Collapse of the Cross-spectral Function

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***Abstract*—Cross-spectral analysis is a mathematical tool for extracting the power spectral density of a correlated signal from two time series in the presence of uncorrelated interfering signals. We demonstrate and explain a set of amplitude and phase conditions where the detection of the desired signal using cross-spectral analysis fails partially or entirely in the presence of a second uncorrelated signal [1-2]. Not understanding when and how this effect occurs can lead to dramatic under-reporting of the desired signal**.

Keywords—anti-correlation; cross-spectrum; collapse; phase inversion; power spectral density

I. The Cross-Power Spectral Density

The cross-spectrum of two signals $x(t)$ and $y(t)$ is defined as the Fourier transform of the cross-covariance function of x and y. However, from the Wiener–Khinchin theorem, it can be implemented far more practically by the metrologist as:

$$X(f)=F\{x(t)\},\, Y(f)=F\{y(t)\},\, \langle S_{xy}(f)\rangle_m=\frac{1}{T}\langle X(f)Y^*(f)\rangle_m, \quad (1)$$

where, $X(f)$ and $Y(f)$ are the truncated Fourier transforms of $x(t)$ and $y(t)$ and $<>_m$ denotes an ensemble of m averages. The cross-power spectral density $S_{xy}(f)$ can thus be determined from the ensemble average of the product of $X(f)$ and the complex conjugate of $Y(f)$. T is the measurement time normalizing the PSD to 1 Hz and '*' indicates complex conjugate. Unlike a normal PSD the cross-PSD is a complex quantity. Suppose we have two signals $x(t)$ and $y(t)$, each composed of four statistically independent, ergodic and random processes $a(t)$, $b(t)$, $c(t)$ and $d(t)$ such that

$$x(t)=a(t)+c(t)+d(t), \quad y(t)=b(t)+c(t)+d(t). \quad (2)$$

We consider $c(t)$ and $d(t)$ to be the desired signal that we wish to recover, and $a(t)$ and $b(t)$ are the uncorrelated interfering signals. If $d(t)$ is also correlated in both $x(t)$ and $y(t)$, then the cross-spectrum converges to the sum of the average spectral densities of $c(t)$ and $d(t)$

$$\langle S_{xy}(f)\rangle=\tfrac{1}{T}\left[\langle CC^*(f)\rangle+\langle DD^*(f)\rangle\right]=S_c(f)+S_d(f). \quad (3)$$

However, if $c(t)$ is correlated in $x(t)$ and $y(t)$ and $d(t)$ is anti-correlated (phase inverted) in x and y as in (4)

$$x(t)=a(t)+c(t)+d(t), \quad y(t)=b(t)+c(t)-d(t), \quad (4)$$

an unexpected outcome occurs. The cross-PSD represented by (5) collapses to zero when $c(t)$ equals $d(t)$

$$\langle S_{xy}(f)\rangle=\tfrac{1}{T}\left[\langle CC^*(f)\rangle-\langle DD^*(f)\rangle\right]=S_c(f)-S_d(f). \quad (5)$$

II. Simulation Results

Mathworks Simulink simulations of the collapse of the cross-spectral function were created using the block diagram shown in Fig. 1. Two noise generators, that can create white or frequency dependent noise slopes were summed and connected to both inputs of the cross-spectral density function. Two switches were provided to allow for the negation (gain = -1) of either one or both signals to one input of the cross-spectrum. Placing only one or the other switch, but not both, into negation creates the collapse of the function (Fig. 2b). If neither or both signals are negated, a normal cross-spectrum occurs (Fig. 2a).

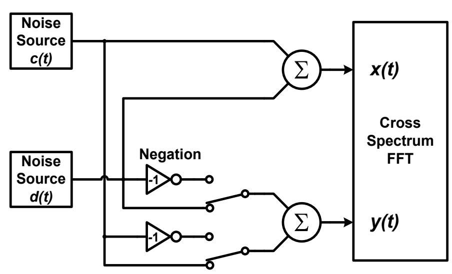

Fig. 1. Block diagram for a Mathworks Simulink simulation. A negation (gain = -1) is switch selectable for creating correlated and anti-correlated inputs to the cross-spectrum.

III. Conclusion

We demonstrated and explained a condition where the detection of a desired signal in cross-spectral analysis fails partially or entirely. If two time series, each composed of the summation of two fully independent signals, are correlated in the first time signal and anti-correlated (phase inverted) in the second, and have the same average spectral magnitude, the cross-spectrum power density between two time series collapses to zero.

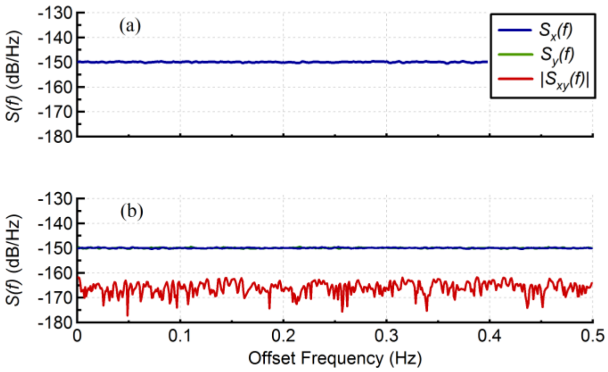

Fig. 2. Mathworks simulation results for the addition of two completely independent noise sources, $c(t)$ and $d(t)$, each with power spectral density of -153 dB/Hz relative to unity. The magnitude of the cross-spectrum for Fig. 2a occurs when $x(t) = y(t) = c(t) + d(t)$ (2) and all three traces coincide. The cross-spectrum for Fig. 2b occurs when $x(t) = c(t) + d(t)$, and $y(t) = c(t) - d(t)$ (4).

REFERENCES

[1] C.W. Nelson, A. Hati, and D.A. Howe, "A Collapse of the Cross-spectral Function in Phase Noise Metrology," *Electronics Letters.*, vol. 49, no. 25, pp. 1640-1641, Dec. 2013.

[2] C. W. Nelson, A. Hati, and D. A. Howe, "A collapse of the cross-spectral function in phase noise metrology," Rev. Sci. Instrum., vol. 85, pp. 024705-7, Feb. 2014.