A Theoretical Analysis of Frequency Uncertainty

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Abstract— **In this work we present a preliminary study for the determination of the uncertainty on the frequency in the time measurements [1, 2]. The first part of this paper is dedicated to the theoretical study of the frequency uncertainty applying the law of the propagation of uncertainty and the spectral analysis approach. In the second part the theoretical results are compared with simulated results. In the final part we use real clock data and time transfer data to verify the analytical results. An accompanying paper in this proceedings "Experimental Analysis of Frequency Transfer Uncertainty" by Thomas E. Parker and Gianna Panfilo presents an experimental analysis of frequency transfer uncertainty [3].**

I. INTRODUCTION

This paper presents theoretical results for estimating the uncertainty of a frequency measurement between remote standards [1, 2, 3]. Remote frequency standards are commonly compared using time links such as GPS common-view, GPS carrier phase and Two-Way Satellite Time and Frequency Transfer (TWSTFT). In these cases, an evaluation of the frequency uncertainty introduced by time difference measurements involved in the remote links is required. The same requirement applies to the calibration of the rate of TAI by a primary frequency standard, as the TAI calculation involves a network of time links. In particular, the uncertainty on the frequency comparison introduced by the time link is calculated in the typical cases of white phase noise, flicker phase noise and white frequency noise affecting the time difference data [4]. The main purpose of this work is to evaluate the autocorrelation functions for all types of noise considered. To obtain the frequency uncertainty we used two different methods:

1. The first method is based on the law of uncertainty propagation [5], applied to the average frequency calculated in a time interval between two time difference measurements. In this case we obtain the autocorrelation functions by means the stochastic processes used to simulate the clock deviation [6, 7].
2. The second is based on the knowledge of the autocorrelation functions given by spectral analysis [8].

By the firs t method we can calculate the uncertainty on the frequency comparison when link data are affected by white phase noise and white frequency noise; by the second method we can also obtain the frequency uncertainty for flicker phase noise. All analytical results are compared with numerical simulations (Monte-Carlo method). Clock data from an HP5071 cesium standards (for white frequency noise) and time transfer data obtained by a double difference process, such as differencing the time series UTC(NIST)–UTC(PTB) obtained by GPS common-view with UTC(NIST)–UTC(PTB) obtained by TWSTFT (for white phase and flicker phase noises) are then used to test the analytical results. Results of this theoretical analysis show that the Allan deviation is not always an accurate estimator of frequency transfer uncertainty when time difference data are involved.

II. DEFINITIONS

The instantaneous output voltage [9, 10] of a precision oscillator can be expressed as

$$V(t) = (V_0 + \varepsilon(t))\sin(2\pi\nu_0 t + \phi(t))$$

where V_0 is the nominal peak voltage amplitude, $\varepsilon(t)$ is the deviation from the nominal amplitude, ν_0 is the nominal frequency, and $\phi(t)$ is the phase deviation from the nominal phase $2\pi\nu_0 t$. Frequency instability of a precision oscillator is defined in terms of the instantaneous, normalized frequency deviation, $y(t)$, as follows:

$$y(t) = \frac{\nu(t) - \nu_0}{\nu_0} = \frac{\dot{\phi}(t)}{2\pi\nu_0}$$

where $\nu(t)$ is the instantaneous frequency (time derivative of the phase divided by 2π), and

$$\dot{\phi}(t) = \frac{d\phi(t)}{dt} .$$

Phase instability, defined in terms of the phase deviation $\phi(t)$, can also be expressed in units of time, as

$$x(t) = \frac{\phi(t)}{2\pi\nu_0} .$$

With this definition, the instantaneous, normalized frequency deviation is

$$y(t) = \frac{dx(t)}{dt} .$$

In particular we define the mean fractional frequency on the time interval τ as reported:

$$\bar{y}(t) = \frac{x(t) - x(t-\tau)}{\tau} \tag{1}$$

III. Theoretical analysis

We want to obtain the uncertainty on (1) and to obtain it we apply two different methods, the first based on the law of the propagation of the uncertainty using the knowledge of the stochastic processes, the second based on the spectral analysis.

A. Law of the prpagation uncertainty

Considering the relation between the phase and the frequency (1) here reported it is possible to apply the law of the propagation uncertainty [5] on $\bar{y}(t)$, and we obtain:

$$u_{\bar{y}(t)}^2 = \frac{u_{x(t)}^2 + u_{x(t-\tau)}^2 - 2u_{(x(t),x(t-\tau))}}{\tau^2} . \tag{2}$$

To apply the relation (2) we have to know the variance and covariance terms reported.

The covariance term [6] for a random process *X(t)* is defined by:

$$\begin{aligned} u_{(X(t_1),X(t_2))} &= Cov(X(t_1), X(t_2)) = \\ &= E((X(t_1) - E(X(t_1)))(X(t_2) - E(X(t_2)))) \end{aligned} \tag{3}$$

but in literature often [7] it speaks about the autocorrelation function given by :

$$R_X(t_1, t_2) = E(X(t_1)X(t_2)) \tag{4}$$

where t_1 and t_2 are arbitrary sampling times and the functional E(·) is the expectation value.

The functions (3) and (4) are identical for zero mean processes. In the case considered in this work the processes are, effectively with zero mean so we consider indifferently both of them . The autocorrelation function tells how well the process is correlated with itself at two different times. If the process is stationary the autocorrelation functions (4) depends only on the time difference $\tau = t_2 - t_1$. Thus, R_X reduces to a function of just the time difference variable τ, that is,

$$R_X(\tau) = E(X(t)X(t-\tau)) \tag{5}$$

where t_1 is now denoted as just t and t_2 is (t-τ). Stationarity assures us that the expectation is not dependent on t. For almost all stationary data, the average values computed over the ensemble at time t_1 will equal the corresponding average values computed over time history record (Ergodic theorem).

To know the variance and covariance terms we have to know the stochastic processes used to model the noises [11, 12]. For this reason we can obtain the uncertainty on the frequency in the case of white phase noise and white frequency noise. To obtain the frequency uncertainty for the flicker phase noise we use the following method.

B. Spectral Analysis

The uncertainty on (1) can be expressed considering the link between autocorrelation functions and the spectral density function by the inverse Fourier transform [13, 14]. The uncertainty can be obtained using the expression (5):

$$u_{\bar{y}(t)}^2 = E\left(\left(\frac{x(t) - x(t-\tau)}{\tau}\right)^2\right) = \frac{2}{\tau^2}\left(R_\alpha(0) - R_\alpha(\tau)\right) = \frac{2I_\alpha(\tau)}{\tau^2} \tag{6}$$

where $I_\alpha(\tau) = R_\alpha(0) - R_\alpha(\tau)$. In this expression we use the functions $I_\alpha(\tau)$ because is a less divergent function instead of $R_\alpha(\tau)$. Also the expression of the Allan variance can be obtained using the functions $I_\alpha(\tau)$:

$$\sigma_y^2(\tau_0) = \frac{4I_\alpha(\tau_0) - I_\alpha(2\tau_0)}{\tau_0^2} . \tag{7}$$

Normalizing the expression (6) of the uncertainty respect to the Allan Variance (7) we obtain the final relation for the uncertainty on the frequency:

$$u_{\bar{y}(t)}^2 = \frac{\dfrac{2I_\alpha(\tau)}{\tau^2}}{\dfrac{4I_\alpha(\tau_0) - I_\alpha(2\tau_0)}{\tau_0^2}} \sigma_y^2(\tau_0) \tag{8}$$

Corresponding to different values of α we have different noises: for *α=-2* the random walk frequency, for *α=-1* the flicker frequency, for *α=0* the white frequency, for *α=1* the flicker phase and for *α=2* the white phase. We consider only the case of the flicker phase noise but it's possible to find the expression for $I_\alpha(\tau)$ for all different noises in [8].

IV. Analytical results

Following the methods presented in the previous section we obtain the analytical expressions for the uncertainty on the frequency in the time transfer link for the white phase noise, white frequency noise and flicker phase noise expressed in terms of the Allan variance.

A. White phase noise

White phase noise can be model by the random numbers independent normally distributed X(t) ~ $N(0,\sigma^2)$ where σ^2 is the variance of this process and the mean is equal to zero. The covariance in the case of the white phase noise is equal to zero and the relation between the Allan variance and σ^2 is given by the following relation $\sigma^2 = \dfrac{\sigma_y^2(\tau_0)\tau_0^2}{3}$.

In this case the uncertainty on the frequency, applying (2) is given by:

$$u_{\bar{y}(t)}^2 = \frac{\sigma^2 + \sigma^2}{\tau^2} = 2\frac{\sigma^2}{\tau^2} = \frac{2}{3}\frac{\sigma_y^2(\tau_0)\tau_0^2}{\tau^2} \tag{9}$$

Considering $\tau_0 = \tau$ we obtain: $u_{\bar{y}(t)}^2 = \frac{2}{3}\sigma_y^2(\tau)$, where $u_{\bar{y}(t)}$ is the standard deviation of the fractional frequency.

B. White frequency noise

To model the white frequency on the phase we use the Brownian motion (or Wiener process). Here we report a briefly description of this process for more details can be seen [6, 11]. The Wiener process indicated by W(t) is defined as a Gaussian Markov process with independent increments whose basic parameters are the drift μ and the diffusion coefficient σ [6, 11]. Considering the definition of the Wiener process, given by the solution of stochastic differential equation:

$$dX_t = \mu dt + \sigma dW(t). \qquad (10)$$

the solution, considering *W(0)=0* of (10) [12] can be written:

$$X_t = \mu t + \sigma W(t) \qquad (11)$$

at any instant the standard Wiener process is described by a Gaussian distribution:

$$X_t \sim N(\mu t, \sigma^2 t).$$

In particular we have that the variance of this process is $Var(X_t)=\sigma^2 t$ and the covariance is $Cov(X_t, X_s)=\sigma^2 \min(t,s)$. The diffusion coefficient σ^2 is linked to Allan variance [16, 17] by the reported relation: $\sigma^2 = \sigma_y^2(\tau_0)\tau_0$. Therefore, considering the values for the variance and the covariance for the white frequency on the phase the uncertainty on the frequency applying (2) in the case of μ=0 is:

$$u_{\bar{y}(t)}^2 = \frac{\sigma^2 t + \sigma^2(t-\tau) - 2\sigma^2(t-\tau)}{\tau^2} = \frac{\sigma^2}{\tau} = \frac{\sigma_y^2(\tau_0)\tau_0}{\tau} \qquad (12)$$

In this case for $\tau_0 = \tau$ we have: $u_{\bar{y}(t)}^2 = \sigma_y^2(\tau)$.

C. Flicker phase noise

To obtain the uncertainty on the frequency for the flicker phase noise we use the relation (8). For the flicker phase noise the function $I_\alpha(\tau)$ is given by [8]:

$$I_1(\tau) = \frac{h_1}{2\pi}\int_{\varepsilon\tau}^{\omega_n\tau}\frac{1-\cos(u)}{u}du = \frac{h_1}{(2\pi)^2}(\gamma + \ln(\omega_n\tau) - \cos\text{int}(\omega_n\tau)) \qquad (13)$$

where the dependence on ω_n is given by the divergence in infinity of the integral, the Cosine Integrals functions [18] which an standard expansion is reported: $\cos\text{int}(x) = \gamma + \ln(x) - \int_0^x \frac{1-\cos(u)}{u}du$ and γ is the Euler's constant equal to 0.57721...

In this case the uncertainty on the frequency, considering the denominator defined by:

$$den = 3\gamma + 3\ln(\omega_n\tau_0) - \ln(2) - 4\cos\text{int}(\omega_n\tau_0) + \cos\text{int}(2\omega_n\tau_0)$$

is:

$$u_{\bar{y}(t)}^2 = \frac{2(\gamma + \ln(\omega_n\tau) - \cos\text{int}(\omega_n\tau))}{den}\frac{\tau_0^2}{\tau^2}\sigma_y^2(\tau_0) \qquad (14)$$

in the case of $\tau_0 = \tau$ this result is:

$$u_{\bar{y}(t)}^2 = \frac{2(\gamma + \ln(\omega_n\tau_0) - \cos\text{int}(\omega_n\tau_0))}{den}\sigma_y^2(\tau)$$

In this case the uncertainty on the frequency depends on ω_n. The ω_n parameter depends on the sampling rate and it is possible to evaluate its numerical value using the analytical expression for the Allan Variance in the case of flicker phase noise [8- 10].

V. SIMULATION RESULTS

In this section we present a comparison between the results for the frequency uncertainty on the time transfer obtained with the simulation techniques and the analytical results presented in the previous section. We use two different methods to simulate the clock deviation behavior. The first is based on the use of the stochastic processes, the second on the use of the fractionally differences. Many details can be found in [12] and [19, 20].

The simulation starts on the phase data, after that the frequency data are obtained following (1) and the standard deviation on each time is calculated to obtain the uncertainty.

We have considered the case t=τ, i.e. the length t of the time series is used to obtain the frequency value.

A. White pahse noise

Considering a time series affected by white phase noise with the Allan variance equal to $\sigma_y(\tau) = 7*10^{-15}/day$ and τ in seconds we obtain the result shown in Figure 1. In this case a log-log plot is reported with τ in seconds. The red line shows the simulation results while the blue line the theoretical results obtained in (9). In this case we have N=500 simulations because the length of the simulation is equal to M=100000 so we had a computational problem with the matrix being too large. We can observe the good agreement between the simulations and the analytical results.

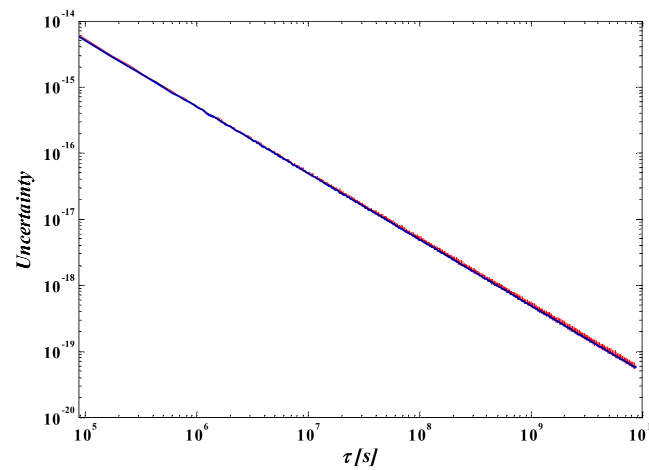

Figure 1. The theoretical result (blue line) for the uncertainty on the frequency for the White phase noise is compared with the simulation results (red star) with $\sigma_y(\tau) = 7*10^{-15}/days$.

B. *White frequency noise*

Considering a time series affected by a white frequency noise with the Allan variance equal to $\sigma_y(\tau) = 4 * 10^{-16} / day$ and τ in seconds we obtain the result shown in Figure 2. In this case a log-log plot is reported with τ in seconds. The red line shows the simulation results while the blue line the theoretical results obtained in (12). In this case we have used N=500 simulations because the length of the simulation is equal to M=100000 so we had a computational problem with the matrix being too large. We can observe the good agreement between the simulations and the analytical results.

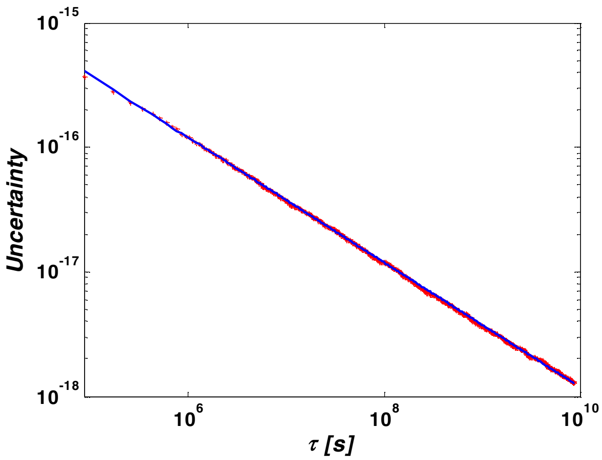

Figure 2. The theoretical results (blue line) for the uncertainty on the frequency for the White frequency noise is compared with the simulation results (red star) with $\sigma_y(\tau) = 4 * 10^{-16} / days$

C. *Flicker phase noise*

Considering a time series affected by flicker phase noise with the Allan variance equal to $\sigma_y(\tau) = 7.5 * 10^{-15} / day$ and τ in seconds we obtain the results reported in Figure 3. In this case the range of ω_n parameter, given by the analytical relation of the Allan Variance in the case of flicker phase noise [8-10], is approximately from $3/\tau_0$ to $4/\tau_0$ where τ_0=86400 s. In this case a log-log plot is reported with τ in seconds. The red line shows the simulation results while the blue line the theoretical results obtained in (14). Also in this case we can observe the good agreement between the simulations and the analytical results.

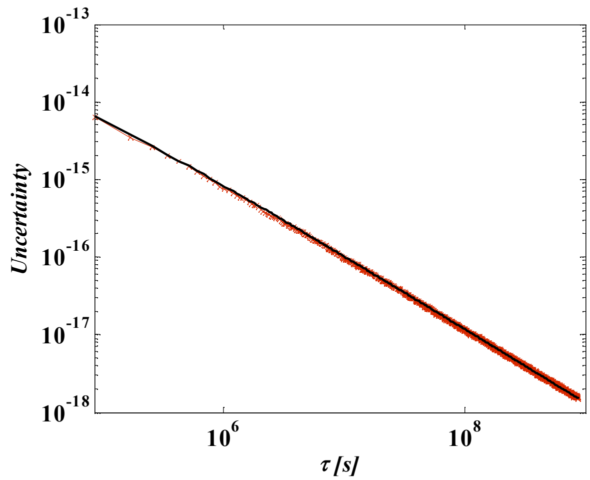

Figure 3. The theoretical results (blue line) for the uncertainty on the frequency for the flicker phase noise is compared with the simulation results (red line) with $\sigma_y(\tau) = 7.5 * 10^{-15} / days$.

VI. THE EXPERIMENTAL DATA

In this part we compared the results about the uncertainty on the frequency obtained with real data (calculated from a standard deviation) with the theoretical results presented in Section 4 (from the Allan deviation). We consider three different data sets obtained from (1) a cesium clock with respect to UTC(NIST), (2) the TWSTFT-GPS CV double differences for the NIST-NPL and (3) similar NIST-PTB data. In the case of the cesium clock data it is dominated by white frequency noise, in the case of time transfer data we have the contribution of the flicker phase and the white phase noises.

A. *White frequency*

In this part we compare the results obtained using the cesium data with respect to UTC(NIST) time scale with the theoretical analysis obtained in (12). The data are reported in Figure 4 with the frequency offset removed.

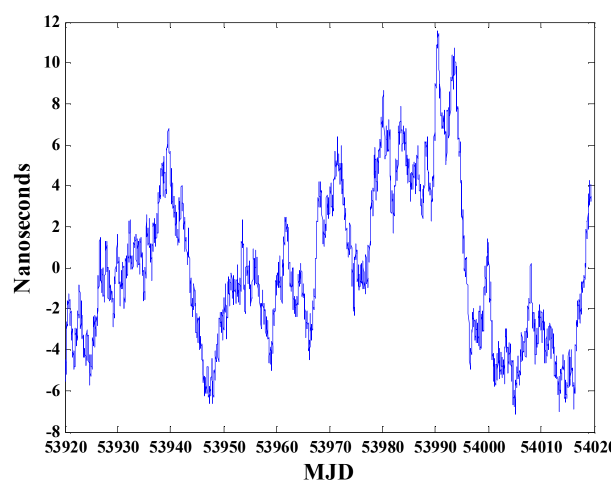

Figure 4. The HP5071-UTC(NIST) data for about 100 days.

We considered 100 days of data with the interval of 12 minutes. The Allan variance of the data shows a clear white frequency noise with $\sigma_y(\tau) = 2.46 * 10^{-13}$ for τ_0 =12 minutes. We can compare the results with the theoretical results considered in (12). The stability analysis for of the HP5071-UTC(NIST) is reported in Figure 5.

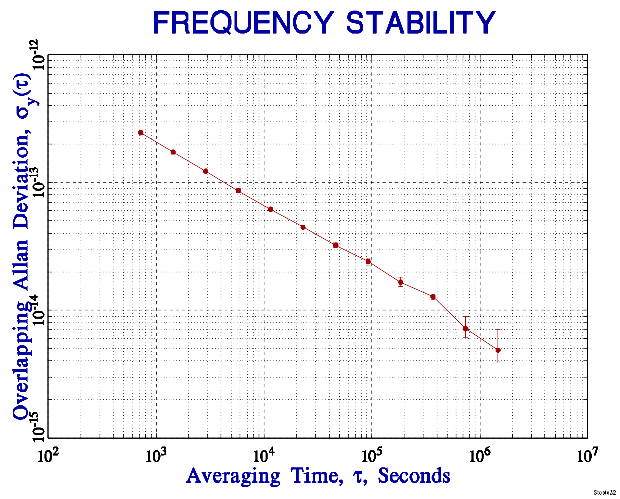

Figure 5. The stability analysis of the HP5071-UTC(NIST)

Using these data we can obtain an experimental evaluation for the frequency uncertainty. The results are reported in Figure 6 considering each time series during 4 hours. We can also choose a longer data set but the number of the data is too small. In this case the theoretical analysis agrees with the real data analysis.

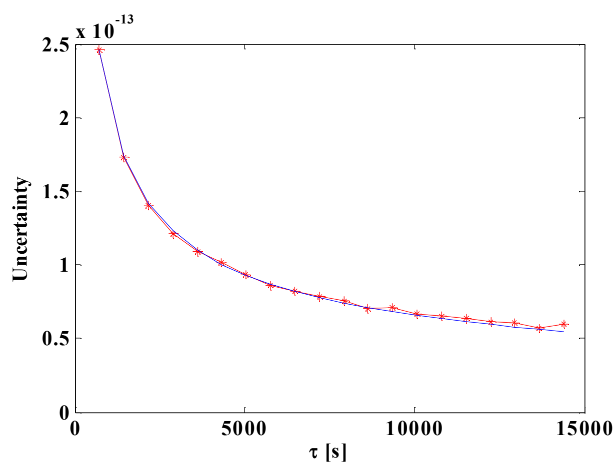

Figure 6. The uncertainty on the frequency obtained from the cesium clock data (red stars) and the theoretical results (blue line) are compared.

B. White phase noise and flicker phase noise

To estimate the uncertainty on the frequency with the flicker noise we have considered the TWSTFT-GPS CV double differences for the NIST-NPL and NIST-PTB data. In this case it is clear from the modified Allan Variance that the data are affected by the white phase noise until 1 day and the flicker phase noise after. The data related to NIST-NPL (blue line) and NIST-PTB (violet line) are reported in Figure 7.

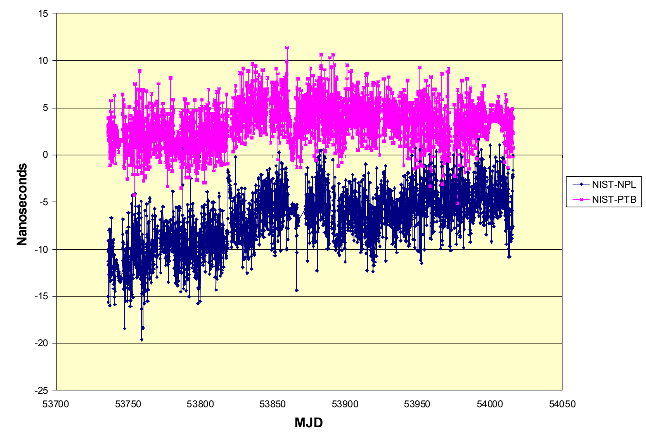

Figure 7. The data related to NIST-NPL (blue line) and NIST-PTB (violet line) TWSTFT-GPS CV are reported

In Figure 8 we report the modified Allan variance related to the NIST-PTB data to show the different noise components.

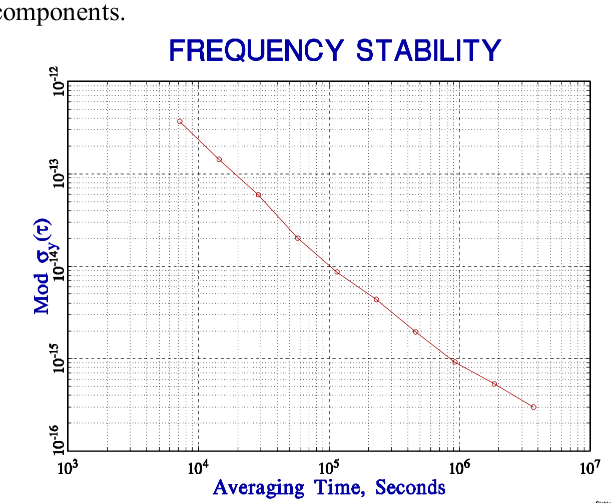

Figure 8. The modified Allan variance for NIST-PTB TWSTFT-GPS CV data.

To evaluate the uncertainty on the frequency we have to consider two components of the noise: the white phase noise with the Allan deviation equal to $\sigma_{y,WPN}(\tau) = 3.7 * 10^{-13}$ for τ_0 =7200 s and the presupposed flicker phase noise with the Allan deviation about equal to $\sigma_{y,FPN}(\tau) = 1.2 * 10^{-13}$ for τ_0 =7200 s. In this case the theoretical uncertainty is given by the combination of these two noises considered independently:

$$u^2_{\bar{y}(t)} = \frac{\tau_0^2}{\tau^2}\left(\frac{2(\gamma + \ln(\omega_n \tau) - \cos\mathrm{int}(\omega_n \tau))}{den}\sigma^2_{y,FPN}(\tau_0) + \frac{2}{3}\sigma^2_{y,WPN}(\tau_0)\right)$$

where:

$$den = 3\gamma + 3\ln(\omega_n \tau_0) - \ln(2) - 4\cos\mathrm{int}(\omega_n \tau_0) + \cos\mathrm{int}(2\omega_n \tau_0).$$

Considering the experimental data we obtain the uncertainty on the frequency and we compare the result with the theoretical expression reported here.

In this case the range of ω_n parameter, given by the analytical relation of the Allan Variance in the case of flicker phase noise [8-10], is approximately from $3/\tau_0$ to $4/\tau_0$ where τ_0=7200 s.

Figure 9 shows the uncertainty on the frequency obtained with the real data (red stars) related to the double differences NIST-PTB, the contribution of the flicker phase noise (blue line), the white phase noise (green line) separately, and both of them combined (black line). Figure 10 shows the same results but using the NIST-NPL data.

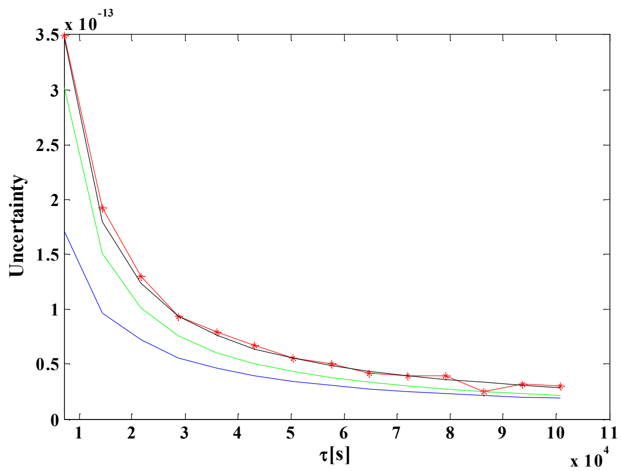

Figure 9. The frequency uncertainty obtained from NIST-PTB data (red stars) and from the theoretical results for flicker phase noise (blue line), white phase noise (green line) and both of them (black line) are reported

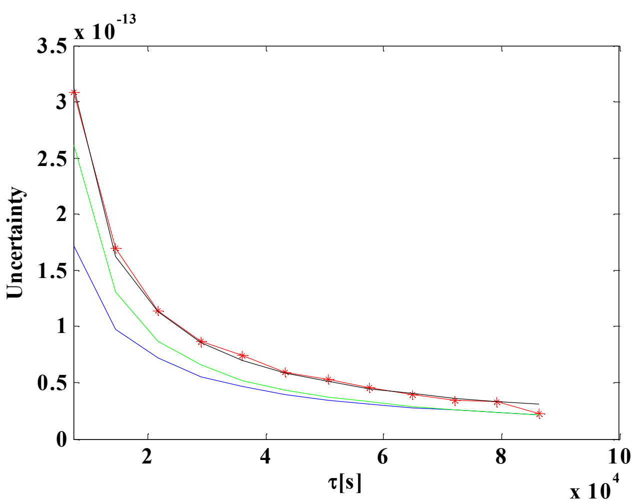

Figure 10. The frequency uncertainty obtained from the NIST-NPL data (red stars) and from the theoretical analyis for flicker phase noise (blue line), white phase noise (green line) and both of them (black line) are reported

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