Optical Fiber Vibration and Acceleration Model

Neil Ashby
Dept. of Physics
University of Colorado
Boulder, CO USA
Email: ashby@boulder.nist.gov

D. A. Howe, J. Taylor, A. Hati, C. Nelson
National Institute of Standards & Technology
Boulder, CO USA

***Abstract*— We derive expressions for the group velocities of transverse electric and transverse magnetic electromagnetic waves in a stretched single-mode fiber. Stretching can occur either as a result of temperature changes of the spool on which the fiber is wound, or as a result of axial vibrations that accelerate and hence deform the spool. Long single-mode fibers are typically used in optoelectronic oscillators, where the group velocity plays a central role in determining the oscillator frequencies. The present idealized calculation assumes there is a fractional length change $\delta l/l$, that results in stress in the fiber. This stress changes the optical properties of the fiber, and hence the group velocities, through its stress-optic coefficients. The principal result of the present calculation is that for optoelectronic oscillators, the main effect on the frequencies comes from the change of length itself rather than from the change in group velocities.**

I. INTRODUCTION

We discuss the effect of fiber length changes, due to vibration, on the frequencies of an optoelectronic oscillator. A typical optoelectronic device consists of a length of fiber into which modulated laser light is injected, an amplifier, a filter and a detector, with feedback arranged so that the device oscillates. A typical such device [1] gives a large number of modes with frequencies

$$\omega_K = \frac{(2K+1)\pi}{\tau_d}\,, \tag{1}$$

where K is an integer, and τ_d is the group delay through the fiber, neglecting small delays through other components of the system. The value of K is selected by the filter. If l is the length of the fiber (assumed to be tightly wound on a cylindrical spool) and v_g is the group velocity of electromagnetic waves down the fiber, then $\tau_d = l/v_g$. We can then construct a model of fractional frequency shifts caused by changes in fiber length, δl. Foremost is the change in length itself–an increase in length will cause the frequency to decrease. Independently, a change in length will result in stresses within the fiber that can change the optical properties of the fiber, resulting in a change in the group velocity. When these changes are small, the fractional frequency shift is a superposition of these two contributions:

$$\frac{\delta f}{f} = \frac{1}{v_g}\Big(\frac{\partial v_g}{\partial(\frac{\delta l}{l})} - v_g\Big)\frac{\delta l}{l}\,. \tag{2}$$

In this paper we investigate the dependence of the group velocity on changes in length of the fiber. The fiber is modelled as a step-index, single-mode cylindrical fiber with cladding having an outer radius much larger than the core radius. The index of refraction of the cladding material is assumed to be slightly less than that of the core, so that total internal reflection occurs in the core. The fiber is assumed to be wound tightly on a cylindrical spool; although the fiber is not actually straight–it is wound around a spool–the radius of curvature of the fiber is very large compared to the radius of the core. The resulting traction and compression stresses are neglected here and the fiber is modelled as though it were straight.

We assume the fiber is in close contact with the spool. We can then identify at least two effects that can change its length. We observe in the laboratory, for example, that when a metallic spool wound with fiber is heated, then before it is possible that the fiber itself could suffer a temperature change, there is a drift in frequency. We attribute this to changes in fiber length due to thermal expansion of the spool. Also, when the spool is clamped to a shake table and subject to a low-frequency acceleration $g(t)$, the spool will be deformed due to internal stresses within the spool material, resulting in length changes in the fiber.

The calculations described here show that for both transverse electric (TE) and transerse magnetic (TM) modes, and for reasonable fiber parameters, dependence of the group velocity on length changes is quite small and can be neglected in a first approximation. In Section II we discuss deformation of the elastic spool material. In Section III we discuss solutions of Maxwell's equations for TE and TM modes in the fiber. Technical details of these solutions are given in the Appendix. Section IV discusses solution of the dispersion relations that result from imposing appropriate boundary conditions at the core-cladding interface. Section V outlines the changes in the calculations when the fiber length changes. Results and conclusions are given in Section VI.

II. FIBER LENGTH CHANGES DUE TO SPOOL DEFORMATION

We model the spool as a solid cylinder with uniform density ρ, Young's modulus E, and Poisson's ratio ν. Suppose the spool is placed on a table with its axis vertical and is deformed under the influence of gravity, g. A horizontal slab of thickness dz must have more pressure on its lower surface than on its upper surface, in order to support the mass of the slab. We assume the relevant component of the stress tensor, σ_{zz}, is uniform across the slab, and that this is the only non-zero

1-4244-0647-1/07/$20.00 ©2007 IEEE

stress tensor component. Then

$$\frac{\partial \sigma_{zz}}{\partial z} = -\rho g\,. \tag{3}$$

The speed of sound in the spool material is assumed to be so high that resonances of the spool itself are very high in frequency. Then if the shake table applies an additional time-dependent acceleration $g(t)$, the spool will very quickly assume another deformed shape described by changes in the stress tensor, given by Eq. (3). Assuming the spool is clamped at the bottom, then

$$\sigma_{zz} = \rho g(t)(h - z)\,, \tag{4}$$

where h is the height of the top of the spool. The relations between the strain and stress tensors can be found in many textbooks [2]. For the simple geometry discussed here, we have for the non-zero components of the strain tensor

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x} = -\frac{\nu}{E}\sigma_{zz}\,, \tag{5}$$

$$\epsilon_{yy} = \frac{\partial u_y}{\partial y} = -\frac{\nu}{E}\sigma_{zz}\,, \tag{6}$$

$$\epsilon_{zz} = \frac{\partial u_z}{\partial z} = \frac{1}{E}\sigma_{zz}\,, \tag{7}$$

where u_x, u_y, u_z are displacements. These equations can immediately be integrated to give a change in radius of the spool:

$$\delta R = \frac{\nu \rho g(t)(h - z)R}{E}\,. \tag{8}$$

If the spool is wound with N_l turns per unit length, then the fractional change of length of fiber that is in contact with the spool can be obtained by integrating with respect to z, giving the result

$$\frac{\delta l}{l} = \frac{\nu \rho g(t) h}{4E}\,. \tag{9}$$

The numerical factor in this result may vary depending on how the spool is clamped. In any case, Eq. (9) contains some reasonable results for this model: the stiffer the spool, the smaller the length change; the denser the spool, the larger the length change because the forces are larger. Eq. (9) also shows a linear dependence on the time-dependent acceleration. Assuming now that length changes are given in terms of reasonable parameters, in the next section we discuss the solution of Maxwell's equations within the deformed fiber.

III. Electromagnetic Waves in a Single-Mode Fiber

The next stage in the development of this model consists of solving Maxwell's equations in a stretched fiber. Assuming $\delta l/l$ (the strain) is given, the elastic constants of the fiber will determine the stresses in the fiber. Then the stress-optic coefficients will determine changes in the dielectric tensor and hence in the indices of refraction. Further, under length changes the fiber radius will change as described by Poisson's ratio, and this has to be accounted for.

A. TE Modes

Our objective is to obtain expressions for the group velocity of electromagnetic waves propagating along the fiber axis, that is, in the z-direction. We assume the angular frequency is ω and the wavenumber is k so that the z-dependence and time-dependence of all fields is

$$e^{i(kz - \omega t)}\,. \tag{10}$$

Cylindrical coordinates are appropriate for this geometry. For the TE mode, there is only an azimuthal component of the electric field, and the magnetic field has radial and axial components (see Appendix); there is one undetermined constant multiplying the fields. Figure 1 shows the geometry of the cylindrical fiber. The outer cladding radius is very large compared to the core radius, and we assume the fields approach zero far outside the core. In the core, the fields are described by ordinary Bessel functions of argument

$$z_c r = \sqrt{\mu \epsilon_c \omega^2 - k^2}\, r\,. \tag{11}$$

In the cladding material, the fields are described by modified Bessel functions $K_n(z_{cl} r)$, where

$$z_{cl} = \sqrt{k^2 - \mu \epsilon_{cl} \omega^2}\,. \tag{12}$$

In these equations, the dielectric constant ϵ and index of refraction n are related by $\epsilon = \epsilon_0 n^2$ in SI units. In order to obtain solutions of the appropriate form, both z_c and z_{cl} must be real. When the frequency is given, as is the case when externally produced laser light enters the fiber, this places limits on the possible values of k:

$$\frac{\omega n_{cl}}{c} \le k \le \frac{\omega n_c}{c}\,. \tag{13}$$

There is one undetermined constant multiplying the field components in the cladding region. At the core/cladding interface, the tangential components of the electric and magnetic fields must be equal. These two conditions give two linear equations in the two undetermined constants, and the determinant of the coefficients of these constants must vanish in order for a solution to exist. The vanishing of the determinant can occur at only one positive value of k for each frequency. This condition is the *dispersion relation*, or relation between frequency and wave number:

$$\zeta_c^2 J_0(\zeta_c) K_1(\zeta_{cl}) + \zeta_{cl}^2 J_1(\zeta_c) K_0(\zeta_{cl}) = 0\,, \tag{14}$$

where

$$\zeta_c = a(1 - \nu\frac{\delta l}{l}) z_c; \quad \zeta_{cl} = a(1 - \nu\frac{\delta l}{l}) z_{cl}\,. \tag{15}$$

Here ν is Poisson's ratio and describes the reduction in core radius that accompanies the lengthening of the fiber. Expressing the dispersion relation in this way shows that the wavenumber corresponding to a given frequency will depend on the fiber length because the core radius will change.

Now imagine solving the dispersion relation, Eq. (14) for two values of frequency that differ by $d\omega$, and that the corresponding values of wavenumber differ by dk. The group velocity is then

$$v_g = \frac{d\omega}{dk} \,. \tag{16}$$

It is remarkable that the form of the dispersion relation, Eq. (14), lends itself to an explicit solution for the group velocity. To express the result, we define the auxiliary functions

$$F_1 = J_0(\zeta_c)K_1(\zeta_{cl}) + \zeta_c \frac{dJ_0(\zeta_c)}{d\zeta_c} K_1(\zeta_{cl}) + \zeta_{cl} \frac{dJ_1(\zeta_c)}{d\zeta_c} K_0(\zeta_{cl})\,; \tag{17}$$

$$F_2 = J_1(\zeta_c)K_0(\zeta_{cl}) + \zeta_c \frac{dK_1(\zeta_{cl})}{d\zeta_c} J_0(\zeta_c) + \zeta_{cl} \frac{dK_0(\zeta_{cl})}{d\zeta_{cl}} J_1(\zeta_c)\,. \tag{18}$$

The group velocity is then

$$v_g = \frac{kc^2}{\omega} \frac{F_1/\zeta_c - F_2/\zeta_{cl}}{n_c^2 F_1/\zeta_c - n_{cl}^2 F_2/\zeta_{cl}} \,. \tag{19}$$

The derivation leading from the dispersion relation to Eq. (19) is exact. However, the dispersion relation is transcendental and cannot be solved explicitly but must be solved numerically for each frequency.

B. TM Modes

For the transverse magnetic mode, there is only an azimuthal component of magnetic field, while there are radial and axial components of the electric field. In the core, again the solutions are described by ordinary Bessel functions, with one undetermined constant multiplying all the fields. In the cladding, the solutions are described by modified Bessel functions that fall off rapidly as the radius increases; there is a second undetermined constant multiplying all the fields. The boundary conditions that must be satisfied are: The tangential components of the electric field, and the normal component of the displacement vector, must be continuous. These give two linear conditions on the two undetermined constants, and again the determinant of the coefficients is a dispersion relation that fixes the wavenumber k, when the frequency is given. The dispersion relation for the TM mode is

$$\zeta_c n_{cl}^2 J_0(\zeta_c)K_1(\zeta_{cl}) + \zeta_{cl} n_c^2 J_1(\zeta_c)K_0(\zeta_{cl}) = 0. \tag{20}$$

This equation can be differentiated to yield an explicit expression for the group velocity of the TM mode. We define two auxiliary functions:

$$G_1 = n_{cl}^2 J_0(\zeta_c)K_1(\zeta_{cl}) + \zeta_c n_{cl}^2 \frac{dJ_0(\zeta_c)}{d\zeta_c} K_1(\zeta_{cl}) + \zeta_{cl} n_c^2 \frac{dJ_1(\zeta_c)}{d\zeta_c} K_0(\zeta_{cl})\,; \tag{21}$$

$$G_2 = n_c^2 J_1(\zeta_c)K_0(\zeta_{cl}) + \zeta_c n_c^2 \frac{dK_1(\zeta_{cl})}{d\zeta_{cl}} J_0(\zeta_c) + \zeta_{cl} n_{cl}^2 \frac{dK_0(\zeta_{cl})}{d\zeta_{cl}} J_1(\zeta_c)\,. \tag{22}$$

Then the group velocity is given by

$$v_g = \frac{d\omega}{dk} = \frac{kc^2}{\omega} \frac{\zeta_{cl} n_c^2 G_1 - \zeta_c n_{cl}^2 G_2}{\zeta_{cl} G_1 - \zeta_c G_2} \,. \tag{23}$$

IV. Numerical Calculations

In order to calculate the group velocity, as in Eq. (19), at some specified frequency, we must have previously solved the dispersion relation to find the corresponding wavenumber k. If done numerically this must be done with care; otherwise there is a considerable loss of accuracy. For example, Newton's method for solving a transcendental equation does not work well here. A method that works is to begin with a trial value of k and a convenient step size, stepping in the right direction until the sign of the dispersion relation changes. Then the step size is accurately divided by some integer–such as 6–and the direction of stepping is reversed. This process works quite well and gives sufficient accuracy for our purposes.

We give here numerical results for an unstretched fiber with reasonable parameters, preparatory to recomputing the group velocity for a stretched fiber (next Section). The values used are

$$n_c = 1.5362\,, \qquad n_{cl} = 1.5306\,, \tag{24}$$

$$a = 8.3 \times 10^{-6}\ \mathrm{m}\,, \tag{25}$$

$$\lambda = 1550\ \mathrm{nanometers}\,. \tag{26}$$

Here λ is the operating wavelength of the laser light. The refractive indices are typical for a single-mode fiber. For the TE mode,

$$k = 6.21627 \times 10^6\ \mathrm{m}^{-1}\,, v_g = 1.9497 \times 10^8 \mathrm{m/s}\,. \tag{27}$$

For the TM mode,

$$k = 6.21625 \times 10^6\ \mathrm{m}^{-1}\,, v_g = 1.9510 \times 10^8 \mathrm{m/s}\,. \tag{28}$$

V. Stretched Fiber

In order to obtain the partial derivative of the group velocity in Eq. (2), the calculations described above must be repeated for a stretched fiber. In this case the important quantity that changes is the radius of the core/cladding interface:

$$a \to a(1 - \nu \frac{\delta l}{l})\,. \tag{29}$$

Also, the refractive indices of the core and possibly the cladding material change. The sequence of calculations here is to first assume the fractional change in length is given. This gives the strain tensor. The stress-optic coefficients (when available) provide the dielectric tensor and hence the changed refractive indices. Then changes will be proportional to the fractional length change. This calculation cannot be carried to completion analytically, but considerable progress can be made when the length change is small.

Let the stress-optic coefficients of the core be c_1 and c_2, and let the Lamè coefficient and bulk modulus of the core be

[3]

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}\,, \tag{30}$$

$$\mu = \frac{E}{2(1+\nu)}\,, \tag{31}$$

where E is Young's modulus of the core and ν is Poisson's ratio for the core. There will be similar relations for the cladding. The principal stresses (there is no shear) in a longitudinally stretched fiber are

$$\sigma_{xx} = \sigma_{yy} = \big(\lambda - 2\nu(\lambda+\mu)\big)\frac{\delta l}{l}\,, \tag{32}$$

$$\sigma_{zz} = (\lambda - 2\nu\lambda + 2\mu)\frac{\delta l}{l}\,, \tag{33}$$

where μ is the core shear modulus. The principal indices of refraction are then:

$$n_x = n_y = n_0 + c_1(\lambda - 2\nu(\lambda+\mu))\frac{\delta l}{l} + c_2(2\lambda + 4\nu\mu)\frac{\delta l}{l}\,, \tag{34}$$

$$n_z = n_0 + c_1(\lambda - 2\nu(\lambda+\mu))\frac{\delta l}{l} + c_2(2\lambda + 4\nu(\lambda+\mu))\frac{\delta l}{l}\,. \tag{35}$$

Thus under stretching, the fiber material becomes anisotropic. These indices of refraction, along with the modified core radius, are inserted in the dispersion relations and the calculation of group velocity is repeated for a small assumed numerical value of the fractional change in length. We give results here for the following reasonable values of the stress-optic coefficients and elastic constants of the core material:

$$c_1 = -1.297\times10^{-12}\mathrm{m}^2/\mathrm{Newton}\,, \tag{36}$$

$$c_2 = -4.835\times10^{-12}\mathrm{m}^2/\mathrm{Newton}\,, \tag{37}$$

$$\nu = 0.17, \tag{38}$$

$$E = 7.3\times10^{-12}\mathrm{Newton}^2/\mathrm{m}\,, \tag{39}$$

$$\mu = 2.62\times10^{-12}\mathrm{Newton}^2/\mathrm{m}\,, \tag{40}$$

$$\lambda = \frac{2\mu\nu}{1-2\nu} = 1.35\times10^{10}\mathrm{Newton}^2/\mathrm{m}\,. \tag{41}$$

Then we obtain for the core:

$$n_{core} = n_c - 0.044375\frac{\delta l}{l}\,. \tag{42}$$

Unfortunately, numbers for the stress-optic coefficients of cladding material do not seem to be available. Therefore in the absence of better information, we shall use a similar result for the cladding:

$$n_{clad} = n_{cl} - 0.044375\frac{\delta l}{l}\,. \tag{43}$$

Proceeding with these assumptions for a stretched fiber, we again compute the group velocities, then take differences to obtain the following final results. For the TE mode,

$$\frac{\partial v_g}{\partial\frac{\delta l}{l}} = 5.614\times10^{6}\mathrm{m/s}\,, \tag{44}$$

and for the TM mode,

$$\frac{\partial v_g}{\partial\frac{\delta l}{l}} = 5.614\times10^{6}\mathrm{m/s}\,, \tag{45}$$

The partial derivatives in Eqs. (44) and (45) are small compared to v_g, given in Eqs. (27) and (28). Hence, from Eq. (2),

$$\frac{\delta f}{f} \approx -\frac{\delta l}{l} \tag{46}$$

for small length fluctuations.

VI. Conclusions

This paper provides a simplified model for computation of changes in the group velocity of transverse electric and transverse magnetic waves propagating down a straight stretched fiber. The principal conclusion reached by this analysis is that changes in the group velocity are quite small, and that we can neglect them in a first approximation so that the fractional frequency shift due to vibration-induced fiber length changes in an optoelectronic oscillator can be adequately approximated by

$$\frac{\delta f}{f} \approx -\frac{\delta l}{l}\,. \tag{47}$$

References

[1] S. Römisch, J. Kitching, E. Ferrè-Pikal, L. Hollberg, and F. L. Walls, IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control **47**(5), 2000, pp 1159-1165.

[2] A. L. Fetter and J. D. Walecka, *Theoretical Mechanics of Particles and Continua*, McGraw-Hill, Inc., New York (1980), Chapter 13.

[3] J. W. Dally and W. F. Riley, *Experimental Stress Analysis*, McGraw-Hill, Inc., New York (1991), Chapter 2.

[4] J. D. Jackson, *Classical Electrodynamics*, Second Edition, John Wiley & Sons, New York (1975), endpapers.

[5] D. Marcuse, *Light Transmission Optics*, Second Edition, Van Nostrand Reinhold, New York (1982), Chapter 8.

Appendix

In this appendix we give a self-contained solution of Maxwell's equations for a clad step-index fiber having azimuthal symmetry, when the fields do not depend on the azimuthal angle ϕ. More general solutions are discussed in many textbooks, e.g., [5], without, however, carrying out the calculation of the group velocity. Here we have made a number of simplifying assumptions. The fiber is assumed to have a uniform circular cross section of radius a (or $a(1-\delta l/l)$), and is assumed to be straight, so that the strain tensor has no shear components. All fields are assumed to have a unique frequency and z-wavenumber, as in Eq. (10). In addition, the fields are assumed not to depend on the azimuthal angle ϕ around the z-axis. Differential operators in cylindrical coordinates can be found in many textbooks. [4]

Transverse Electric Mode. Since the fields only depend on radius r and axial coordinate z, Maxwell's equations simplify. In terms of components in the radial, z, and azimuthal directions the transverse electric mode solutions in the core reduce

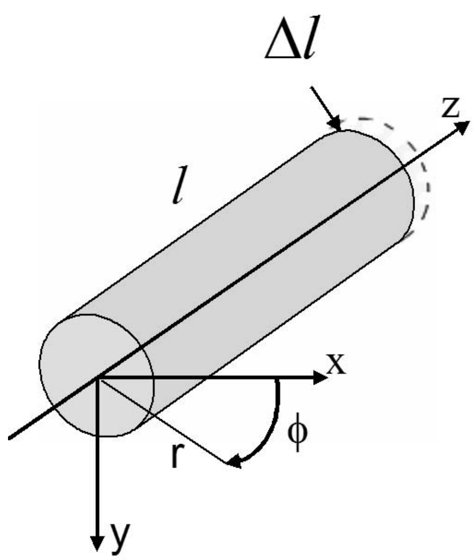

Fig. 1. Coordinate system used to express solutions to Maxwell's equations for a circular fiber of radius a. Ordinary polar coordinates in the $x - y$ plane are also used.

to

$$E_r = E_z = B_\phi = 0\, r, ; \tag{48}$$
$$E_\phi = C \frac{dJ_0(z_c r)}{dr}\, r, , \tag{49}$$
$$B_r = -\frac{k}{\omega} C \frac{dJ_0(z_c r)}{dr}\,, \tag{50}$$
$$B_z = \frac{i z_c^2}{\omega} C J_0(z_c r)\,, \tag{51}$$

where the z-dependence, given by Eq. (10), has been suppressed, z_c is defined in Eq. (11), and J_0 is the ordinary Bessel Function of order 0. The constant C is an overall amplitude that at this point is undertermined since Maxwell's equations are linear.

In the clad material, the fields must fall off rapidly as the distance from the core increases. We assume here that the outer radius of the cladding is sufficiently large that leakage of fields past the cladding is negligible. Then the solutions in the cladding region are

$$E_r = E_z = B_\phi = 0\,, \tag{52}$$
$$E_\phi = D \frac{dK_0(z_{cl} r)}{dr}\,, \tag{53}$$
$$B_r = -\frac{k}{\omega} D \frac{dK_0(z_{cl} r)}{dr}\,, \tag{54}$$
$$B_z = -\frac{i z_{cl}^2}{\omega} D K_0(z_{cl} r)\,. \tag{55}$$

Here K_0 is the modified Bessel function of order 0, and D is an overall undetermined constant. In order to satisfy the boundary conditions at the core/cladding interface, D and C must be related. The relevant boundary conditions in the present case are continuity of the tangential component of E_ϕ, and continuity of the tangential component of H_z. These conditions give two equations

$$\begin{aligned} C \frac{dJ_0(z_c r)}{dr} - D \frac{dK_0(z_{cl} r)}{dr} &= 0\,, \\ -C z_c^2 J_0(z_c r) - D z_{cl}^2 K_0(z_{cl} r) &= 0\,. \end{aligned} \tag{56}$$

The arguments of the Bessel functions must be evaluated at the radius a, or $a(1 - \nu \delta l / l)$ for the stretched fiber.

The determinant of the coefficients of C and D in Eqs. (56) must vanish in order to obtain a self-consistent solution. This condition can be satisfied by only one positive value of the wavenumber k for a given frequency, and thus gives the dispersion relation for TE modes. (See main text for discussion.) There is then one relation between C and D and one overall undetermined field amplitude.

Transverse Magnetic mode. In this case the only non-zero component of the magnetic field is in the azimuthal direction. In the core, the solutions to Maxwell's equations are

$$B_r = B_z = E_\phi = 0\,, \tag{57}$$
$$B_\phi = \mu \epsilon_c C \frac{dJ_0(z_c r)}{dr}\,, \tag{58}$$
$$E_r = \frac{k}{\omega} C \frac{dJ_0(z_c r)}{dr}\,. \tag{59}$$
$$\tag{60}$$

where again the z-dependence, given by Eq. (10), has been suppressed; ϵ_c is the dielectric constant of the core, $\epsilon_c = \epsilon_0 n_c^2$.

The solutions in the cladding region are

$$B_r = B_z = E_\phi = 0\,, \tag{61}$$
$$B_\phi = \mu \epsilon_{cl} D \frac{dK_0(z_{cl} r)}{dr}\,, \tag{62}$$
$$E_r = \frac{k}{\omega} D \frac{dK_0(z_{cl} r)}{dr}\,, \tag{63}$$
$$E_z = \frac{i z_{cl}^2}{\omega} D K_0(z_{cl} r)\,, \tag{64}$$

and $\epsilon_{cl} = \epsilon_0 n_{cl}^2$. The boundary conditions that must be satisfied in this case are: continuity of the tangential components of the electric field, and radial components of the displacement vector. The determinant of the coefficients of C and D gives the dispersion relation, and differentiation of the dispersion relation gives an explicit expression for the group velocity, as discussed in Section III above.