

Phase inversion and collapse of cross-spectral function

C.W. Nelson, A. Hati and D.A. Howe

Cross-spectral analysis is a mathematical tool for extracting the power spectral density of a correlated signal from two time series in the presence of uncorrelated interfering signals. A set of conditions is demonstrated and explained where the detection of the desired signal using cross-spectral analysis fails partially or entirely in the presence of a second uncorrelated signal. Not understanding when and how this effect occurs can lead to dramatic under-reporting of the desired signal. Theoretical and simulated demonstrations of this effect are presented.

Introduction: The detection of a signal in the presence of interfering noise always presents a challenge. The power spectral density (PSD) of an ergodic and stationary signal can be determined from the cross-spectrum even in the presence of interfering noise. If we can create two reproductions of a desired signal, and the interfering noise in each copy is not correlated, the average of the cross-spectrum can be used to estimate the PSD of the desired signal even when the intensity of the interfering noise is dominant. Cross-spectral analysis has been used to improve the sensitivity of the measurements in the field of modulation noise metrology for nearly 50 years [1–4]. The past 20 years have seen extensive use of cross-spectral analysis in the laboratory [5–7]. It is generally understood that when a desired signal has some level of correlation with an interferer, occasionally some level of cancellation of signals can be observed. In this Letter, we demonstrate and explain a set of conditions where the detection of the desired signal using cross-spectral analysis collapses partially or entirely in the presence of a second uncorrelated interfering signal.

Cross-power spectral density: The cross-spectrum of two signals $x(t)$ and $y(t)$ is defined as the Fourier transform of the cross-covariance functions of x and y . However, from the Wiener-Khinchin theorem, it can be implemented far more practically by the metrologist as

$$\begin{aligned} X(f) &= F\{x(t)\} \\ Y(f) &= F\{y(t)\} \\ S_{xy}(f) &= \frac{1}{T} \langle Y(f)X^*(f) \rangle \end{aligned} \quad (1)$$

where $X(f)$ and $Y(f)$ are the Fourier transforms of $x(t)$ and $y(t)$ and $\langle \rangle$ denotes an ensemble average. The cross-PSD $S_{xy}(f)$ can thus be determined from the ensemble average of the product of $X(f)$ and the complex conjugate of $Y(f)$. T is the measurement time normalising the PSD to 1 Hz and ‘*’ indicates the complex conjugate. Unlike a normal PSD, the cross-PSD is a complex quantity. An excellent detailed description of the cross-spectrum can be found in [8]. Although two-sided Fourier transforms and spectra are typically preferred for theoretical discussions, we will be using single-sided representations exclusively in this Letter. Suppose, we have two signals $x(t)$ and $y(t)$, each composed of four statistically independent, ergodic and random processes $a(t)$, $b(t)$, $c(t)$ and $d(t)$ such that

$$\begin{aligned} x(t) &= a(t) + c(t) + d(t) \\ y(t) &= b(t) + c(t) + d(t) \end{aligned} \quad (2)$$

We consider $c(t)$ and $d(t)$ to be the desired signals that we wish to recover, and $a(t)$ and $b(t)$ are the uncorrelated interfering signals. The Fourier transforms of these signals are represented by the corresponding capitalised variables as

$$\begin{aligned} X(f) &= A(f) + C(f) + D(f) \\ Y(f) &= B(f) + C(f) + D(f) \end{aligned} \quad (3)$$

and the corresponding cross-PSD is represented by

$$|S_{xy}(f)| = \frac{1}{T} \langle X(f)Y^*(f) \rangle = S_c(f) + S_d(f) \quad (4)$$

The cross-terms in (4) average to zero and the cross-spectrum results in the addition of $S_c(f)$ and $S_d(f)$. In practice, when the cross-spectrum is calculated, the contributions of $a(t)$ and $b(t)$ are reduced by the square root of the observation time. The cross-PSD is unable to discern

between the two correlated signals and converges to the combination of both. However, if $c(t)$ is correlated in $x(t)$ and $y(t)$ and $d(t)$ is anti-correlated (phase inverted) in x and y as in (5), an unexpected outcome occurs

$$\begin{aligned} x(t) &= a(t) + c(t) + d(t) \\ y(t) &= b(t) + c(t) - d(t) \end{aligned} \quad (5)$$

The corresponding Fourier transforms and the cross-PSD are represented by

$$S_{xy}(f) = \frac{1}{T} [\langle CC^*(f) \rangle - \langle DD^*(f) \rangle] = S_c(f) - S_d(f) \quad (6)$$

What (6) tells us is that at any frequency f where the average magnitude of signal $C(f)$ is equal to that of signal $D(f)$, the magnitude of the cross-spectrum collapses to zero. Any contribution of the desired signal $c(t)$, or the interferer $d(t)$, to the cross-spectral density is eliminated. This occurs even though signals $c(t)$ and $d(t)$ are completely uncorrelated. If the PSD of the two signals are exactly equal, the amount of observed cancellation is limited to \sqrt{N} , where N is the number of averages. If $C(f)$ and $D(f)$ have the same shape or slope against frequency, entire octaves or decades of spectrum can be suppressed and be grossly under-reported. If the PSD of C and D are not exactly equal, a partial cancellation still occurs.

Simulation results: Mathworks Simulink simulations of the collapse of the cross-spectral function were created using the block diagram shown in Fig. 1. Two noise generators that can create white or frequency dependent noise slopes were summed and connected to both inputs of the cross-spectral density function. Two switches were provided to allow for the negation (gain = -1) of either one or both signals to one input of the cross-spectrum. Placing only one or the other switch, but not both, into negation creates the collapse of the function (Fig. 2b). If none or both signals are negated, a normal cross-spectrum occurs (Fig. 2a). Finally, the collapse due to the interaction of two differently sloped noise types creates a notch as shown in Fig. 3.

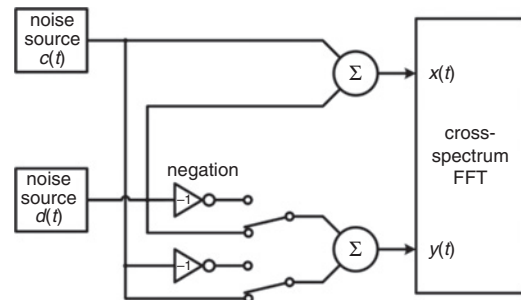


Fig. 1 Block diagram for Mathworks Simulink simulation

Negation (gain = -1) is switch selectable for creating correlated and anti-correlated inputs to cross-spectrum

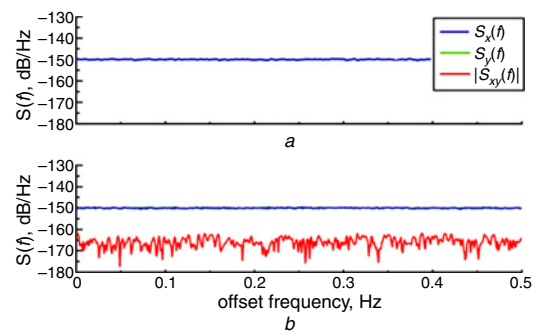


Fig. 2 Mathworks simulation results for the addition of two completely independent noise sources, $c(t)$ and $d(t)$, each with PSD of -153 dB/Hz relative to unity

a Cross-spectrum occurs when $x(t) = y(t) = c(t) + d(t)$ and all three traces coincide
 b Cross-spectrum occurs when $x(t) = c(t) + d(t)$, and $y(t) = c(t) - d(t)$
 Both Figures are for 1024 point fast Fourier transform (FFT) and 1000 averages. The amount of cancellation is 15 dB and is proportional to the square root of the number of averages

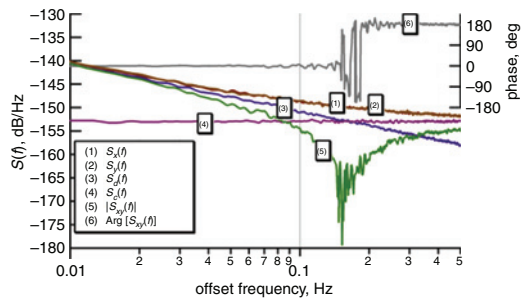


Fig. 3 Mathworks simulation results for addition of two independent noise sources, $c(t)$ and $d(t)$, with different frequency dependence

Signal $S_c(f)$ has PSD of -153 dB/Hz relative to unity
 Signal $S_d(f)$ has f^{-1} slope and intersects signal $S_c(f)$ at frequency of 0.164 Hz
 Cross-spectrum is calculated with $x(t) = c(t) + d(t)$, and $y(t) = c(t) - d(t)$. Both figures are for 1024 point FFT and 1000 averages

Conclusion: We demonstrate and explain a condition where the detection of a desired signal in cross-spectral analysis fails partially or entirely. If two time series, each composed of the summation of two fully independent signals, are correlated in the first time signal and anti-correlated (phase inverted) in the second, and have the same average spectral magnitude, the cross-spectrum power density between the two time series collapses to zero. These two conditions may occur only at localised offset frequencies or over a wide range of frequency of the cross-spectrum. The anti-correlation of one of the signals relative to the other may be caused by phase inversion, negation, time-delay or some other mechanism. In the future, we will report in detail the effect of this phenomenon in phase noise metrology and propose solutions for its mitigation.

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One or more of the Figures in this Letter are available in colour online.

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