# CLOCK ERROR STATISTICS AS A RENEWAL PROCESS

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## Abstract

A model ensemble specifying the distribution of clock readings about their average is given which leads to an integral equation of renewal type. Renewal processes are often described by the random times of replacement of mechanical or electrical components. In our case, clocks in the ensemble may repeatedly read correctly and "renew" themselves. Solutions of the equation are discussed and are related to other error statistics.

## Introduction

The initial purpose of this paper is to present and discuss a mathematical description of the probabilistic behavior of a clock subject to random influences; such a clock is conveniently regarded as a typical member of a statistical ensemble of identical independently running clocks. A model of this ensemble may be specified to some extent in terms of the probability density distribution or marginal distribution of its clock readings about the ensemble average and a conditional transition probability between pairs of readings. More detailed specifications are given in the next sections. The model was suggested in part by certain experimental studies (Barnes and Allan, 1966, Barnes, 1967, and Barnes and Allan, 1967), but it is not the purpose of this paper to discuss or present details of these experiments; rather, it is the purpose simply to deduce properties and behaviors of the model so suggested.

The time indicated on a clock, that is, its reading, is simply proportional to an accumulation (following some initial epoch) of repetitive events, called cycles, which are generated by the clock. To get the indicated time, one divides this cycle count by some average measure of frequency, which specifies the rate of occurrence of the cycles. Incorporated in a more sophisticated clock is a way of measuring its frequency (or rate) in terms of a frequency standard and constantly adjusting it (perhaps on paper) so that its time reading is an accumulation of cycles divided by frequency. Or, if the detailed trend during each cycle can be observed, and the phase measured during the cycle, then the reading of the clock is expressed by

$$\tau = \int_0^{\varphi} \frac{d\varphi'}{\omega(\varphi')} \tag{1}$$

where $\varphi$ is the measured phase, or $2\pi$ times the number of elapsed cycles, and $\omega$ is the measured angular frequency ($2\pi$ times the cycle repetition rate).

The integral is often evaluated continuously, to an approximation, by the clock mechanism itself; or it may be evaluated "after the fact" by a computer process, as is usually the case for the most accurate clocks (Barnes, Allan and Andrews, 1965).

Now, if the measurements of phase (counting of cycles) and frequency were perfectly precise, instantaneous, and continuous, and the frequency standard were perfectly accurate, and the integral could be evaluated with perfect accuracy, and the clock were read with perfect precision, then the reading, $\tau$, would be exactly the local proper, or inherent, time for the clock. Moreover, it is a basic assumption of (non-quantum) physics that all such clocks in any local neighborhood, if compared with perfect precision, and under ideal conditions, would read the same proper time interval, $\tau$.

None of these conditions is, of course, fulfilled in practice, and it is the random errors and fluctuations introduced into $\tau$ which produce the probabilistic behavior treated herein. Because of these effects, the readings of two clocks in the same neighborhood will differ even if set to read the same value initially. But it is conceivable that, from time-to-time, the clocks might again exhibit identical readings, though the interval between such coincidences would fluctuate in a random fashion. Similarly, if one clock were compared with an indefinitely large number of clocks (an ensemble) in its neighborhood whose initial readings were all equal, its later reading might agree with the average reading, t, of all these clocks occasionally, but usually it would be relatively fast or slow. When it agrees, we shall say that the clock is reading "correctly." The time, T, between such correct readings is known in the probabilistic "theory of recurrent events" as a "waiting time" (Feller, 1957a and 1966a). It is often a bonafide random variable with a probability distribution function; one of our purposes is to determine such probability distributions of T for various kinds of clock characteristics. We shall see that this function is directly related to the transition probability rate between pairs of correct readings on the condition that the clock did not read correctly between the readings.

It is known that the distributions of such waiting times are characteristic of what are called "renewal processes" (Feller, 1966b and Mandelbrot, 1967). This general class of processes has been studied often in connection with the use of equipment items which age randomly and need to be replaced or renewed from time to time. Well-known examples are light bulbs, employees, electronic equipment, automobiles, and baseball players. A more direct analogy with clock statistics may be found in processes which possess a measurable characteristic that may recur automatically in a random way. The average or expected time between successive returns to par value for a given stock is certainly an important vital statistic. Similarly, for the average time between successive correct clock readings.

During any given small interval in which a record is kept of the performance of a clock ensemble, there is a probability that a clock will have a reading which agrees at some instant with the ensemble average. The limiting value of the ratios of this quantity to the interval length as the length approaches zero will be known as the correct-reading rate of the ensemble. It is exactly analogous to the probability per unit time that the fluctuating water level behind a dam accumulating water at a constant average rate, goes through the expected value.

Such a description as this will be seen to lead to a certain integral equation (or summation equation, in the discrete case), a type of "renewal equation" (Feller, 1957c and 1966c). It relates the two probabilistic functions of time (i. e., the correct-reading rate, and the probability distribution of time between successive correct readings) whose trends are deemed significant in the study and classification of clock reading statistics. Hence, either of these functions and the marginal distribution specify the model for our purposes, i. e., to arrive at a partial statistical description which, it seems to us, should be incorporated in any more complete study. A relation between the marginal distribution and the correct-reading rate could lead to some important information about the cause of the fluctuations.

The second purpose of the paper is to determine and exhibit for illustration some pairs and classes of solution functions satisfying the renewal equation and to study their properties. Cases of particular importance, since they apparently relate to types of clock reading behaviors designated by the terms "white-phase noise", "random walk of phase noise", and "flicker walk of phase noise" (Allan, 1966 and Cutler and Searle, 1966), are studied; this leads to explicit formulas describing the behavior of the ensemble for large and small times.

Difficulties of this mode of description associated with certain cases of flicker walk of phase noise and, more generally, inverse correct-reading rates with trends which are proportional to powers of time greater than or equal to unity, are noted. In such cases a more general mode of description involving probability densities and transition probability rates for clocks which are not reading correctly is useful in clarifying concepts. It is introduced in more detail in the next section. Another direction for generalization of the description utilizes a statistical mechanical representation of the ensemble in a two-dimensional phase space, in which the initial clock running rate and its reading are independent random variables. This has not yet been thoroughly investigated.

The results and methods of this paper should be regarded as preliminary steps in helping establish a general theory of clocks, a task now being undertaken in many laboratories and research establishments. Certain results are new, as are the point-of-view and suggestions for future work. Since the integral equation studied is of standard renewal type, many consequences of studies of such processes are immediately applicable (Feller, 1966e and 1966h).

## Formal Specification of the Ensemble Type

Let us consider a statistical ensemble of identically constructed independently running clocks contained in a small enough spatial region so that comparisons of their simultaneous readings, $\tau$, can be made, effectively, in an instantaneous fashion. Assume that, at any instant, the average reading, t, of the ensemble exists, as well as its variance, $\sigma^2$. As mentioned, we shall for convenience refer to the ensemble average time, t, as the "correct" time. Let S denote the random clock reading variable, whose values are $\tau$.

Then

$$t = E\left\{S\right\} \tag{2a}$$

$$\sigma^2 = E\left\{(S-t)^2\right\} \tag{2b}$$

where the symbol, $E\{\ \}$, denotes the operation of forming the ensemble average, or expectation, to be presently specified, explicitly. Let the random variable, $\Delta$, whose values are $\xi$, denote the deviation of an ensemble clock reading from the average, so that

$$\Delta = S - t$$

and, hence

$$E\left\{\Delta\right\} = 0. \tag{2c}$$

We shall also suppose in general that the dispersion, $\sigma$, depends on "the time", t,

$$\sigma = \sigma(t). \tag{3a}$$

It is further specified that, at one particular instant, all the clocks are set to have the same reading, t = 0, so that, at this instant

$$\sigma(0) = 0. \tag{3b}$$

Moreover, they have been adjusted prior to the initial instant, to run at the same nominal rates. Nevertheless, thereafter each diverges from t in its readings, because each reading and rate is subject to random statistical fluctuations, which produce the hypothesized distributions as well as other statistical characteristics to be described.

Experiments at the National Bureau of Standards in which the frequencies of quartz oscillators were measured repeatedly against a standard, suggest an empirical model. When such an oscillator was used to drive a clock, it was shown that the statistical ensemble of records of clock readings could be described empirically, in part, by the probability density ("marginal distribution density") function

$$\Phi(\xi, t) = \frac{1}{\sqrt{2\pi}\,\sigma(t)}\, e^{-\frac{\xi^2}{2\sigma^2(t)}}, \quad t > 0 \tag{4a}$$

with the additional condition that, since all sample records were made to agree initially,

$$\Phi(\xi, 0) = \delta(\xi). \tag{4b}$$

For the "white-phase noise" case, rarely, if ever encountered, $\sigma$ would be a constant. Figure 1 illustrates how the function, $\Phi$, describes the dispersion of an ensemble for a case where $\sigma$ increases with time.

The quantity, $\Phi(\xi, t)$ should be considered as a basic element specifying the ensemble behavior. In fact, $\Phi(\xi, t)d\xi$ is the probability of finding a clock at time, t, with its reading deviation from t, or error, lying between $\xi$ and $\xi + d\xi$. That is, the probability distribution for $\xi$ is

$$\int_{-\infty}^{\xi} \Phi(\xi', t)d\xi' = P_r\,\{\Delta \leqq \xi; t\}. \tag{5}$$

Similarly, in an obvious way, $\Phi$ specifies the distribution for the clock reading, S. It is clear how $\Phi$ can be used in calculating certain expectation values, and has the properties required by Equations (2).

Other probability functions are needed in describing the ensemble process. To introduce these we define a probability associated with the deviation value, $\xi$, at time t, and the time interval dt following t. It is the probability that a clock has the deviation value $\xi$ at least once in time dt, on the condition that it had zero deviation at t = 0 and that it did not deviate by $\xi$ between the initial instant and time t. We shall call this probability $P(\xi, t|0,0;\xi)dt$. To reiterate, this is the probability that, given a zero deviation initially, the first occurrence of the given value, $\xi$, for the deviation, takes place in the time interval dt, following time t.

$P(\xi, t|0, 0;\xi)$ is a special case of a conditional probability rate function, $P(\xi, t|\xi', t';\xi)$; like $\Phi(\xi, t)$ this more general quantity, too, is fundamental in specifying the ensemble properties. Eventually, it will be related to the probability distribution of intervals between pairs of correct readings. So, for present completeness and later use we explicitly define

$$P(\xi, t|\xi', t';\xi)dt$$

exactly to be the conditional transition probability that a clock, which had a given deviation, $\xi'$, at time t', and did not deviate by $\xi$ between t' and t, is later observed between t and t + dt, to go through the deviation, $\xi$.

Symbolically, we may write

$$P(\xi, t|\xi', t';\xi)dt = P_r\{\Delta = \xi \text{ at least once on } \overline{t, t+dt} | [\Delta = \xi' \text{ at } t'] \cap [\Delta \neq \xi \text{ on } \overline{t', t}]\} \quad (6)$$

where $\overline{a, b}$ represents the interval from a to b, including b, but not a.

Now, imagine that, as the ensemble clocks disperse from their initial δ-function density distribution, one of the clocks is observed to go through a given deviation, $\xi$, in the time interval dt following t. It can do this in one of the following possible, independent ways: (See Figure 2.)

(1) its deviation might not equal $\xi$ between 0 and t;

(2) its deviation might equal $\xi$ just once between 0 and t;

(3) its deviation might equal $\xi$ exactly twice between 0 and t, etc.

As we have seen, there is a probability, $P(\xi, t|0, 0;\xi)dt$, for the first way. Call this $p_o dt$. There is also a probability, $p_1 dt$ (which will be calculated in Appendix I), for the second way, and a probability, $p_2 dt$, for the third, etc. The net probability that a clock takes on, or is read to have, the deviation $\xi$, at some instant in the interval dt, following t defines what we shall call the "error-reading rate function", $R(\xi, t)$. In symbols

$$R(\xi, t)dt = \sum_{i=o}^{\infty} p_i dt. \quad (7)$$

After evaluating $p_i$ it can be seen that the series in Equation (7) converges for all $(\xi, t) \neq (0, 0)$. The definition of R depends also on the fact that the ensemble clocks originally were all set to read zero. Because of this, the function, R, could have been written $\psi(\xi, t|0, 0)$. If the initial deviation of a clock had been $\xi'$ at $t'$, then we would write $\psi(\xi, t|\xi', t')$ which would emphasize that $\psi dt$ is the probability of transition from $\xi'$, $t'$ to $\xi, t$ without the condition of no deviation being equal to $\xi$ between $t'$ and t. Thus

$$\psi(\xi, t|\xi', t')dt = P_r\{\Delta = \xi \text{ at least once on } \overline{t, t+dt} | \Delta = \xi' \text{ at } t'\}. \quad (8)$$

For short, we shall call $\psi$ the "error-reading rate beginning at $(\xi', t')$."

For the sake of more complete generality we also generalize the error probability density function, $\Phi$, at this point. Let $\Theta(\xi, t|0, 0)d\xi = \Phi(\xi, t)d\xi$ be the probability of a clock deviation, $\Delta$, being between $\xi$ and $\xi + d\xi$ at time t, given that its initial error at $t' = 0$ was zero. Then we define

$$\Theta(\xi, t|\xi', t')d\xi = P_r\{\xi < \Delta \leq \xi + d\xi, \text{ at } t | \Delta = \xi' \text{ at } t'\}. \quad (9)$$

The "error probability density", $\Theta$, could have one of various forms, like

$$\Theta(\xi' - \xi, t, t') \tag{10}$$

for example, but this is a matter to be determined empirically or on the basis of theory, or as a basic statistical hypothesis. Obviously, even though all the ensemble clocks are specified to have "paths" proceeding from the origin, they would not all read $\xi$ sometime during a given range dt at $\xi, t$. An example is shown marked "A" in Figure 2. But from the definition of the time told by a clock, each path from the origin must proceed to the right (increasing t), be single-valued in $\xi$ at each t, and cross each "vertical" line t = constant once, for $t > 0$.

These are rather mouthfilling probabilistic definitions of the conditional transition probability rate, P, the error-reading rate, $\psi$, and the error probability density, $\Theta$, all beginning at $(\xi', t')$. But, they are really needed in order to develop the theory sufficiently and carefully. It will be seen that, in spite of the seeming complication, their use assists in understanding the conditions of validity of a more simplified, yet basically important, situation for which $\xi = 0$, treated in the next section.

## A Basic Relationship

To foreshadow the later development, it is instructive to derive, at this point, an important relationship which illustrates the essential simplicity of the notions involved. Following this, it will then be necessary to return to the more general case and give additional detailed and carefully reasoned arguments to support the further development of the theory. According to an idea of one of us (J. A. B.), many features of the clock ensemble of the kind considered here can be characterized most simply in terms of two other functions, $\rho(t)$ and $r(t)$, specifically defined for transitions between correct readings, as follows: As in the introduction, assume that the random deviations of the clock readings from their average are time varying in such a way that each clock might read the correct time, again and again, according to some probability law. That is, if a clock reads the value $t'$ at the instant when the ensemble average is $t'$, there is a conditional probability rate, $p(t|t') \geq 0$ that the clock will again read the "correct" time, t, but not read the correct value at any intervening instant. Moreover, it is convenient for the moment to assume that this probability rate is <u>stationary</u>, in the wide sense that p depends only on the difference, $t - t'$, that is,

$$p(t|t') = \rho(t-t'). \tag{11a}$$

Stochastically, then, we shall regard

$$\rho(t - t')dt \tag{11b}$$

as the conditional probability that a clock in the ensemble has a correct reading between times t and $t + dt$, on the condition that it did read the correct time $t'$, at an earlier time $t'$, but did not read the correct time between $t'$ and t.

This conditional probability is related to a probability rate function, $r(t)$. The differential

$$r(t')dt' \tag{12}$$

is to be regarded as the probability that an ensemble clock has a correct reading when the ensemble is observed between the times $t'$ and $t' + dt$. We shall call $r(t)$ the "correct-reading" rate of the ensemble.

Because of the hypothesized initial $\delta$-function distribution, it is only after the instant $t' = 0$, that a clock can again read "correctly." Subsequently, the probability that a clock reads correctly at time $t'$ and again shows the correct time t at a later moment, t, but not in between, is

$$r(t')\rho(t-t')dtdt', \quad t \geqq t'.$$

Now, at time $t \geqq 0$, the total probability that an ensemble clock reads correctly sometime in the interval dt must equal the probability

$$\rho(t)dt$$

that it reads correctly during dt after time t, having also read zero at t = 0 (but not in between), plus the probability that it did read correctly in any of the intervals dt' at all times, t', between 0 and t, but did not read correctly between t' and t, that is,

$$\int_0^t r(t')\rho(t-t')dt'dt.$$

Thus, we arrive at the integral equation relating r(t) and ρ(t):

$$r(t) = \rho(t) + \int_0^t r(t')\rho(t-t')dt'. \tag{13}$$

This is the simple result sought. It is one form of a standard "renewal equation", as in the theory of probability (Feller, 1966c and 1967).

It is, of course, possible to proceed from this point strictly analytically and investigate the possible solution pairs, ρ(t) and r(t), of Equation (13). But in order to be able to use this basic relation with more assurance, and not make unwarranted inferences about extending its range of application, we first derive, under less restrictive conditions, more general equations involving the functions, Θ, ψ, and P introduced previously. It will then be shown how Equation (13) may be made to follow; we shall see what the conditions are under which (13), its consequences, and the assumptions leading to it remain valid.

## Generalizations

Difficulties were at one time encountered in the attempted application of Equation (13) to some cases of considerable importance. One such case is characterized by a reading rate, $r(t)$, which is inversely proportional to the time, so that the convolution integral in (13) diverges. Many other, even more extremely divergent situations, have been found to be of interest.

A way out of these difficulties has been found, but at the cost of a somewhat more complex formulation. Yet, as was asserted, this too is compensated by a greater generality yielding a clarification of the "setting" of the description.

The derivation in the preceding section offers us a guide when viewed in the context of the definitions of the various probability functions, P, $\psi$, and $\Theta$. The quantity

$$P(0,t\,|\,0,t';0)dt$$

according to the section before the last is the probability of finding an ensemble clock to have a correct reading during the time interval dt following t on the conditions that it read correctly at the earlier time $t'$, but did not read correctly between $t'$ and t. But the statements already made about the probability, $p(t|t')dt$ (<u>without</u> the specialization to stationarity), show that

$$p(t\,|t') = P(0,t\,|\,0,t';0)\,. \tag{14a}$$

Similarly, one notes that

$$r(t) = \psi(0,t\,|\,0,0) = R(0,t) \tag{14b}$$

is an equally valid identification. This formally establishes a connection and furnishes more motivation to generalize the basic idea summed up in Equation (13). Briefly, the direction of this generalization may be indicated in the form of a question: "What has the fact that clocks are often read to have the same or different errors many times between different moments to do with its conditional probabilistic description?"

Final motivation comes from thinking about the obvious but important fact that clocks ordinarily run continuously. Again, we ask: "How can this property be described probabilistically?"

A simple answer has recently been found for the last question. Consider the probability that a clock, which had a deviation $\xi'$ at time $t'$, is later found to have the error $\xi$ in a time interval dt, following t. As we know, this is the error-reading probability, $\psi(\xi,t\,|\,\xi',t')dt$. But the fact is that any clock in its transition from the reading $(\xi',t')$ to the reading $(\xi,t)$ must also be found at an intermediate time, $t''(t'<t''<t)$, with some error in the range $d\xi''$ near $\xi''$. But the probability of the transition from $(\xi',t')$ to $(\xi'',t'')$ in

the range, $d\xi''$, and thence to the error $\xi$ sometime in the interval dt following t is

$$\Theta(\xi'', t''|\xi', t')d\xi''\psi(\xi, t|\xi'', t'')dt.$$

"Adding" all possibilities yields the fundamental relation

$$\psi(\xi, t|\xi', t') = \int_{-\infty}^{\infty}\Theta(\xi'', t''|\xi', t')\psi(\xi, t|\xi'', t'')d\xi'' \tag{15}$$

between the error-reading rate and the error probability density, beginning at any reading. If one sets $(\xi', t') = (0, 0)$, so that initially the clock read correctly, as depicted in Figure 3, we find the more special relation

$$R(\xi, t) = \int_{-\infty}^{\infty}\Phi(\xi'', t'')\psi(\xi, t|\xi'', t'')d\xi'', \quad (0<t''<t) \tag{16}$$

between the error-reading rate, the error-reading rate beginning at any reading, and the marginal distribution density. In view of the relation between R and $\psi(\xi, t|0, 0)$, it may best be regarded as an equation restricting $\Phi$, given $\psi$, although, under certain conditions, the converse problem might be posed. Equation (15) answers, in a general way, the question concerning continuity.

An answer to the first motivating question may also be developed. As before, consider the probability of a transition from an arbitrary initial moment, $t'$, with error $\xi'$, to the error $\xi$ observed in dt, after t. This is, of course, $\psi(\xi, t|\xi', t')dt$. Now such a transition can come about in two ways: First, the clock may not assume the deviation $\xi$ between $t'$ and t, with probability $P(\xi, t|\xi', t';\xi)dt$. Or, second, it may read $\xi$ once, or twice, or thrice, etc., during the range of times $t'$ to t. But in this case, there is some time, $t''$, after which, during $dt''$, it may be observed to read $\xi$, and then does not read $\xi$ until the interval dt following t. (See Figure 4, where, however, $\xi' = t' = 0$.) The probability of this second occurrence is clearly

$$\psi(\xi, t''|\xi', t')dt''\cdot P(\xi, t|\xi, t'';\xi)dt.$$

Forming the total probability of the union of all such possible events at $t''$, together with that of the first way, gives us the second fundamental equation

$$\psi(\xi, t|\xi', t') = \int_{t'}^{t}P(\xi, t|\xi, t'';\xi)\psi(\xi, t''|\xi', t')dt'' + P(\xi, t|\xi', t';\xi). \tag{17}$$

If one is furnished with information leading to P, so that it satisfies certain conditions (Lovitt, 1950), Equation (17) may be solved to yield $\psi$, from which in turn a probability density, $\Phi$, should be found which satisfies (16). Such a general solution of (17), along the lines of the discussion leading to the

## Specializations

Equations (15) and (17) are necessarily valid provided one assumes the existence of the functions $\Theta$, $\psi$, and P for the stochastic processes of interest, because the equations follow directly from the definitions. Consequently, they are also of great generality and subsume many, many more specialized situations. For example, we see immediately that Equations 14 (a and b) lead to the correct-reading renewal Equation (13), if one additionally imposes the wide-sense stationarity property of Equation (11a).

Other assumed identifications, like

$$\bar{s} \cdot \Phi(0,t) = R(0,t) = r(t) \tag{18a}$$

where $\bar{s}$ is a dimensionless proportionality constant, lead to important consequences. This relation is suggested by the intuition that, if a clock is likely to be reading nearly correctly, then it might be proportionately likely that it will read correctly in any short period thereafter. An investigation by Rice (Rice, 1944 and 1945) points to the reasonableness of such an assumption. An immediate result is that the dispersion, $\sigma$, for the normal Gaussian diffusion case is related to the correct-reading rate, r(t) in the simple fashion:

$$\sigma(t) = \frac{\bar{s}}{\sqrt{2\pi}\, r(t)} . \tag{18b}$$

One can show explicitly that this relation is not inconsistent with the conditions (15) and (16); nor is the more general possibility

$$\bar{s} \cdot \Phi(\xi, t) = R(\xi, t), \tag{18c}$$

which we may call a "proportional error-reading rate" process, inconsistent with (15) or (16).

The detailed investigation of the not unreasonable conditions under which Equations (18) are valid is, however, left for a later paper, in which the situation for transitions between incorrect readings is investigated more extensively. We assume in the remainder of this paper the applicabilty of (18).

The use of (18b) does raise some problems, however, as was mentioned in the introduction. For if the dispersion, $\sigma(t)$, is proportional to a power of t, say $t^{1-\beta}$, then, when $\beta<0$, and we use the relation of $\sigma(t)$ to the correct-reading rate, r(t), the convolution integral in Equation (13) has no meaning.

The range of validity of the stationarity condition (11a) is another question for investigation. Let us first extend it, as follows: In Equation (17), set $\xi' = t' = 0$, use (18c) and assume that the conditional probability rate, P, has the wide-sense stationarity property, and is expressible in terms of a new function, $P_{\xi'''}$:

$$P(\xi, t \mid \xi''', t''; \xi) = P_{\xi'''}(t-t'' \mid \xi). \tag{19}$$

This specialization leads to the "stationary proportional error-reading rate" equation

$$\bar{s} \cdot \Phi(\xi, t) = \int_0^t \bar{s} \cdot \Phi(\xi, t'') P_\xi (t-t'' \mid \xi) dt'' + P_0(t \mid \xi). \tag{20}$$

Now, $P_{\xi'}(t \mid \xi)dt$ is the probability that a time interval t is taken for the error transition

$$(\Delta = \xi') \rightarrow (\Delta = \xi) \text{(sometime in dt)},$$

given that $\Delta \neq \xi$ during the interval t. The corresponding random variable, T, the time taken for the transition $\xi' \rightarrow \xi$, could therefore possess a probability distribution, specified by the density $P_{\xi'}(t \mid \xi)$; or it may be a "defective" random variable (Feller, 1966d). (See Equations 21c and d.). In any event, it is reasonable to assume that

$$\int_0^\infty P_{\xi'}(t \mid \xi) dt = f(\xi' \mid \xi) \tag{21a}$$

exists, and is less than or equal to unity. This assumption can be regarded as part of the specification of a "stationary proportional error-reading rate" process. Here $f(\xi' \mid \xi)$ is a new function.

Equation (20) itself is in the form of the general renewal equation with a parameter, $\xi$. As such, it is still very general. Some of its properties will be investigated in the later paper. There are excellent indications, because of the exponential "convergence" factor in the Gaussian form of $\Phi$ (Equation (4a)) that this equation will be applicable to situations where r(t) diverges badly as $t \rightarrow 0$.

Naturally, as $\xi \rightarrow 0$, we identify $P_0(t \mid 0)$ with $\rho(t)$, the probability density that T = t for transitions between successive correct readings. For this case, Equation (20) reduces to the correct-reading renewal Equation (13). Because $\rho(t)$ is a probability density, the condition (21a) specializes to the form

$$\int_0^\infty P_0(t \mid 0) dt = \int_0^\infty \rho(t) dt = f(0 \mid 0) \leq 1 \tag{21b}$$

so that, by definition, if T is a proper random variable

$$f(0 \mid 0) = 1 \tag{21c}$$

while, if it is a defective random variable,

$$f(0\,|\,0) < 1. \tag{21d}$$

In this last case, one interprets the quantity $(1-f(0\,|\,0))$ as the probability that a clock never again reads correctly after the initial setting.

We do not present, in this paper, certain discrete summation equations, or their solutions, which approximate (13). These matters are discussed elsewhere (Feller, 1957b); some pertinent results we have obtained will be forwarded on request.

Our immediate goal will be to treat those special cases for which the dispersion, $\sigma(t)$, has a particular form of time dependence, namely

$$\sigma(t) = \bar{\xi}\,\frac{\Gamma(\beta)}{\sqrt{2\pi}}\,(kt)^{1-\beta}, \quad 0<\beta\leq 1. \tag{22}$$

where k, $\bar{\xi}$, and $\beta$ are constants, and $\Gamma$ is Euler's $\Gamma$-function. A reason for this is that several noise types of considerable interest (many of electronic origin) exhibit stochastic behaviors included in this range. Later, we shall be interested in comparing their power spectral behaviors in the Fourier frequency domain, as classified in the literature (Cutler and Searle, 1966), with corresponding temporal characteristics specified by (22).

Noise types have come to be called by special names, in analogy to general "random walk" phenomena. We propagate this convenient terminology (see Table I) by referring to the cases:

$\beta = 1$ as White

$\beta = \frac{1}{2}$ as Random Walk

$\beta = 0$ as Flicker Walk

$\beta = -\frac{1}{2}$ as Random Run

$\beta = -1$ as Flicker Run

where the terms specifically apply to the type of noise, in this case to fluctuations in the clock-reading error, $\Delta$, i. e., its relative "phase" (Barnes and Allan, 1966).

This completes our discussion of some specializations of the general theory, and forms the setting for the discussions and solutions presented in the next sections.

## Renewal Process Properties for Correct-Reading Situations

We now turn to a consideration of some of the implications of the renewal Equation (13) taken in conjunction with the "normalization" condition (21b). The correct-reading statistics associated with this equation are therefore representative of a self-renewing process, and accordingly can be classified into two types. Equation (21c) specifies the first type, so-called ordinary or non-terminating processes. The dispersion behaviors characterized by Equation (22) are representative of this kind, as we shall see immediately, and in the analytic treatment in the next section. Additional cases, termed transient or terminating processes, are specified by the inequality (21d). A clock which exhibited such a transient correct-reading behavior would have a probability

$$q = 1 - f(0|0) \qquad (21e)$$

of never again reading correctly, after the initial instant.

Space precludes any attempt here to discuss or even list all the renewal theorems which might be invoked; in any event, they are readily available (Feller, 1966e and 1966h). It is desirable, however, to quote some results. Many of the theorems refer to the asymptotic behaviors of solutions of (13) and (21b). Our analytic treatment, utilizing perhaps more familiar methods, will then serve as a confirmation and amplification of some of the probabilistic results.

Of particular importance is the quantity U(t), defined by

$$U(t) \equiv 1 + \int_0^t r(t')dt'. \qquad (23a)$$

Since it is greater than unity, it is certainly not a probability distribution. It is interpreted consistently, however, as the expected number of correct readings up to time t, including the original setting (Feller, 1966e). Now, integration of both sides of Equation (13) from 0 to $\infty$, with an interchange of the order of integration on t' and t, and using (21c) leads easily to the result

$$U(\infty) = 1 + \int_0^{\infty} r(t)dt = \infty \qquad (23b)$$

characteristic of non-terminating renewal processes. We immediately check that this is valid for the dispersion trends in Equation (22), in view of the assumed relation (18b). For terminating processes,

$$U(\infty) = 1/q < \infty \qquad (23c)$$

and the probability is unity that a clock behaving this way fails to read correctly after a finite interval of time. Moreover, in this case the expected number of correct readings (renewal epochs) including the initial one, is finite and equal, in fact to $U(\infty)$; thus, we see the probabilistic significance of Equation (21c) or Equation (23b) in differentiating terminating from non-terminating processes. Again, for a terminating process (Feller, 1966g), the probability distribution of the "time until no more correct readings occur", that is, the total duration of the process (denoted by the random variable, D) is

$$\frac{U(t)}{U(\infty)} = P_r\{D \le t\}. \tag{24}$$

U(t) may be a useful measure of the quality of clocks, even though for the non-terminating situations for which (23) holds, there is no probability distribution for D. This should not be too surprising. As we have seen in earlier sections, r(t) is a probability rate; r(t)dt is the probability of a correct reading in the interval dt following the epoch t after any number of correct readings could have occurred previously, given an initially correct one at t = 0. So, for ordinary processes the expectation of correct-reading is so great, that there is no distribution of time until no more correct readings occur. The interpretation of U (t) is correct and of general use, however.

Although r(t) is not a probability density over the sample space of values of its argument, the function, $\rho(t)$ is such a density. Indeed, the probability distribution

$$F(t) = \int_0^t \rho(t')dt' \tag{25a}$$

measures the probability that the random variable T (the time between successive correct readings) is less than or equal to t. Clearly, the average time between correct readings

$$E\{T\} = \int_0^\infty t\rho(t)dt = \overline{T} \tag{25b}$$

is an important statistic.

There are some special processes, termed "arithmetic", such that all points of increase of F are among integral multiples of some interval. The largest interval value with this property, $t_A$, is called the span of F.

The basic renewal theorem (Feller, 1966e) states that if F is not arithmetic, then, as $t \to \infty$,

$$U(t) - U(t-h) = \int_{t-h}^t r(t')dt' \to h/\overline{T} \tag{26a}$$

for every fixed h. If F is arithmetic, this is true provided h is a multiple of the span, $t_A$. It can be shown that (26a) is equivalent to stating that, for the non-arithmetic case:

$$r(t) \rightarrow \frac{1}{\overline{T}} \int_0^{\infty} \rho(t)dt = \frac{f(0|0)}{\overline{T}} \tag{26b}$$

as $t \rightarrow \infty$. The power of application of the renewal theory becomes immediately obvious. For our special cases of interest, except for "white" phase noise for which $\beta = 1$, $\sigma(t) \rightarrow \infty$, and hence (if $\overline{s}$ is finite), $r(t) \rightarrow 0$, as $t \rightarrow \infty$ (see Equation (18b)). Then, the form (26b)of the basic theorem shows that

$$\overline{T} = \infty . \tag{27a}$$

In words, "The expected length of time between successive correct clock readings is infinite except for the case of white phase noise." In this case, since $r(t) = \frac{\overline{s}}{\sqrt{2\pi}\sigma}$ is a constant, and $f(0|0) = 1$,

$$\overline{T} = \frac{\sqrt{2\pi}\sigma}{\overline{s}} = \xi / \overline{s} . \tag{27b}$$

In view of the interpretation of $U(t)$, the basic renewal theorem asserts also that, if F is non-arithmetic, the expected number of correct readings within any time interval of fixed length, h, approaches $h/\overline{T}$, in the limit of very long times. Since, for white deviation noise, r is a constant, this limit is already reached in any finite time. But this statement then coincides with the definition of a simple Poisson process. Thus, the probability of an interval value, t, between successive correct readings for white correct-reading noise should vary exponentially. We shall verify this in the next section.

Finally, it is of interest to note without discussion two further important results of renewal theory. We first define the variance of the times between successive correct readings:

$$\sigma_T^2 = \int_0^{\infty} t^2 \rho(t)dt - \overline{T}^2. \tag{28}$$

Then, one can show (Feller, 1966f) that

$$0 \leq U(t) - \frac{t}{\overline{T}} \rightarrow \frac{\sigma_T^2 + \overline{T}^2}{2\overline{T}^2} \tag{29}$$

as $t \rightarrow \infty$. Thus, if $\sigma_T$ and $\overline{T}$ exist, then for long times the number of correct readings, $N_t$, up to time t (excluding the initial moment), has a normal-

Gaussian distribution with expectation $t/\overline{T}$ and variance equal to

$$t\,\sigma_T^2/\overline{T}^3.$$

Unfortunately, we seem to be often faced with situations for which $\overline{T}$ and $\sigma_T$ do not exist. There have been investigations of such cases, some of which we have not yet had time to assimilate (Feller, 1949 and Mandelbrot, 1967). Others have used alternative approaches and approximations to noise processes which involve statistical procedures where only finite samples and statistics occur (Halford, 1968). At any rate, we now turn to an analytic treatment of Equation (13). This will amplify and verify some of the statements made in this section.

## Some Solutions and Their Properties

We have specified the dispersion of our ensemble clocks according to Equation (22), and the correct-reading rate is determined by (18b). Thus, we are left with the problem of determining $\rho(t)$ given

$$r(t) = \frac{\overline{s}}{\overline{\xi}} \frac{(kt)^{\beta-1}}{\Gamma(\beta)} = \frac{\beta\overline{s}}{\overline{\xi}} \frac{(kt)^{\beta-1}}{\Gamma(1+\beta)} . \tag{30}$$

The magnitudes of the proportionality constants are either to be determined empirically, or by a more detailed physical theory.

Of course, the converse theoretical problem is often encountered. Indeed, we have indicated a quite general form of solution for the general error-reading rate $R(\xi, t)$ in Equation (7), as explicitly developed in Appendix I. Similarly, one can express $r(t)$ as a sum of repeated autoconvolutions of $\rho(t)$:

$$r(t) = \rho(t) + \int_0^t dt'\rho(t-t')\rho(t') + \int_0^t dt'\rho(t-t')\int_0^{t'} dt''\rho(t'-t'')\rho(t'') + \cdots . \tag{31}$$

The interpretation of the nth term of this series is obvious in view of what is said in Appendix I. It is the probability density of a correct reading following time t given a correct one initially and (n-1) correct ones in the interval $\overline{0t}$. Hence, (31) simply is a formal statement of the definition of $r(t)$ and is the unique solution of (13), given $\rho(t)$.

Even so, it is instructive to verify (31) by analytic methods, for this points the way toward finding $\rho(t)$ explicitly, given $r(t)$. Therefore, take the one-sided Laplace transform of both sides of Equation (13), using the general property that the Laplace transform of the convolution of two functions is the product of their Laplace transforms. Denote by $\tilde{r}(\lambda)$ the ordinary Laplace transform

$$\tilde{r}(\lambda) \equiv \int_0^\infty e^{-\lambda t} r(t)dt \tag{32}$$

of $r(t)$. Similarly, for $\tilde{\rho}(\lambda)$. Then, we have

$$\tilde{r} = \frac{\tilde{\rho}}{1-\tilde{\rho}} \tag{33a}$$

or

$$\tilde{\rho} = \frac{\tilde{r}}{1+\tilde{r}} . \tag{33b}$$

The connection of (33a) with (31) is quickly grasped. Expansion of (33a) in powers of $\tilde{\rho}$ yields

$$\tilde{r}(\lambda) = \sum_{n=1}^{\infty} \tilde{\rho}(\lambda)^n. \tag{33c}$$

But this is just the Laplace-transform of (31) and supplies its formal confirmation. Given r(t), $\tilde{r}(\lambda)$ can be found, from which $\tilde{\rho}$ follows, using (33b). The complex inverse Laplace transformation (Churchill, 1944)

$$\rho(t) = \frac{1}{2\pi i} \lim_{\delta\to\infty} \int_{\gamma-i\delta}^{\gamma+i\delta} e^{zt} \frac{\tilde{r}(z)}{1+\tilde{r}(z)} dz \tag{34}$$

(as in Appendix II) or some equivalent more convenient procedure yields the probability density that the time is t between correct readings. A few solution pairs have been determined in this way; they are available on request.

Now, more specially, from (30), (32) and (33b), we have

$$\tilde{r}(\lambda) = \frac{\mu}{\lambda^{\beta}}, \quad \tilde{\rho}(\lambda) = \frac{\mu}{\mu+\lambda^{\beta}} \tag{35}$$

where $\mu = \bar{s}(k^{\beta-1}/\bar{\xi})$. The general inversion integral, Equation (34), can be evaluated to find $\rho(t)$. This leads, as shown in Appendix II, to the representation of $\rho(t)$ in various integral forms:

$$\rho(t) = \begin{cases} \dfrac{\sin \pi\beta}{\pi\mu t^{1+\beta}} \displaystyle\int_0^{\infty} e^{-\nu} \frac{\nu^{\beta} d\nu}{1+ \frac{2\cos\pi\beta}{\mu}\left(\frac{\nu}{t}\right)^{\beta} + \frac{1}{\mu^2}\left(\frac{\nu}{t}\right)^{2\beta}} & (36a) \\ \text{useful for large t (i.e., } t\to\infty\text{), and } \beta \neq 1, \text{ or} & \\ \dfrac{\mu t^{\beta-1}}{2\pi i} \displaystyle\int_0^{\infty} e^{-\nu} \frac{d\nu}{\nu^{\beta}} \left[ \frac{e^{i\pi\beta}}{1+\mu\left(\frac{t}{\nu}\right)^{\beta} e^{i\pi\beta}} - \frac{e^{-i\pi\beta}}{1+\mu\left(\frac{t}{\nu}\right)^{\beta} e^{-i\pi\beta}} \right] & (36b) \\ \text{useful for small t (i.e., } t\to 0\text{), or} & \\ \mu e^{-\mu t} \quad \text{for } \beta = 1. & (36c) \end{cases}$$

It is helpful to discuss a few limiting situations.

First, consider the asymptotic form for very large t-values. Using Equation (36a) and expanding the integrand in powers of $(\nu/t)^\beta$, followed by integration with respect to $\nu$ (justifiable by applying a suitable form of Watson's Lemma), we find, for large t,

$$\rho(t) \cong \frac{\sin \pi\beta}{\pi t} \left\{ \frac{\Gamma(\beta+1)}{\mu t^\beta} - 2 \cos \pi\beta \frac{\Gamma(2\beta+1)}{(\mu t^\beta)^2} + 0 \left[ (\mu t^\beta)^{-3} \right] \right\} \quad (37)$$

which even yields $\rho(t) \cong 0$ for $\beta = 1$ (and large t, of course). This asymptotic result is of considerable interest, in particular for $\beta = \frac{1}{2}$, which corresponds to r(t) varying as $t^{-\frac{1}{2}}$, so that the standard deviation or dispersion of clock errors varies as $t^{\frac{1}{2}}$; we see that $\rho(t)$ varies as $t^{-3/2}$ in agreement with Chandrasekhar (Chandrasekhar, 1943). This is the case known as "random walk", or "Brownian motion." Other cases are also of interest; currently, the case in which $\beta \to 0$, known as "flicker noise" of clock rate or "flicker walk" of clock error, is receiving a great deal of attention (Cutler and Searle, 1966 and Cutler, 1967). As mentioned before, the case $\beta = 1$ corresponds to "white noise" and (36c) confirms the remarks concerning its relation to the Poisson process, made in the previous section.

For large t-values, the ratio $\rho/r$ has the asymptotic behavior

$$\frac{\rho(t)}{r(t)} \cong \frac{\Gamma(1+\beta)}{\Gamma(1-\beta} \frac{1}{(\mu t^\beta)^2} + 0 \left[ (\mu t^\beta)^{-3} \right] \quad (38a)$$

for $\beta \neq 1$; explicitly we have

$$\frac{\rho(t)}{r(t)} = e^{-\mu t} \quad (38b)$$

for $\beta = 1$.

Next, consider the behavior of $\rho(t)$ for small values of t. From Equation (36b) we find, after expansion of the integrands in powers of $(t/\nu)^\beta$ and integration with respect to $\nu$, that

$$\rho(t) = \frac{\beta}{t} \sum_{n=0}^{\infty} (n+1)(-)^n \frac{(\mu t^\beta)^{n+1}}{\Gamma[1+\beta(n+1)]} \cong \frac{\beta \mu t^{\beta-1}}{(1+\mu t^\beta)^2} \quad (39)$$

The series converges even for $\beta = 1$ to the correct value, and shows the behavior near $\beta = 0$. The trend of $\rho(t)$ near $t = 0$ for $\beta = \frac{1}{2}$ is as $t^{-\frac{1}{2}}$ just as for r(t). Indeed, the ratio $(\rho/r)$ has the interesting behavior

$$\lim_{t\to 0}\left(\frac{\rho(t)}{r(t)}\right) = \frac{\beta\Gamma(\beta)}{\Gamma(1+\beta)} = 1. \tag{40}$$

This is not so for large t, and should only be applied for $\beta>0$, as t approaches zero.

Note that the function describing the behavior of $\rho$ for small t and small $\beta$ also gives the correct asymptotic behavior for large t and small $\beta$ -- hence one guesses that it may be a close approximation for all t. This statement is strengthened by the observation that

$$\int_0^\infty \frac{\beta\mu t^{\beta-1}dt}{(1+\mu t^\beta)^2} = 1 \tag{41}$$

in agreement with (21c).

As a matter of fact, the expression given in (39)

$$\overline{\rho}(t) = \frac{\beta\mu t^{\beta-1}}{(1+\mu t^\beta)^2} \tag{42a}$$

is a fairly good approximation to $\rho$, even for values of $\beta$ approaching unity. It has the correct behavior, for both large and small t, for $\beta = \frac{1}{2}$, and for small t, with $\beta = 1$. An even better expression, since it yields a more refined approximation to $\rho(t)$ for positive $\beta\leq 1$, and all t, is

$$\overline{\overline{\rho}}(t) = \frac{\beta\mu t^{\beta-1}}{\Gamma(1+\beta)}\left[1+\frac{\mu t^\beta}{\Gamma(1+\beta)}\sqrt{\frac{\pi\beta}{\sin\pi\beta}}\right]^{-2} + \left(1-\sqrt{\frac{\sin\pi\beta}{\pi\beta}}\right)\beta^n\mu e^{-\beta^n t} \tag{42b}$$

with an adjustable parameter, n. It is easy to see that $\int_0^\infty \overline{\overline{\rho}}dt = 1$. This may bear further investigation. In using formulas (36) - (42) in conjuntion with (30), one should remember that $\mu = \overline{s}(k^{\beta-1}/\overline{\xi})$.

Table I summarizes the limiting trends for large and small t-values, with other entries corresponding to important special fluctuation or noise types. It shows a comparison of these trends with power spectral behaviors of certain common noise processes. Assuming ergodicity, the power spectral density, $S_\Delta(f)$, is defined in terms of the temporal trend of a sample error process, $\Delta(t)$, by

$$S_\Delta(f) = 2\int_0^\infty R_\Delta(t)\cos 2\pi f t\,dt \tag{43a}$$

where $R_\Delta(t)$ is the auto-correlation function of the sample, computed from

$$R_\Delta(t) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \Delta(t')\Delta(t'+t)dt'. \qquad (43b)$$

In the present paper, no statistical relation has been established by logical derivation between the power spectral density functions so defined and the quantities $\sigma$, $\rho$, and r, of primary interest here. However, other studies (Allan, 1966) have delved into this question, and have related the trends of $S_\Delta(f)$ to those of $\sigma(t)$. Some results are summarized in Table I, and enable the comparisons of trends to be made. The correspondences shown depend specifically on the "stationary proportional correct-reading rate" process hypothesis. In many cases of interest, as the Table indicates, the power spectral density in the Fourier frequency domain has been a statistic applicable to processes otherwise found difficult to analyze.

In the Table the exponents, $\alpha$, of the absolute Fourier frequency $|f|$, in the expressions listed for the power spectral density, $S_\Delta(f)$, and the corresponding temporal exponents, $\beta$, are apparently related by

$$\alpha = 2\beta - 3 \qquad (44)$$

for $\alpha \leq -2$. This may, however, be fortuitous. Analogous relations between Fourier frequency and temporal (averaging time) behaviors were obtained by D. Allan in the foregoing reference in connection with the statistical analysis of atomic clock and oscillator data.

## Conclusions

From the foregoing analyses we conclude that the renewal Equation (13) relating the distribution of times between correct readings to the correct-reading rate, leads to a theoretical description of use for white-noise and random-walk-noise clock reading processes. It is indicative of the correct relationship for the limiting case of flicker-walk of phase, but is inadequate to describe lower order noises, that is, $\beta < 0$, whose dispersion rates accelerate from zero. It is likely that a more general approach, utilizing more fully the general probability functions $\psi$, P, and $\Theta$, introduced initially, will be needed. As a minimum requirement, investigation of the more general renewal Equation (20), applicable to non-zero reading deviations, seems in order. Perhaps the most important result of the present investigation is the recognition that many important clock processes are stochastic processes of standard renewal type.

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## APPENDIX I

### Solution of the Error-Reading Rate Equation (17)

We find a solution for $\psi(\xi, t|\xi', t')$ by the method of iteration and successive approximations. Let the zeroth approximation be

$$\psi_o(\xi, t|\xi', t') = P(\xi, t|\xi', t';\xi).$$

Then, let the next approximation be

$$\psi_1(\xi, t|\xi', t') = P(\xi, t|\xi', t';\xi) + \int_{t'}^{t} P(\xi, t|\xi, t'';\xi)\psi_o(\xi, t''|\xi', t')dt''$$

and, in general, $(n = 1, \cdots, \infty)$, let

$$\psi_n(\xi, t|\xi', t') = P(\xi, t|\xi', t';\xi) + \int_{t'}^{t} P(\xi, t|\xi, t'';\xi)\psi_{n-1}(\xi, t''|\xi', t')dt''.$$

Solution of this integro-difference equation yields

$$\psi_1(\xi, t|\xi', t') = P(\xi, t|\xi', t';\xi) + \int_{t'}^{t} P(\xi, t|\xi, t'';\xi)dt''P(\xi, t''|\xi', t';\xi)$$

and, for $n = 2, 3, \cdots \infty$,

$$\psi_n(\xi, t|\xi', t') = P(\xi, t|\xi', t';\xi) + \int_{t'}^{t} P(\xi, t|\xi, t'';\xi)dt''P(\xi, t''|\xi', t';\xi) +$$

$$+ \int_{t'}^{t}\int_{t'}^{t''} P(\xi, t|\xi, t'';\xi)dt''P(\xi, t''|\xi, t''';\xi)dt'''P(\xi, t'''|\xi', t';\xi) +$$

$$+ \sum_{j=2}^{n} \int_{t'}^{t}\int_{t'}^{t''} \cdots \int_{t'}^{t^{(j+1)}} P(\xi, t|\xi, t'';\xi)dt'' \cdots P(\xi, t^{(j)}|\xi, t^{(j+1)};\xi)dt^{(j+1)}P(\xi, t^{(j+1)}|\xi', t';\xi).$$

Proof that $\psi_n$ converges to a unique solution, $\psi$, as $n \to \infty$, if P satisfies certain conditions, will be omitted here for brevity (Lovitt, 1950). If we set $\xi' = t' = 0$, then, as $n \to \infty$, $\psi_n(\xi, t|0, 0)$ approaches the error-reading rate function, $R(\xi, t)$, as defined in (7). We see that the probability, $p_1 dt$, has the form

$$p_1 dt = \int_{0}^{t} P(\xi, t|\xi, t'';\xi)dt''P(\xi, t''|0, 0;\xi)dt.$$

It is the probability that a clock, initially reading correctly, will have a deviation, $\xi$, in the time interval, dt, following t, having had the error, $\xi$, just once between 0 and t. From this statement, the interpretation of each of the terms in the infinite series for $\psi(\xi, t|\xi', t')$ becomes obvious. The meaning of each of the conditional probabilities, $\psi_n(\xi, t|\xi', t')dt$ is also clear.

## APPENDIX II

<u>The Laplace Inversion Integral for $\rho(t)$ (Equations (34) and (35))</u>

If we substitute the expression for $\tilde{r}(\lambda)$ from Equation (35) into the inversion integral of Equation (34), and allow $\lambda$ to be a complex variable z whose argument is restricted to its principal value

$$-\pi < \arg z \leq \pi,$$

then we are faced with the problem of evaluating

$$\rho(t) = \frac{\mu}{2\pi i} \lim_{\delta \to \infty} \int_{\gamma - i\delta}^{\gamma + i\delta} e^{zt} \frac{dz}{\mu + z^{\beta}}, \text{ for } \beta > 0.$$

The zeros of the denominator in the integrand occur at

$$z = z_n^{\pm} = \mu^{(1/\beta)} e^{\pm i\pi(2n+1)/\beta}$$

for $n = 0, 1, 2, \cdots$. But the restriction on the argument of z means that only those n-values (or poles) need be considered for which

$$2n + 1 \leq \beta.$$

Two cases arise, which will be considered separately:

(A) $\beta = 2m + 1$, an odd integer, and

(B) $\beta$ is not an odd integer.

<u>Case (A)</u> <u>$\beta = 2m + 1$</u>.

The number of poles of the integrand is $2m + 1$, since for $n = m$, only the real zero at

$$z_m^{+} = \mu^{1/\beta} e^{i\pi} = -\mu^{1/\beta}$$

lies in the z-plane as defined, while $z_m^{-}$ does not. For the other poles, $0 \leqq n \leqq \frac{\beta - 1}{2} = m$.

Choose a rectangular contour C consisting of:

(1) a line segment $L^{+}$, parallel to the imaginary axis on which

$$z = \gamma + iy, \ -\delta \leqq y \leqq \delta,$$

(2) a line segment $M^{+}$, parallel to the real axis on which

$$z = x + i\delta, \ \gamma \geqq x \geqq -\gamma',$$

(3) a line segment $L^-$, parallel to the imaginary axis on which

$$z = -\gamma' + iy, \; +\delta \geqq y \geqq -\delta,$$

and

(4) a line segment $M^-$, parallel to the real axis on which

$$z = x - i\delta, \; -\gamma' \leqq x \leqq \gamma$$

where $(\gamma, \gamma', \delta)$ are all positive and large enough so that C contains all the poles of the integrand in its interior. We know that the contour integral

$$\frac{1}{2\pi i} \oint_C e^{zt} \frac{dz}{\mu + z^\beta}$$

equals the sum of the residues of the integrand at its (2m+1) poles.

Now, the integrals:

$$\int_{M^\pm} = \mp e^{\pm i\delta t} \int_0^\gamma e^{xt} \frac{dx}{\mu + (x \pm i\delta)^\beta} \pm e^{\pm i\delta t} \int_0^{-\gamma'} e^{xt} \frac{dx}{\mu + (x \pm i\delta)^\beta}$$

$$\int_{L^-} = \int_\delta^{-\delta} e^{-\gamma' t} e^{iyt} \frac{i\,dy}{\mu + (-\gamma' + iy)^\beta}$$

all clearly vanish, as $\delta$, $\gamma'$ approach $\infty$. Hence, we have, in this limit,

$$\rho(t) = \mu \sum \text{residues at the } (2m+1)\text{-poles.}$$

The residue at $z_n^\pm$ is

$$\exp(t\mu^{1/\beta} e^{\pm i\pi(2n+1)/\beta} \Big/ \beta\mu^{(\beta-1)/\beta} e^{\pm i\pi(2n+1)(\beta-1)/\beta}),$$

so in this case

$$\rho(t) = \frac{\mu^{1/\beta}}{\beta} \left[ e^{-\mu^{(1/\beta)} t} + \sum_{n=0}^{\frac{\beta-3}{2}} 2e^{\mu^{(1/\beta)} t \cos \pi(2n+1)/\beta} \cos\left\{ \mu^{1/\beta} t \sin(\pi(2n+1)/\beta) + -\pi(2n+1)(\beta-1)/\beta \right\} \right].$$

For $\beta = 1$, we have the "white noise situation",

$$\rho = \mu e^{-\mu t}.$$

Case (B). $(2m - 1 < \beta < 2m + 1)$

The number of poles of the integrand is 2m, where (m - 1) is the largest value of n for which $2n + 1 < \beta$. Clearly $0 \leqq n \leqq m - 1$.

Let us introduce a cut along the negative real axis of the z-plane, even though the integrand is continuous across this cut when $\beta$ is a positive even integer.

The new integration contour, C, consists of the previous rectangle (except that $L^-$ is cut at y = 0) plus the following:

(1) two line segments, $Q^{\pm}$ parallel to the x-axis, on which, for sufficiently small $\epsilon > 0$,

$$z = \sigma e^{\pm \pi} \pm i\epsilon$$

where

$$\gamma' \leqq \sigma \leqq a,$$

and

(2) a circular contour, A, of radius "a" around the origin (cut at the negative axis) on which

$$z = ae^{i\theta}$$

and

$$-\pi + \eta \leqq \theta \leqq \pi - \eta$$

where $\eta$ is chosen so as to connect the circle with $Q^{\pm}$. Again, the contour integral

$$\frac{1}{2\pi i} \oint_{C'} e^{zt} \frac{dz}{\mu + z^{\beta}}$$

equals the sum of the polar residues.

The integral:

$$\int_A = \int_{-\pi+\eta}^{\pi-\eta} \exp(ae^{i\theta}t) \frac{aie^{i\theta}d\theta}{\mu + a^{\beta}e^{i\beta\theta}}$$

clearly vanishes as $a \to 0$.

The integrals along $Q^{\pm}$ are:

$$\int_{Q^+} + \int_{Q^-} = \int_{\gamma'}^{a} e^{[i\epsilon-\sigma]t} \frac{e^{+i\pi}d\sigma}{\mu+[\sigma e^{i\pi}+i\epsilon]^{\beta}} + \int_{a}^{\gamma'} e^{[i\epsilon-\sigma]t} \frac{e^{-i\pi}d\sigma}{\mu+[\sigma e^{-i\pi} - i\epsilon]^{\beta}} .$$

As $\epsilon$ approaches zero, these integrals become equal to

$$\int_{a}^{\gamma'} e^{-\sigma t} d\sigma \left[\frac{1}{\mu+\sigma^{\beta} e^{+i\pi\beta}} - \frac{1}{\mu+\sigma^{\beta} e^{-i\pi\beta}}\right],$$

and the integral along $L^-$ has its former value.

Let $\gamma'$ and $\delta$ become infinite as before, and let "a" approach zero.

Then, in this limit

$$\rho(t) = \mu \sum (\text{residues at the 2m-poles}) + I(t)$$

where

$$I(t) = \frac{\mu}{2\pi i} \int_{0}^{\infty} e^{-\sigma t} d\sigma \left[\frac{1}{\mu+\sigma^{\beta} e^{-i\pi\beta}} - \frac{1}{\mu+\sigma^{\beta} e^{i\pi\beta}}\right] .$$

If we set $\nu = \sigma t$, combine the split terms, and factor out $\mu^2$, we easily find

$$I(t) = \frac{\sin \pi\beta}{\pi\mu t^{\beta+1}} \int_{0}^{\infty} e^{-\nu} d\nu \frac{\nu^{\beta}}{1+\frac{2\cos \pi\beta}{\mu}\left(\frac{\nu}{t}\right)^{\beta} + \frac{1}{\mu^2}\left(\frac{\nu}{t}\right)^{2\beta}}$$

as in the text, Equation (36a), while a simple rearrangement of factors in the split terms leads immediately, with this substitution, to Equation (36b). In this we assume that $0 < \beta < 1$, so that the residue sum does not contribute. When $\beta = 1$, we get the white noise form for $\rho(t)$, Equation (36c), and if $\beta > 1$ and not an odd integer, we have the formula

$$\rho(t) = 2\frac{\mu^{1/\beta}}{\beta} \sum_{n=0}^{m-1} e^{\mu^{1/\beta} t \cos\left(\pi\frac{2n+1}{\beta}\right)} \cos\left\{\mu^{1/\beta} t \sin\left(\pi\frac{2n+1}{\beta}\right) - \pi\frac{\beta-1}{\beta}(2n+1)\right\} + I(t).$$

Note that the constant, $\mu$, equals $\bar{s} k^{\beta-1}/\bar{\xi}$.

The writers express their thanks to Dr. S. Jarvis for assistance with the contour integrations in this Appendix.

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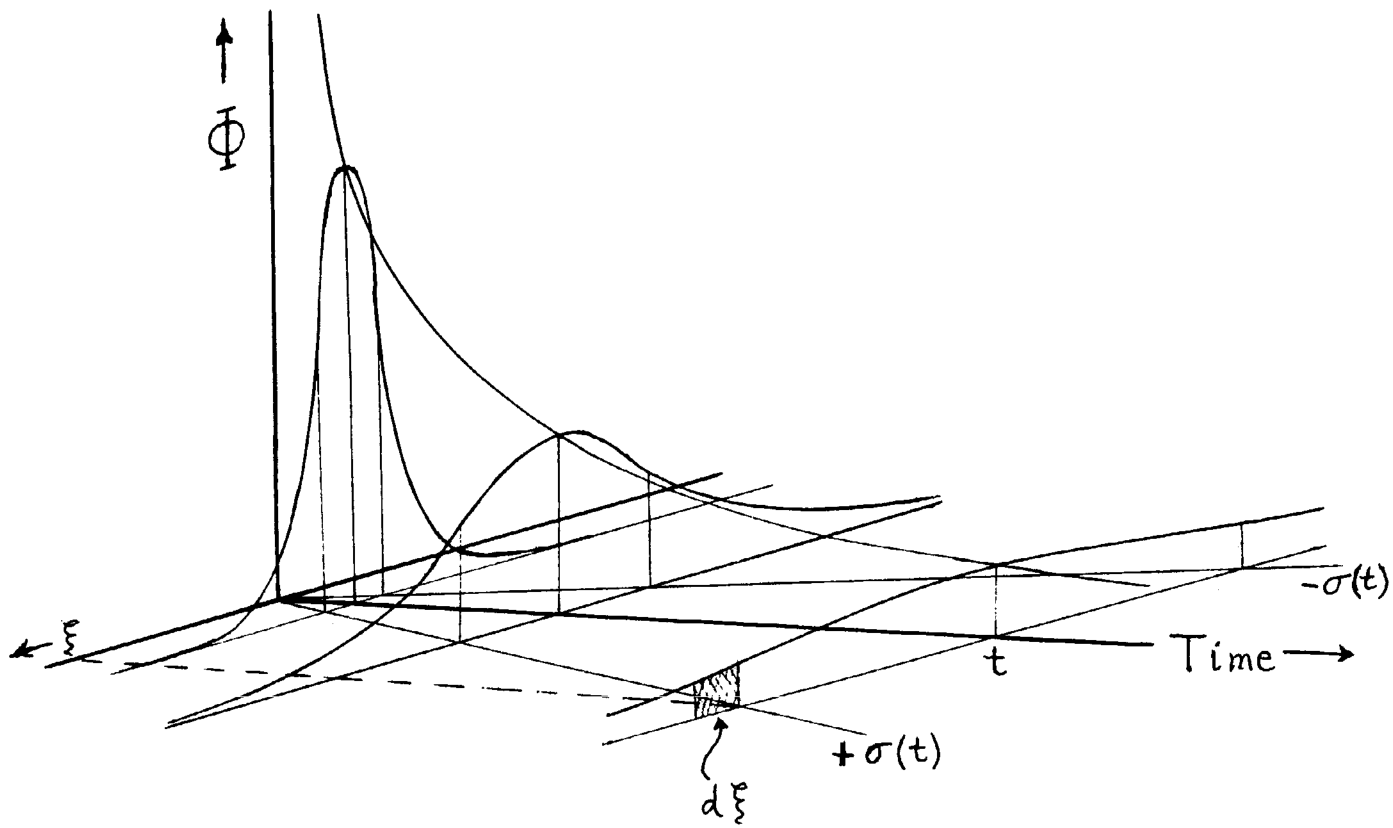
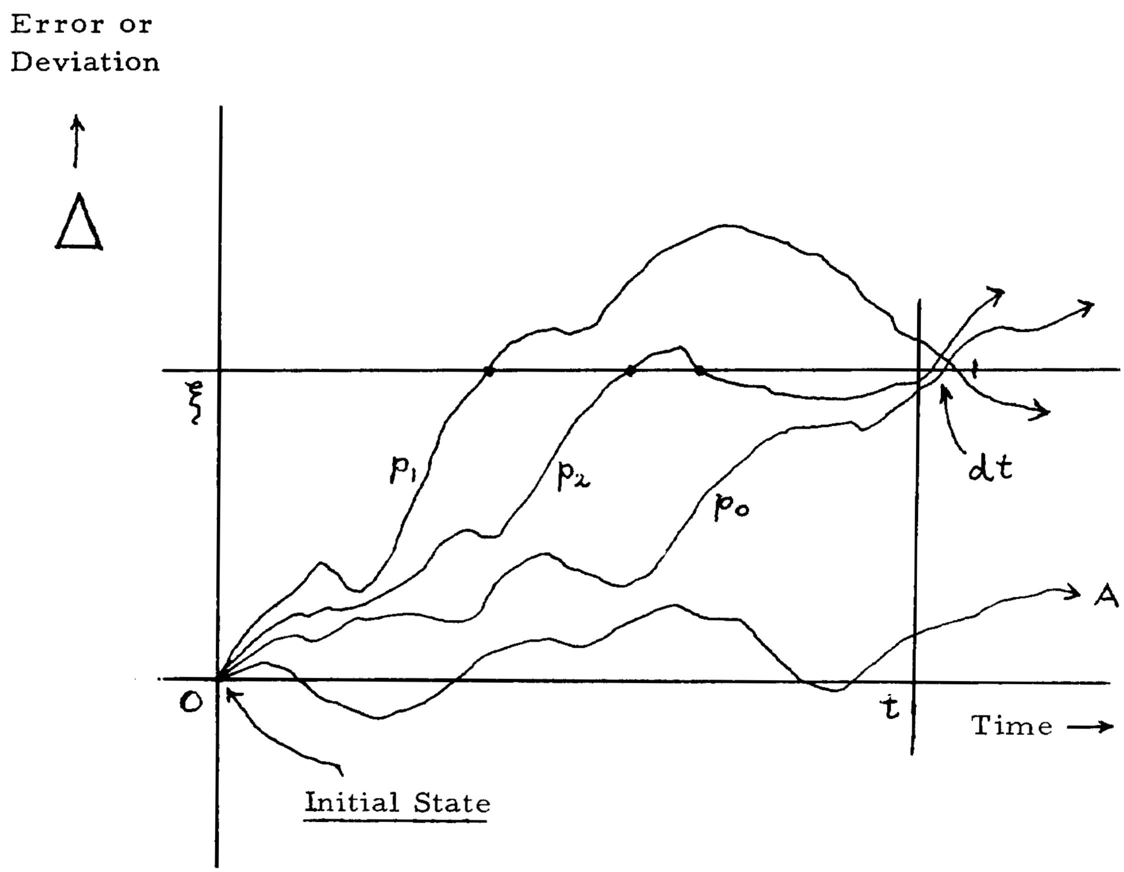



Figure 1. The Marginal Distribution Density or Probability Density Function, $\Phi$, illustrating the dispersion of a clock ensemble with time-dependent variance, $\sigma^2(t)$.




$$R(\xi, t)dt = \psi(\xi, t \mid 0, 0)dt = P_r\{\Delta=\xi \text{ at least once in } dt \mid \Delta=0 \text{ at } t=0\} = \sum_{i=0}^{\infty} p_i dt$$

Figure 2. The Definition of the Error-Reading Rate, $R(\xi, t)$, illustrated by clock-reading paths. Each path is assumed continuous and single-valued on t. The reading of a clock with deviation $\xi$ at t is, of course, $\xi + t$. Most clocks have a sample path like that of "A", which does not cross the given value $\xi$ during the interval dt.

definition of R (Equation (7)), is presented in Appendix I. Equation (17) is a generalization of (13); it answers the question concerning $\psi$, the "error-reading rate, beginning at $\xi', t'$", and the conditional transition probability rate, P.

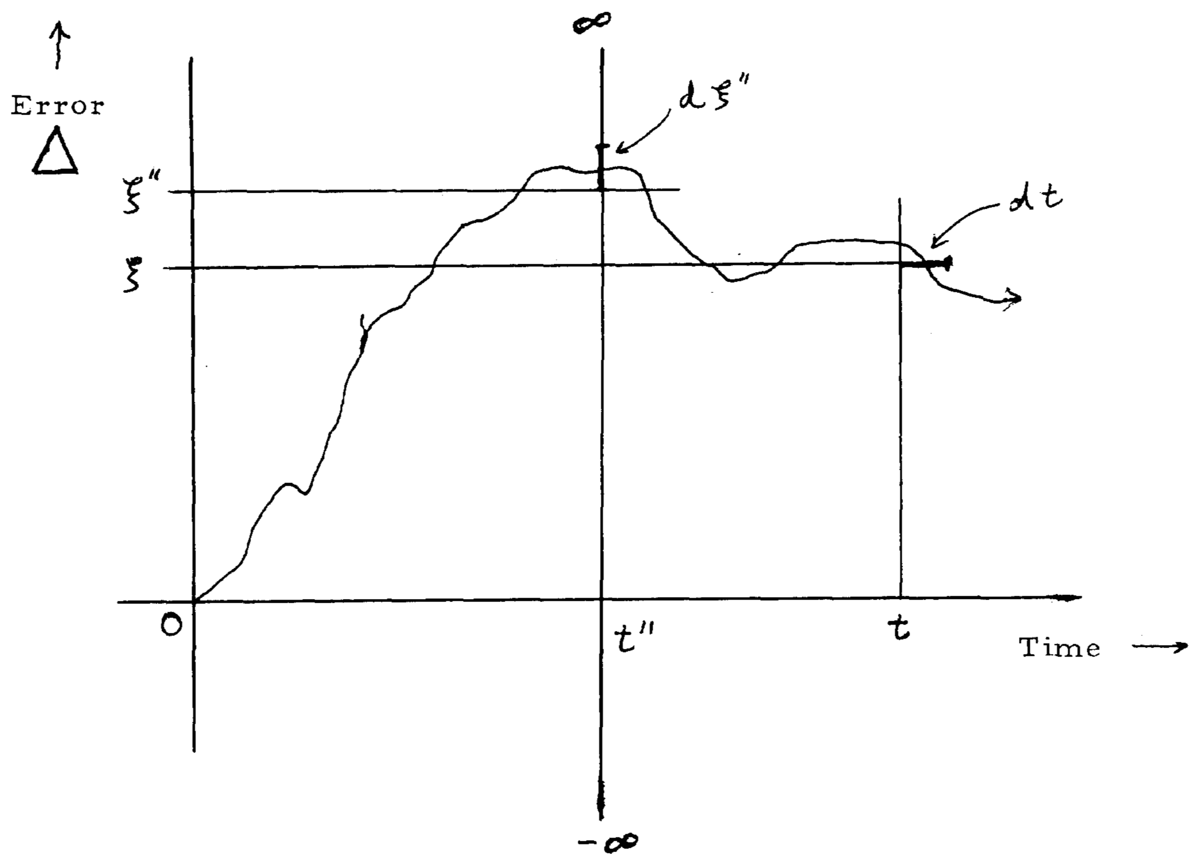



Figure 3. The Continuity of Clock-Readings relates the marginal distribution to the error-reading rate and the error-reading rate beginning at an arbitrary reading.

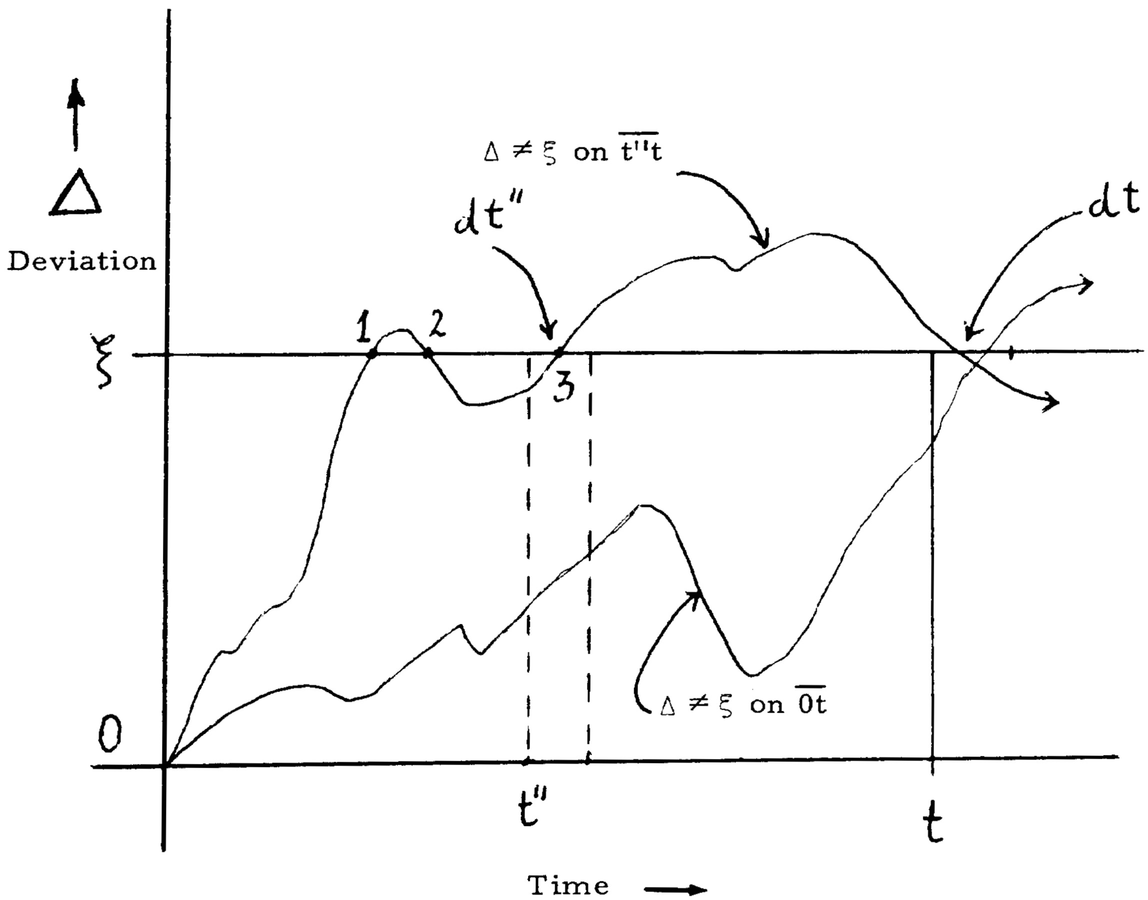



Figure 4. Types of Clock Transitions beginning with a correct initial reading.

## TABLE I
## TRENDS IN SPECTRAL AND STATISTICAL BEHAVIOR OF CLOCKS FOR STATIONARY PROPORTIONAL CORRECT-READING RATE PROCESSES

| NOISE TYPE | PHASE SPECTRUM: | | DISPERSION: | CORRECT READING | PROBABILITY DENSITY, $\rho(t)$, of t BETWEEN CORRECT READINGS: | | |
|---|---|---|---|---|---|---|---|
| | $\alpha$ | $S_\Delta(f)$ | $\sigma(t)$ | RATE: $r(t)$ | $t \to 0$ | $t \to \infty$ | $\beta$ |
| WHITE PHASE | 0 | $\lvert f \rvert^{o}$ | $t^{o}$ | $t^{o}$ | $e^{-t/\overline{\xi}}$ | $e^{-t/\overline{\xi}}$ | 1 |
| FLICKER OF PHASE | -1 | $\lvert f \rvert^{-1}$ | $(\ln (t/\overline{\xi}))^{\frac{1}{2}}$ | $(\ln (t/\overline{\xi}))^{-\frac{1}{2}}$ | $(\ln (t/\overline{\xi}))^{-\frac{1}{2}}$ | $t^{-2}$ | * - |
| RANDOM WALK OF PHASE | -2 | $\lvert f \rvert^{-2}$ | $t^{\frac{1}{2}}$ | $t^{-\frac{1}{2}}$ | $t^{-\frac{1}{2}}$ | $t^{-3/2}$ | $\frac{1}{2}$ |
| FLICKER WALK OF PHASE | -3 | $\lvert f \rvert^{-3}$ | $t$ | $t^{-1}$ | $t^{-1}$ | $t^{-1}$ | 0 |
| RANDOM RUN OF PHASE | -4 | $\lvert f \rvert^{-4}$ | $t^{3/2}$ | $t^{-3/2}$ | . . — — . . | . . — — . . | $-\frac{1}{2}$ |
| $0 < \beta \quad 1$ | $\alpha$ | $\lvert f \rvert^{\alpha}$ | $t^{1-\beta}$ | $t^{\beta-1}$ | $t^{\beta-1}$ | $t^{-\beta-1}$ | $\beta$ |

Note the comparative trends of the exponents, $\alpha$ and $\beta$, which define respectively the power spectral behavior with Fourier frequency, f, and the temporal dispersion and probabilistic behavior with running time, t.

*The behavior of $\rho(t)$ for this important special situation was supplied by D. Halford. In this case we may take the process to start at a small finite time, $\overline{\xi}$.