generating a spectral density that has the desired frequency characteristics. For ordinary power-law noises, the exponent λ is a multiple of $1/2$, but any value is possible. The frequency noise could then be Fourier transformed to the time domain, producing a time series that could be used to simulate one of the stability variances. The time between successive simulated time measurements is denoted by τ_0. A natural frequency cutoff occurs at $f_h = 1/(2\tau_0)$. The time measurements are assumed to be made at the times $k\tau_0$, where k is an integer, and the time errors or residuals relative to a reference clock are denoted by x_k. The total length of time of the entire measurement series is $N\tau_0$. The possible frequencies that occur in the Fourier transform of the time residuals are

$$f_m = \frac{m}{N\tau_0}; \quad -\frac{N}{2} + 1 \le m \le \frac{N}{2}. \tag{1}$$

For a set of noise amplitudes in the frequency domain to represent a real series in the time domain, the amplitudes must satisfy the reality condition

$$w_{-m} = (w_m)^*. \tag{2}$$

Thus, if $w_m = u_m + iv_m$, where u_m and v_m are independent uncorrelated real random numbers, then $(w_m)^* = u_m - iv_m$. N random numbers are placed in $N/2$ real and $N/2$ imaginary parts of the positive frequency spectrum and then (2) is used to populate the negative frequency part of the spectrum. Since the frequencies $\pm 1/(2\tau_0)$ represent essentially the same contributions, we shall assume that $v_{N/2}$ does not occur. We shall assume that the variance of the noise amplitudes is such that

$$\langle (w_m)^* w_n \rangle = \langle u_m^2 + v_m^2 \rangle = 2\sigma^2 \delta_{mn}; \quad m \ne 0, N/2. \tag{3}$$

Also, in order to avoid division by zero, we shall always assume that the Fourier amplitude corresponding to zero frequency vanishes. This means that the average of the time residuals will be zero, and has no effect on any variance that involves time differences. This model does not account for drift.

The kth term in the discrete Fourier transform of white phase noise generated by the random variable w_m is proportional to the kth member of the time series for white PM [3], [4]

$$x_k \propto \sum_{m=-N/2+1}^{N/2} e^{-\frac{2\pi imk}{N}} w_m. \tag{4}$$

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We multiply each frequency component by $|1/f_m|^\lambda$ and insert a proportionality constant K to determine the level of the noise, and give the sequence of time residual measurements the physical dimensions of time. This will generate the desired power-law form of the spectral density or of the phase noise when K and λ are appropriately chosen. The simulated time series will be represented by

$$X_k = K \sum_{m=-N/2+1}^{N/2} \frac{1}{|f_m|^\lambda} e^{-\frac{2\pi imk}{N}} w_m. \tag{5}$$

To show how (5) gives rise to commonly used expressions for spectral density, we shall compute the single-sided spectral density arising from the time series. The average (two sided) spectral density of time residuals is obtained from a single term in (5)

$$\overline{s}_x(f_m) = \frac{K^2}{|f_m|^{2\lambda}} \left\langle \frac{w_m (w_m)^*}{\Delta f} \right\rangle = \frac{K^2 N \tau_0 (2\sigma^2)}{|f_m|^{2\lambda}} \tag{6}$$

where $\Delta f = 1/N\tau_0$ is the spacing between successive allowed frequencies. We double (6) to obtain the average one-sided spectral density: $\overline{S}_x(f_m) = 2\overline{s}_x(f_m)$. The average one-sided spectral density of fractional frequency fluctuations is given by the well-known relation

$$\overline{S}_y(f_m) = (2\pi f_m)^2 \overline{S}_x(f_m). \tag{7}$$

The average one-sided spectral density of fractional frequency fluctuations is then

$$\overline{S}_y(f_m) = 2(2\pi f_m)^2 \frac{K^2 N \tau_0 (2\sigma^2)}{|f_m|^{2\lambda}} = h_\alpha f_m^\alpha \tag{8}$$

for positive frequencies, where $S_y(f) = h_\alpha f^\alpha$ is usually used for a single type of power-law noise [5]. Therefore, we must set

$$\lambda = 1 - \frac{\alpha}{2}; \quad \alpha = 2 - 2\lambda \tag{9}$$

and

$$K = \sqrt{\frac{h_\alpha}{16\pi^2 \sigma^2 (N\tau_0)}}. \tag{10}$$

In a particular simulation run in which one set of random variables (u_m, v_m) is generated, resulting in a time series as in (5), a spectral density will generally deviate from its average value. We denote such particular values by omitting the overbar, whereas an average is denoted by relations, such as

$$\overline{S}_x(f_m) = \frac{h_\alpha f_m^\alpha}{(2\pi f_m)^2}. \tag{11}$$

The averaged spectral density of fractional frequency fluctuations is then

$$\overline{S}_y(f_m) = (2\pi f_m)^2 \overline{S}_x(f_m) = h_\alpha f_m^\alpha. \tag{12}$$

The spectral density of phase fluctuations is [5]

$$S_\phi(f_m) = \frac{\nu_0^2}{f_m^2} S_y(f_m) \tag{13}$$

and its averaged value is

$$\overline{S}_\phi(f_m) = \frac{\nu_0^2}{f_m^2} \overline{S}_y(f_m) = \frac{\nu_0^2}{f_m^2} h_\alpha f_m^\alpha. \tag{14}$$

A particular simulation run will result in values of phase noise at offset frequency f_m given by

$$\mathcal{L}(f_m) = \frac{\nu_0^2}{2 f_m^2} h_\alpha f_m^\alpha \frac{|w_m|^2}{2\sigma^2}. \tag{15}$$

Here, the factor of 1/2 arises when the IEEE definition of phase noise $\mathcal{L}$ is used. The average value of the phase noise at a given frequency will be obtained by replacing $|w_m|^2$ by its average value, $2\sigma^2$, so the average value of the phase noise will correspond to the definition

$$\overline{\mathcal{L}} = \langle \mathcal{L}(f) \rangle = \frac{1}{2} \frac{\nu_0^2}{f^2} \overline{S}_y(f) = \frac{1}{2} \frac{\nu_0^2}{f^2} h_\alpha f_m^\alpha. \tag{16}$$

III. Probabilities

We are going to employ a representation of the delta function of the form [6]

$$\delta(L - \mathcal{L}(f)) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega(L - \mathcal{L}(f))}. \tag{17}$$

The probability of observing a particular value $\mathcal{L}$ of the phase noise at offset frequency f_m can be obtained from

$$\begin{aligned} P(L) &= \int \delta\Big(L - \mathcal{L}(f_m)\Big) \frac{du_m dv_m}{2\pi\sigma^2} e^{-\frac{u_m^2 + v_m^2}{2\sigma^2}} \\ &= \int \frac{d\omega}{2\pi} e^{i\omega\left(L - \frac{\nu_0^2}{4\sigma^2 f_m^2} \overline{S}_y(f)(u_m^2 + v_m^2)\right)} \\ &\quad \times \frac{du_m dv_m}{2\pi\sigma^2} e^{-\frac{u_m^2 + v_m^2}{2\sigma^2}} \end{aligned} \tag{18}$$

where it is understood that all quantities refer to a fixed frequency. We shall drop the subscript m where it is simpler to do so. The delta function in the first line of (18) constrains the normally distributed random variables to values which satisfy the condition that a particular value of phase noise is of interest, and the exponential representation of the delta function (17) allows the integrals to be performed. An intermediate result is obtained by performing the integrals over the random variables

$$\begin{aligned} P(L) &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{i\omega L}}{\left(\sqrt{1 + \frac{i\omega \nu_0^2 \overline{S}_y(f)}{2f^2}}\right)^2} \\ &= \frac{2f^2}{i\nu_0^2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{i\omega L}}{\omega - \frac{2if^2}{\nu_0^2 \overline{S}y(f)}}. \end{aligned} \tag{19}$$

The integration contour can be deformed to encircle the simple pole in the complex ω plane and the final result is

$$P(L) = \frac{2f^2}{\nu_0^2 \overline{S}_y(f)} e^{-\frac{2f^2}{\nu_0^2 \overline{S}_y(f)} L}. \tag{20}$$

The average value of $\mathcal{L}(f)$ obtained from this probability is

$$\overline{\mathcal{L}}(f) = \int_0^\infty L P(L) dL = \frac{1}{2} \frac{\nu_0^2}{f^2} \overline{S}_y(f). \tag{21}$$

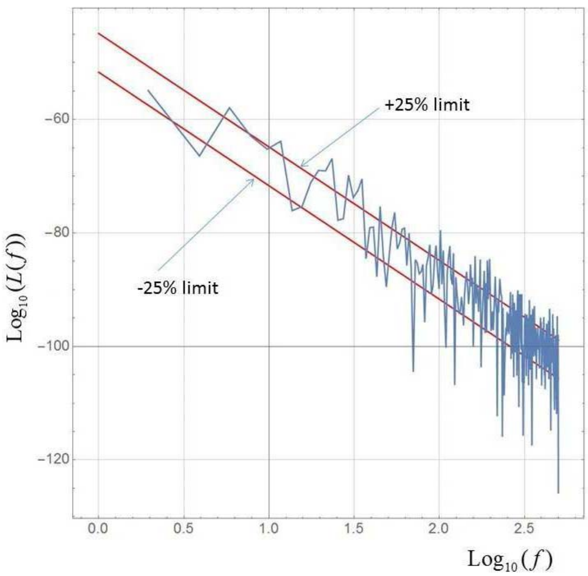

Fig. 1. Two million white FM measured data points were divided into 4096 files each with 512 points. For one of these files, the phase noise, having 256 data items, was computed and compared with the limits given in (24) and (25). The mean and rms deviations of the numbers of points between the limits for the set of 4096 files was 126.5 ± 8.5.

The rms deviation from the mean of $\mathcal{L}(f)$ is a crude measure of the confidence in the average value, and is

$$\sqrt{\int_0^\infty L^2 P(L)dL - \overline{\mathcal{L}}^2} = \frac{1}{2}\frac{\nu_0^2}{f^2}\overline{S}_y(f). \tag{22}$$

For any given power law for the spectral density of frequency, the $\pm 25\%$ limits on the probability can be determined from (20). The cumulative probability will be

$$\Phi(\mathcal{L}) = \int_0^{\mathcal{L}} P(L)dL = 1 - e^{-\frac{2f^2\mathcal{L}}{\nu_0^2\overline{S}_y(f)}}. \tag{23}$$

Equating the cumulative probability to 0.25 or 0.75 gives the values of corresponding to the $\pm 25\%$ limits of observing around the average. For the value corresponding to -25%

$$0.250 = 1 - e^{-\frac{2f^2\mathcal{L}}{\nu_0^2\overline{S}_y(f)}}$$

and solving for the phase noise

$$\mathcal{L}|_{-25\%} = 0.2877\frac{\nu_0^2}{2f^2}\overline{S}_y(f) \tag{24}$$

similarly

$$\mathcal{L}|_{+25\%} = 1.3863\frac{\nu_0^2}{2f^2}\overline{S}_y(f). \tag{25}$$

Fig. 1 shows a comparison between the $\pm 25\%$ limits calculated above and the probabilities obtained from actual data. Further analysis of the data is discussed in Section VIII.

IV. Probability Distributions for Other Noise Measures

Because of the linear relationships between phase noise and spectral densities, such as in (11)–(14), probability densities can easily be derived for the spectral densities $S_x(f)$, $S_y(f)$, and $S_\phi(f)$. Thus

$$\begin{aligned} P(S_x) &= \frac{1}{\overline{S}_x}e^{-\frac{S_x}{\overline{S}_x}} = \frac{(2\pi f)^2}{h_\alpha f^\alpha}e^{-\frac{(2\pi f)^2 S_x}{h_\alpha f^\alpha}} \\ P(S_y) &= \frac{1}{\overline{S}_y}e^{-\frac{S_y}{\overline{S}_y}} = \frac{1}{h_\alpha f^\alpha}e^{-\frac{S_y}{h_\alpha f^\alpha}} \\ P(S_\phi) &= \frac{1}{\overline{S}_\phi}e^{-\frac{S_\phi}{\overline{S}_\phi}} = \frac{f^2}{\nu_0^2 h_\alpha f^\alpha}e^{-\frac{f^2 S_\phi}{\nu_0^2 h_\alpha f^\alpha}}. \end{aligned} \tag{26}$$

V. Averaging Multiple Runs

In this section, we consider what happens to the probability when M independent measurements of the phase noise are averaged. The probability of obtaining an average value $\langle\mathcal{L}\rangle$ can be expressed in terms of the independent values that are averaged, using a representation for the δ-function. We use the subscript M to denote the average over independent simulation runs. Then

$$\begin{aligned} P(\langle\mathcal{L}\rangle_M) = \int \frac{d\omega}{2\pi}e^{i\omega\left(\langle\mathcal{L}\rangle_M - \frac{1}{M}\sum_{K=1}^M L_K\right)}dL_1 dL_2 \ldots dL_M \\ \times P(L_1)P(L_2)\ldots P(L_M) \end{aligned} \tag{27}$$

where L_K denotes the value of the phase noise obtained in the Kth simulation run. From (20), one of the independent integrals in the product in (27) is of the form

$$\int_0^\infty dL\frac{2f^2}{\nu_0^2\overline{S}_y(f)}e^{-\frac{2f^2 L}{\nu_0^2}\overline{S}_y(f) - \frac{i\omega L}{M}} = \frac{1}{\frac{i\omega}{M} + \frac{2f^2}{\nu_0^2}\overline{S}_y(f)}. \tag{28}$$

There are M such factors, so

$$P(\langle\mathcal{L}\rangle_M) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{e^{i\omega\langle\mathcal{L}\rangle_M}}{\left(\frac{i\omega}{M} + \frac{2f^2}{\nu_0^2\overline{S}_y(f)}\right)^M}. \tag{29}$$

The contour integral can be closed in the upper half ω plane, where it encircles the pole of order M on the imaginary axis at

$$\omega = \frac{2iMf^2}{\nu_0^2\overline{S}_y(f)}. \tag{30}$$

The result of the contour integration is

$$P(\langle\mathcal{L}\rangle_M) = \frac{(\langle\mathcal{L}\rangle_M)^{M-1}}{(M-1)!}\left(\frac{2Mf^2}{\nu_0^2\overline{S}_y(f)}\right)^M e^{-\frac{2Mf^2}{\nu_0^2\overline{S}_y(f)}\langle\mathcal{L}\rangle_M}. \tag{31}$$

This is a chi-squared distribution with $2M$ degrees of freedom. If M is large compared with unity, a chi-squared distribution can be approximated by a normal distribution, which in this case is approximately given by

$$\begin{aligned} P(\langle\mathcal{L}\rangle_M) &= \sqrt{\frac{M}{2\pi}}\left(\frac{2f^2}{\nu_0^2\overline{S}_y(f)}\right) \\ &\quad\times e^{-\frac{M}{2}\left(\frac{2f^2}{\nu_0^2\overline{S}_y(f)}\right)^2\left(\langle\mathcal{L}\rangle_M - \frac{\nu_0^2\overline{S}_y(f)}{2f^2}\right)^2}. \end{aligned} \tag{32}$$

From this expression, it is straightforward to see that the average of $\mathcal{L}_M$ is

$$\langle\mathcal{L}\rangle_M = \int_0^\infty \overline{\mathcal{L}}P(\overline{\mathcal{L}})d\overline{\mathcal{L}} = \frac{\nu_0^2\overline{S}_y(f)}{2f^2} \tag{33}$$

that is, there is no dependence on M. However, the rms deviation from the mean is reduced by a factor of $M^{-1/2}$

$$\left\langle\overline{\mathcal{L}}^2 - <\overline{\mathcal{L}}>^2\right\rangle = \int_0^\infty \overline{\mathcal{L}}^2 P(\overline{\mathcal{L}})d\overline{\mathcal{L}} - \left(\frac{\nu_0^2 \overline{S}_y(f)}{2f^2}\right)^2 = \left(\frac{\nu_0^2 \overline{S}_y(f)}{2f^2}\right)^2 \frac{(M+1)!}{M(M!)} - \left(\frac{\nu_0^2 \overline{S}_y(f)}{2f^2}\right)^2 = \frac{1}{M}\left(\frac{\nu_0^2 \overline{S}_y(f)}{2f^2}\right)^2. \quad (34)$$

This might well have been expected for white FM, which involves simple Gaussian noise, but it has been shown here to be valid for any one of the power-law noises.

VI. Superposition of Power Noise Processes

Probabilities, such as are given in (20), may be generalized to an arbitrary superposition of independent random noise processes. At a given offset frequency f_m, let w_m^α be the random variable that contributes to the time residual from the random power-law noise process described by the parameter α. The superposition of time residuals will be the sum over the processes that contribute

$$X_k = \frac{1}{4\pi\sigma\sqrt{N\tau_0}} \sum_\alpha \sum_{m=-N/2+1}^{N/2} \frac{\sqrt{h_\alpha f_m^\alpha}}{|f_m|} e^{-\frac{2\pi i m k \tau_0}{N\tau_0}} w_m^\alpha \quad (35)$$

where we have used (9) and (10). The contribution to the time series X_k from a given frequency f_m is

$$X_k(f_m) = \frac{1}{4\pi\sigma\sqrt{N\tau_0}} e^{-\frac{2\pi i m k \tau_0}{N\tau_0}} \sum_\alpha \frac{\sqrt{h_\alpha f_m^\alpha}}{|f_m|} w_m^\alpha. \quad (36)$$

The two-sided spectral density of time fluctuations is the absolute square of this quantity divided by the resolution bandwidth, $1/(N\tau_0)$

$$s_x(f_m) = \frac{1}{(4\pi\sigma)^2}\left|\sum_\alpha \frac{\sqrt{h_\alpha f_m^\alpha}}{|f_m|} w_m^\alpha\right|^2. \quad (37)$$

For a one-sided spectral density, we must multiply by 2 and take $f_m > 0$. Then, the simulated phase noise will be

$$\mathcal{L}(f_m) = \frac{1}{2}\frac{\nu_0^2}{f_m^2}(2\pi f_m)^2 2\left[\frac{1}{(4\pi\sigma)^2}\left|\sum_\alpha \frac{\sqrt{h_\alpha f_m^\alpha}}{|f_m|} w_m^\alpha\right|^2\right] = \frac{\nu_0^2}{4\sigma^2 f_m^2}\left[\left(\sum_\alpha \sqrt{h_\alpha f_m^\alpha} u_m^\alpha\right)^2 + \left(\sum_\alpha \sqrt{h_\alpha f_m^\alpha} v_m^\alpha\right)^2\right]. \quad (38)$$

As the random variables take on their values during the simulation runs, the probability of observing a value of phase noise will be given by integrating over all the possibilities, subject to the condition that the phase noise is constrained by expression (38)

$$P(\mathcal{L}) = \int \delta\left(\mathcal{L} - \frac{\nu_0^2}{4\sigma^2 f_m^2}\left[\left(\sum_\alpha \sqrt{h_\alpha f_m^\alpha} u_m^\alpha\right)^2 + \left(\sum_\alpha \sqrt{h_\alpha f_m^\alpha} v_m^\alpha\right)^2\right]\right) \times \prod_\alpha \left(\frac{du_m^\alpha dv_m^\alpha}{2\pi\sigma^2} e^{-\left(\frac{(u_m^\alpha)^2 + (v_m^\alpha)^2}{2\sigma^2}\right)}\right). \quad (39)$$

To evaluate this expression, we first observe that the result will be independent of the variance σ; only ratios, such as u_m/σ and v_m/σ, occur. Therefore, we can set $\sigma = 1$, simplifying the expression. The integrations will be performed by diagonalizing the squared quantities occurring in the exponent due to representation (34). Furthermore, we shall do this in such a way that the Gaussian form of the probability distributions remains essentially unchanged. We need an orthogonal transformation of variables in order to accomplish this, as discussed in [3] and [4]. There is a further simplification since if the diagonalization can be accomplished for the variables u_m^α, the same procedure will work for v_m^α. Then, defining

$$K_m = \frac{\nu_0^2}{(2f_m)^2} \quad (40)$$

the probability becomes

$$P(\mathcal{L}) = \int \frac{d\omega}{2\pi} \prod_\alpha \frac{du_m^\alpha dv_m^\alpha}{2\pi} e^{-\left(\frac{(u_m^\alpha)^2 + (v_m^\alpha)^2}{2}\right)} \times e^{i\omega\left(\mathcal{L} - K_m\left[\left(\sum_\alpha \sqrt{h_\alpha f_m^\alpha} u_m^\alpha\right)^2 + \left(\sum_\alpha \sqrt{h_\alpha f_m^\alpha} v_m^\alpha\right)^2\right]\right)}. \quad (41)$$

This integral can be completely evaluated. First, we note that the dependence on the real and imaginary contributions from the random variables separates so that integrals can be done separately. Consider the contribution from the real part

$$I = \int e^{-i\omega K_m\left(\sum_\alpha \sqrt{h_\alpha f_m^\alpha} u_m^\alpha\right)^2} \prod_\alpha \left(\frac{du_m^\alpha}{\sqrt{2\pi}} e^{-\frac{(u_m^\alpha)^2}{2}}\right). \quad (42)$$

Obviously, the contribution from the imaginary parts will be similar so the final result will involve the square of the abovementioned integral.

Let us define the symmetric real matrix by means of the elements

$$H_{\alpha\beta} = \left(h_\alpha h_\beta f_m^\alpha f_m^\beta\right)^{1/2}. \quad (43)$$

Then, we look for eigenvalues ϵ and column eigenvectors $\psi_\beta(\epsilon)$ of $H_{\alpha\beta}$ by solving

$$\sum_\beta H_{\alpha\beta}\psi_\beta(\epsilon) = \epsilon\psi_\alpha(\epsilon). \quad (44)$$

Study of the eigenvalue problem for a matrix, such as (43), shows that there is only one nonzero eigenvalue, which simplifies the calculation. We find for this eigenvalue

$$\epsilon = \sum_\alpha h_\alpha f_m^\alpha = \overline{S}_y(f_m). \quad (45)$$

To prove that there is only one nonzero eigenvalue, consider a matrix of the form

$$\begin{pmatrix} a_{11}-\epsilon & \sqrt{a_{11}a_{22}} & \sqrt{a_{11}a_{33}} & \dots & \sqrt{a_{11}a_{nn}} \\ \sqrt{a_{22}a_{11}} & a_{22}-\epsilon & \sqrt{a_{22}a_{33}} & \dots & \sqrt{a_{22}a_{nn}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sqrt{a_{nn}a_{11}} & \sqrt{a_{nn}a_{22}} & \sqrt{a_{nn}a_{33}} & \dots & a_{nn}-\epsilon \end{pmatrix}. \quad (46)$$

Multiply the first row by $(a_{22}/a_{11})^{1/2}$ and subtract it from the second row. The second row is reduced to

$$\begin{pmatrix} \epsilon & -\epsilon & 0 & 0 & 0 & \dots & 0 \end{pmatrix} \quad (47)$$

that is, the second row is reduced to only two nonzero elements. Similarly, if we multiply the first row by $(a_{jj}/a_{11})^{1/2}$ and subtract it from the jth row, that row will be reduced to only two nonzero elements. Then, if we expand the determinant along the rows $1, 2, 3, \ldots, n$, it is obvious that there will be a factor ϵ^{n-1} in the expansion of the determinant. Thus, there will be $n-1$ zero eigenvalues, and there can only be one nonzero eigenvalue. To obtain an expression for the nonzero eigenvalue, consider the product of all the diagonal elements. It will be of the form

$$\begin{aligned}(-\epsilon)^n + (-\epsilon)^{n-1}&(a_{11} + a_{22} + \cdots + a_{nn} + 0)\\ &= (-\epsilon)^{n-1}(-\epsilon + \mathrm{Tr}(a_{ij})).\end{aligned} \tag{48}$$

since products, such as $(-\epsilon)^{n-2}$, $(-\epsilon)^{n-3}$,...and so on, cannot occur. Thus, the nonzero eigenvalue is just the trace of matrix (43). We can proceed with the argument given that only one nonzero eigenvalue will contribute.

In preparation for diagonalization of the quadratic forms that occur in the exponent of (42), we define some new column vectors by means of

$$\begin{aligned}U^T &= \{u_m^2, u_m^1, u_m^0, \ldots\}\\ C^T &= \{\sqrt{h_2 f_m^2}, \sqrt{h_1 f_m^1}, \sqrt{h_0 f_m^0}, \ldots)\}\end{aligned} \tag{49}$$

where the upper indices label the power-law noise type. Then, we can write

$$\sum_\alpha \sqrt{h_\alpha f_m^\alpha} u_m^\alpha = U^T C = C^T U. \tag{50}$$

Therefore

$$\frac{\nu_0^2}{4 f_m^2}\Big(\sum_\alpha \sqrt{h_\alpha f_m^\alpha} u_m^\alpha\Big)^2 = K_m U^T C C^T U = K_m U^T H U. \tag{51}$$

Then, the integral (42) becomes

$$I = \int e^{-i\omega K_m U^T H U} \prod_\alpha \left(\frac{du_m^\alpha}{\sqrt{2\pi}} e^{-\left(\frac{(u_m^\alpha)^2}{2}\right)}\right). \tag{52}$$

The orthogonal transformation O that diagonalizes the quadratic form in the exponent of (52) is obtained by writing the normalized eigenvectors of H in successive columns, say in order of increasing eigenvalue. Then, it follows that:

$$HO = OE; \qquad O^T H O = E \tag{53}$$

where E is a diagonal matrix consisting of the eigenvalues, and we have used the fact that the inverse of an orthogonal matrix is just the transpose

$$O^{-1} = O^T. \tag{54}$$

Thus, $H = OEO^T$ and

$$U^T H U = U^T O E O^T U. \tag{55}$$

We can then introduce a linear combination of random variables with the same orthogonal transformation

$$V = O^T U. \tag{56}$$

The argument of the Gaussian distribution remains essentially a sum of squares

$$\sum_\alpha (u_m^\alpha)^2 = U^T U = U^T O O^T U = V^T V. \tag{57}$$

Also, the element of integration over the space of random variables remains essentially unchanged because

$$\begin{aligned}dV_1 dV_2 \cdots = \left|\det \frac{\partial V}{\partial U}\right| dU_1 dU_2 \ldots &= |\det(O)| dU_1 dU_2 \ldots\\ &= dU_1 dU_2 \ldots.\end{aligned} \tag{58}$$

The expression in the exponent of (52) becomes

$$K_m U^T C C^T U = K_m \sum_\ell \epsilon_\ell V_\ell^2. \tag{59}$$

The integral that we seek is then

$$I = \int e^{-i\omega K_m \sum_\ell \epsilon_\ell V_\ell^2} \prod_i \Big(\frac{dV_i}{\sqrt{2\pi}} e^{-\frac{V_i^2}{2}}\Big). \tag{60}$$

Now, only one eigenvalue contributes. The factors from all the zero eigenvalues just integrate out, and the result is

$$I = \frac{1}{\sqrt{1 + 2i\omega K_m \epsilon}}. \tag{61}$$

There is one such factor from the real part, and an equal factor from the imaginary part. The probability obtained from (41) is, therefore

$$P(\mathcal{L}) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left(\frac{e^{i\omega\mathcal{L}}}{1 + 2i\omega K_m \epsilon}\right). \tag{62}$$

This contour integral may be evaluated by closing the contour around the simple pole in the upper half plane, yielding for any superposition of power-law noises

$$P(\mathcal{L}) = \frac{2f^2}{\nu_0^2 \sum_\alpha h_\alpha f_m^\alpha} e^{-\frac{2f^2}{\nu_0^2 \sum_\alpha h_\alpha f_m^\alpha}\mathcal{L}} \tag{63}$$

which is the generalization of (20).

VII. Random Walk

To give a specific example, we discuss the case of random walk FM. In this case, in the limit of continuous frequencies

$$S_y(f) = h_{-2} f^{-2} \tag{64}$$

so

$$P(\mathcal{L}(f)) = \frac{2f^4}{\nu_0^2 h_{-2}} e^{-\frac{2f^4 \mathcal{L}(f)}{\nu_0^2 h_{-2}}}. \tag{65}$$

The ±25% limits on the probability can be determined from this expression. These limits are

$$\{0.14384\nu_0^2(h_{-2}/f^4), 0.693147\nu_0^2(h_{-2}/f^4)\}. \tag{66}$$

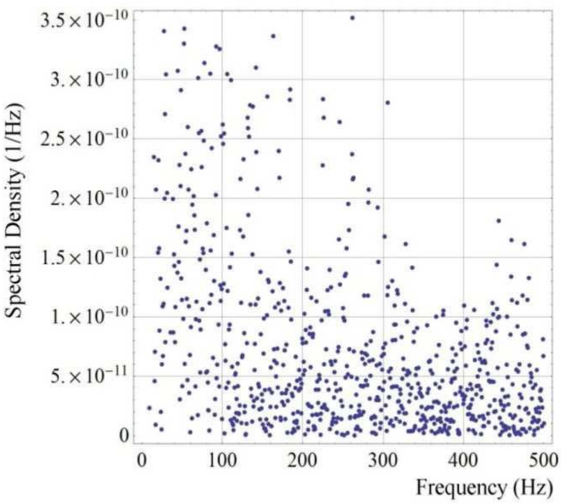

Fig. 2. Flicker-like spectral density data from an amplifier. The spectral density is given in units of rms voltage/$\sqrt{\text{Hz}}$; the frequency range is 5–500 Hz offset from the carrier.

VIII. Comparison With Experimental Data

In this section, we apply these methods to the calculation of the probability distribution for a power-law spectral density and apply it to actual measurements. From (7)–(9), before averaging, a single-sided spectral density at a particular frequency is simulated as

$$S_y(f_m) = \frac{h_\alpha f_m^\alpha}{2\sigma^2}\left(u_m^2 + v_m^2\right). \quad (67)$$

Following the procedure of (18), the probability of finding a value S will be:

$$P(S) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega\left(S - \frac{h_\alpha f_m^\alpha}{2\sigma^2}\left(u_m^2+v_m^2\right)\right)} \times e^{\frac{\left(u_m^2+v_m^2\right)}{2\sigma^2}} \frac{du_m dv_m}{2\pi\sigma^2}. \quad (68)$$

The integrals over the random variables are straightforward; no additional diagonalization is necessary. Each integral gives the same factor so the probability is

$$P(S) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\left(\frac{e^{i\omega S}}{1 + i\omega h_\alpha f_m^\alpha}\right) = \frac{e^{-S/\left(h_\alpha f_m^\alpha\right)}}{h_\alpha f_m^\alpha}. \quad (69)$$

For comparison of this probability distribution with measurements, the Metrology Group at NIST in Boulder provided 1024 independent measurements of amplifier noise as a function of frequency offset from the carrier; one such run is shown in Fig. 2. Usually many such runs are averaged and the average spectral density is presented as the measurement result. In Fig. 2, the frequency spacing between points is $\Delta f = 0.61875$ Hz. The conversion from time residuals to spectral density is not known, but by selecting a small range of frequencies and collecting data from many such independent runs, the number of times a spectral density occurs within the range of a particular bin can be reduced to a probability and compared with (69). When these data are plotted on a log–log scale, it is found that the spectral density is proportional to f^α where $\alpha = -0.83 \pm .070$, so the noise process is similar to flicker noise but the slope varies. We shall compare the distributions of spectral density with simulations at the unique slope -0.83. The probability function (69) is then determined except for one overall constant, h_α, which is the unknown conversion between time residuals and spectral density (the density was provided as rms voltage per root hertz). Fig. 3 shows plots of the theoretical number of occurrences, proportional to (69), at two frequencies, together with a histogram of the number of occurrences constructed from the data. The values of spectral density were collected into bins of width 4.415×10^{-12} Volts$^2/\sqrt{\text{Hz}}$ for plotting as a histogram. The probability function (69) can easily be used to find the mean, median, and 25% and 75% confidence levels of the spectral density distribution and hence to compared with the data. Fig. 4 provides these theoretical plots, with no additional adjustment of parameters. The corresponding values obtained from the data are given at several frequencies. The simulated probability distribution begins to disagree with the data at frequencies lower than about 70 Hz; use of a value $\alpha = -0.80$ improves the agreement significantly at low frequencies.

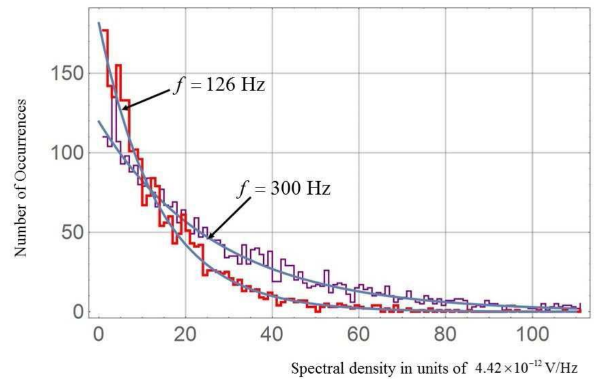

Fig. 3. Histograms of occurrences of spectral densities of flicker-like noise at two offset frequencies; smooth lines are simulated results (69).

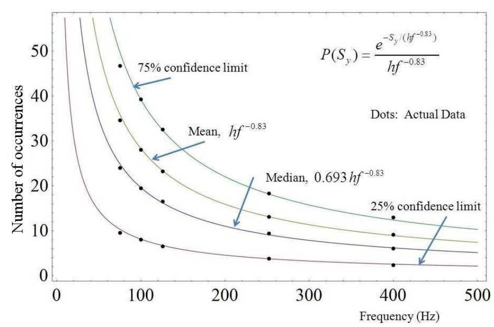

Fig. 4. Comparison between probability function (69) and experimental data. The dots are computed directly from the data at frequencies 75, 100, 126, 252, and 400 Hz; 200 data files were used.

IX. Summary and Conclusion

A method for simulation of power-law noise based on generation of white phase noise in the frequency domain, then division of each frequency component by an appropriate power

TABLE I
PRINCIPAL RESULTS FOR PROBABILITIES

Probability	Expression	Equation
$P(S_y)$	$\frac{1}{h_\alpha f^\alpha} e^{-\frac{S_y}{h_\alpha f^\alpha}}$	(26)
$P(S_x)$	$\frac{1}{h_\alpha f^\alpha} e^{-\frac{S_x}{h_\alpha f^\alpha}}$	(26)
$P(S_\phi)$	$\frac{f^2}{\nu_0^2 h_\alpha f^\alpha} e^{-\frac{f^2 S_\phi}{\nu_0^2 h_\alpha f^\alpha}}$	(26)
$P(L)$	$\frac{2f^2}{\nu_0^2 \overline{S}_y(f)} e^{-\frac{2f^2 L}{\nu_0^2 \overline{S}_y(f)}}$	(26)
$P(\langle\mathcal{L}\rangle_M)$	$\sqrt{\frac{M}{2\pi}} \left(\frac{2Mf^2}{\nu_0^2 \overline{S}_y}\right)^M e^{-\frac{2Mf^2\overline{S}_y}{\nu_0^2}}$	(31),(32)
$P(\mathcal{L})$	$\frac{2f^2}{\nu_0^2 \sum_\alpha h_\alpha f^\alpha} e^{\frac{-2f^2\mathcal{L}}{\nu_0^2 \sum_\alpha h_\alpha f^\alpha}}$	(63)

of frequency, has been applied in this paper to simulation of spectral density and phase noise. The main results are summarized in Table I. The simulation method does not require transformation to the time domain, and is simpler to apply than applications previously discussed [3], [4] to Allan and Hadamard variances. Probabilities for observation of values of phase noise $\mathcal{L}(f)$ and related spectral densities have been derived and applied to estimate confidence intervals for various forms of power-law noise. Theoretical predictions compare well to flicker-like experimental noise data obtained from an amplifier, showing good agreement between simulations and experiment over a wide range of frequencies.

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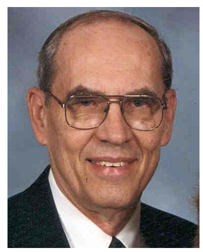

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