Metrologia 44 (2007) 91–96 doi:10.1088/0026-1394/44/1/014

Uncertainty of a frequency comparison with distributed dead time and measurement interval offset*

Dai-Hyuk Yu[1], **Marc Weiss**[2] **and Thomas E Parker**[2]

[1] Korea Research Institute of Standards and Science, PO Box 102, Yuseong, Daejeon 305-600, Korea
[2] National Institute of Standards and Technology, 325 Broadway, Boulder, CO 80305, USA

Received 11 May 2006
Published 2 February 2007
Online at stacks.iop.org/Met/44/91

Abstract
A theory is presented for estimating the uncertainty of a frequency comparison in the presence of distributed dead time or measurement interval offset using an extension of the method of Douglas and Boulanger (1997 *Proc. 11th European Frequency and Time Forum* pp 345–9). The uncertainties due to the distributed dead time and lumped dead time with mixed power law noise type are calculated and compared. It is shown that the use of distributed measurements of frequencies can greatly reduce the uncertainty as compared with that of lumped measurements. When a measurement interval offset is present, two different methods are possible for the frequency estimation and uncertainty evaluation. We compare and discuss the different results for the different methods.

1. Introduction

Most laboratory built primary standards are operated as frequency standards and not as continuously running clocks. The operation of a primary frequency standard such as a caesium fountain continuously for tens of days is a difficult task. Thus the dead time or the measurement interval offset generally cannot be avoided when we compare the frequency of a primary frequency standard to that of a spatially remote standard or to International Atomic Time (TAI). This is also true for the same standard compared to itself at different times. This comparison process generally involves a stable but not necessarily accurate, secondary, or transfer, frequency standard such as a hydrogen maser, a maser ensemble or TAI. A primary frequency standard measures the frequency of the transfer standard, and comparisons to other standards with dead time or to itself with a measurement interval offset depend on the stability of the transfer standard. High precision frequency comparison, therefore, needs accurate knowledge of the uncertainty caused by the dead time and/or measurement interval offset.

Azoubib *et al* presented a method of comparing the frequencies from several standards or the same standard at different times with that of TAI [2]. They focused mainly on the frequency comparison with measurement interval offset, and each of the frequency measurements was assumed not to have dead time in it or an unknown dead time. Douglas and Boulanger performed a similar analysis for the dead time or measurement interval offset and presented a simple method to calculate the uncertainty due to a transfer process from one time interval to another [1]. Before this work, they developed a formalism to calculate the local oscillator's contribution to the uncertainty of a caesium fountain running in a quasi-continuous fashion [3]. This method was used for the analysis of time scale work [4] and the frequency transfer uncertainty for the hydrogen maser calibrated with a caesium fountain [5–7]. Examination of multiple calibration runs and a drifting secondary frequency standard were also treated in these works. Parker *et al* obtained an approximate solution that estimated uncertainty in the presence of uniformly distributed dead time [8].

In this paper, we present a simple method to extend the previous study [1] of calculating the uncertainty due to any form of dead time or measurement interval offset and compare the differences between the different configurations of dead time or measurement interval offset.

2. Theory

Assume the live frequency measurements in an interval A are performed at N different times with $(N-1)$ dead times in between. We can calculate a weighted mean of the frequencies using all the live measurements. If we want to compare the measured and calculated frequency with that at another time interval B (which may either contain all the live measurements or has no overlap with them), the uncertainty caused by the transfer process must also be calculated. The uncertainty U can be expressed as the mean-square difference between these two frequencies:

$$U^2 = \left\langle \left(\sum_{i=1}^{N} a_i y_i - y_\mathrm{T} \right)^2 \right\rangle, \qquad (1)$$

where y_i and a_i are the ith measured frequency and normalized weight, respectively, in the interval A, and y_T is the unknown 'true' frequency for an interval B. It can easily be shown that equation (1) can be expanded into equation (2). We assume that the signal is stationary and the interchange of the order of the integration in calculating autocovariance gives the same value. This step allows the results of [1] to be extended to the case with the distributed live measurements:

$$U^2 = \sum_{i=1}^{N} a_i \langle (y_i - y_\mathrm{T})^2 \rangle - \frac{1}{2} \sum_{i,j=1}^{N} a_i a_j \langle (y_i - y_j)^2 \rangle. \qquad (2)$$

As seen in equation (2), the uncertainty variance is the weighted sum of variances of each live interval to transfer to interval B, minus the weighted sum of covariance terms of all possible combinations between the two live measurements. The uncertainty can easily be calculated with proper assignment of the weights a_i and the analytical solutions for the uncertainties $\langle (y_i - y_\mathrm{T})^2 \rangle$ and $\langle (y_i - y_j)^2 \rangle$ found in [1, 3–7]. The detailed assumptions such as the band limit of the upper and lower frequency cut-off to get the analytical solutions are also explained in the references. We can simply use the same weights for all the live measurements, but this will not produce the optimum frequency estimate with the minimum uncertainty. Given any configuration of live measurements, the weights a_i to minimize the uncertainty can be calculated by the Lagrangian multiplier method. We define the function F by

$$F = U^2 + \lambda \left(\sum_{i=1}^{N} a_i - 1 \right). \qquad (3)$$

By using equation (2), the minimum of U^2 is reached when

$$\frac{\partial F}{\partial a_i} = \langle (y_i - y_\mathrm{T})^2 \rangle - \sum_{j=1}^{N} a_j \langle (y_i - y_j)^2 \rangle + \lambda = 0, \qquad (4.1)$$

$$\frac{\partial F}{\partial \lambda} = \sum_{i=1}^{N} a_i - 1 = 0. \qquad (4.2)$$

Optimum weights for minimizing the uncertainty can easily be calculated by solving the above $(N+1)$ linear equations with $(N+1)$ variables. As is well known for the specific case of white frequency modulation (FM) noise, the solution shows that only the total measurement live time matters, and the weight of each measurement is proportional to the measurement interval, i.e. inversely proportional to the Allan variance, as follows:

$$a_i = \tau_i \bigg/ \sum_{j=1}^{N} \tau_j = (1/\sigma_y^2(\tau_i)) \bigg/ \sum_{j=1}^{N} (1/\sigma_y^2(\tau_j)), \qquad (5)$$

where τ_i is the ith measurement interval. However, the optimum weights do not exactly follow equation (5) when there are other power law noise types.

3. Uncertainty with distributed dead time

To gain a better understanding of the practical effects of the dead time, we use a reference time scale with a specific mixed noise type used in [8] throughout this paper. The noise characteristics, $\sigma_y(\tau)$, of the reference time scale are assumed to be defined in terms of the three basic noise types (white FM, flicker FM and random-walk FM). Allan deviations of the three noise types are $4 \times 10^{-16} \tau^{-1/2}$ for white FM, 4×10^{-16} for flicker FM and $1.3 \times 10^{-16} \tau^{1/2}$ for random-walk FM, where τ is expressed in days. These noise levels are representative of a small ensemble of active hydrogen masers. Here it is assumed that the individual maser frequency drift parameters are included in the ensemble algorithm, and therefore the ensemble frequency has negligible drift. This is the case with the NIST ensemble. Negligible drift is also present in the TAI. If a single maser with drift is used as a reference it may be necessary to symmetrize the dead time about the centre of the measurement interval in order to minimize the effect of the linear component of the drift.

First, to test the theory, we generated an ensemble of 1000 simulated time series data sets with the given mixed noise type from above. We used a software (Stable 32) to generate the data sets and the noise generation is known to be based on the works by Kasdin and Walter [9] and Greenhall [10]. We calculated two different frequencies for a given total accumulated measurement time for each data set. One is an estimated frequency obtained by averaging the frequencies of live measurements and the other is the 'true' average frequency found from the end points of the data set. We obtained the variance by averaging the differences between the two frequencies squared for all the simulated time series data sets. We also calculated the theoretical values from equation (2) and compared the simulated and theoretical variances. We tested the uncertainties due to lumped and distributed dead time. We assume that interval B is of 60 days and all the measurements are performed within this interval B. All the live time is at the beginning of interval B for the lumped dead time case, and the live and dead times are 1 day, respectively, and alternate from the beginning for the distributed dead time case. A 3 day measurement example is shown in figure 1. The same weights for all 1 day live measurements are used. As shown in figure 2, the theoretical and 'true' results from the simulated data are in good agreement, confirming that the

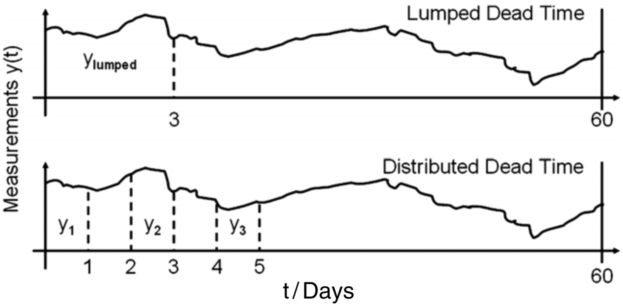

Figure 1. 3 day measurement example with lumped and distributed dead time around the beginning.

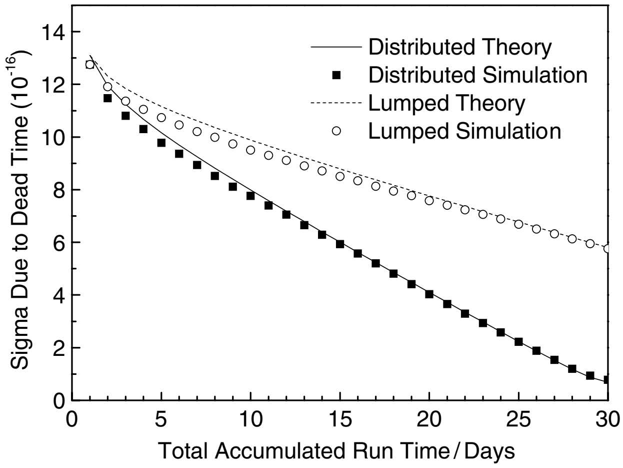

Figure 2. Frequency uncertainty due to dead time in a 60 day evaluation interval (live measurements around the beginning).

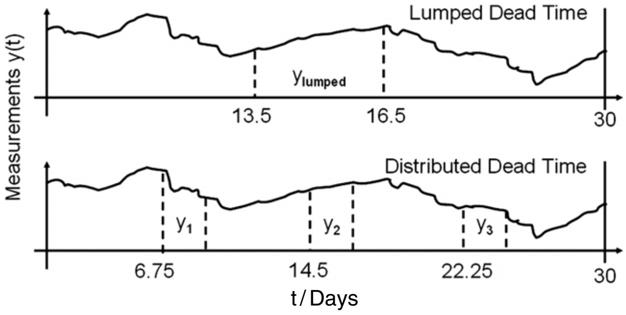

Figure 3. 3 day measurement example with lumped and distributed dead time around the centre.

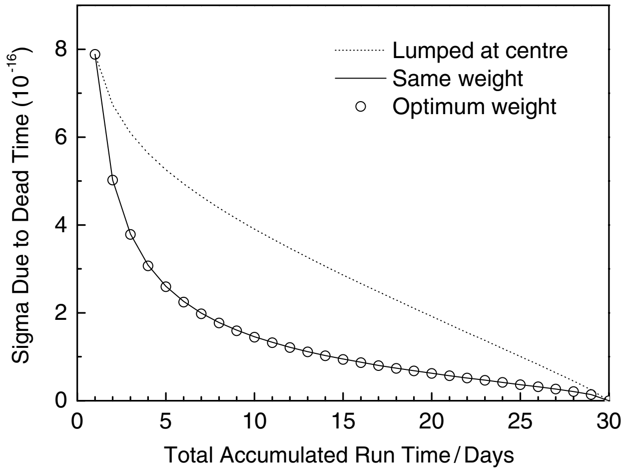

Figure 4. Fractional frequency uncertainty due to dead time in a 30 day evaluation interval (live measurements around the centre).

analytical approach is correct. The slight discrepancy between the analytical and 'true' results in figure 2 at low run time values is most likely due to the difficulty in generating true simulated flicker frequency noise. Also we should note that the uncertainty due to distributed dead time is always smaller than that due to the lumped case as estimated in [8]. We also carried out similar comparisons for individual noise types and found good agreement there as well.

Since live measurements are usually centred in the middle of the 30 day reporting interval for the evaluation of a primary frequency standard, we compared the theoretically calculated uncertainty due to lumped dead time centred at the 30 day interval and the distributed case of each 1 day live time distributed symmetrically with the same dead time from the end and the in-between intervals along the reporting interval. A 3 day measurement example is shown in figure 3. The results are shown in figure 4. The dotted line is for the lumped case and the solid line is for the distributed case with the weights proportional to each separate live measurement interval. Since all the live measurements have the same 1 day interval, the solid line is the uncertainty with equal weights. The circle points are for the distributed case with the weights optimized to minimize the uncertainty. The difference between the lumped and distributed cases is remarkable. The uncertainty due to the distributed dead time can be less than one third of that of the lumped case for a given total accumulated run time. In other words, for example, we need over 25 days of evaluation in the lumped case to have the uncertainty less than 1.0×10^{-16}. However, 15 days of distributed run time is enough in the distributed case to reduce the uncertainty below the 1.0×10^{-16} level. Figure 5 shows the optimum weights for various total accumulated measurement times. As can be seen more clearly for a relatively small number of measurements, the measurements done at either end have the larger weights and those at the centre show lower values. For the total measurement time of 6 days, each of the two centre measurements has a weight corresponding to 85% of that at either edge. As for the uncertainty, the weight optimization does not appreciably reduce the uncertainty for the considered mixed noise type. The maximum difference between the two distributed cases was 1.5×10^{-18}. This means that weight proportional to the interval is nearly optimum in practical applications.

4. Uncertainty with measurement interval offset

We also investigated the effect of a measurement interval offset on the uncertainty of frequency transfer. As shown in figure 6, we assume that the frequency was measured during the interval A and is compared with itself at another interval B. Live measurements are distributed in the interval A in the same way, as shown in figure 3. Intervals A and B are assumed to have the same 30 day duration. The same specific reference time scale used in the above section is also used for this section. The uncertainty due to the frequency transfer to

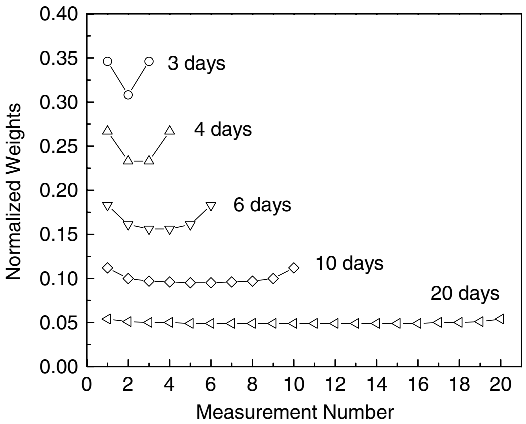

Figure 5. Normalized weights to minimize the uncertainty due to dead time in a 30 day evaluation interval (live measurements around the centre).

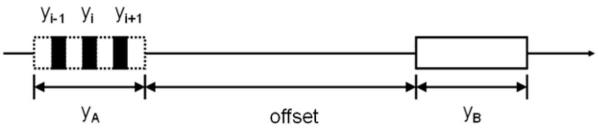

Figure 6. Measurement configuration of the frequency and the uncertainty with measurement interval offset. Intervals A and B have the same duration.

the later interval B with measurement interval offset can be calculated in two different ways.

4.1. Method 1

Measurements done in the interval A can be used to estimate frequencies for the whole interval A or B by use of equation (1). Estimated frequencies are generally different for intervals A and B since optimum weights obtained from equations (4.1)–(4.2) are different for different target intervals. A direct frequency estimate for the target interval B will be considered in method 2 below. Usually we have a frequency estimate and its uncertainty for the target interval A by use of the measurements of y_i with dead time calculated even before the interval B arrives. If we want to know the uncertainty in using this frequency estimate for the interval A to transfer the frequency to the later interval B, we should consider the uncertainty due to the measurement interval offset between intervals A and B as well as the uncertainty due to dead time within the interval A. It seems that this is the usual way of the frequency transfer process. In this case, the uncertainty variance u_1 can be expressed by use of equation (1) as follows:

$$u_1^2 = \left\langle \left(\sum_{i=1}^{N} a_{\mathrm{A}i} y_i - y_{\mathrm{B}} + y_{\mathrm{A}} - y_{\mathrm{A}} \right)^2 \right\rangle = u_{\mathrm{A}}^2 + u_{\mathrm{O}}^2 + u_{\mathrm{C}}, \tag{6.1}$$

$$u_{\mathrm{A}}^2 = \left\langle \left(\sum_{i=1}^{N} a_{\mathrm{A}i} y_i - y_{\mathrm{A}} \right)^2 \right\rangle, \tag{6.2}$$

$$u_{\mathrm{O}}^2 = \langle (y_{\mathrm{A}} - y_{\mathrm{B}})^2 \rangle, \tag{6.3}$$

$$u_{\mathrm{C}} = 2\left\langle \left(\sum_{i=1}^{N} a_{\mathrm{A}i} y_i - y_{\mathrm{A}} \right)(y_{\mathrm{A}} - y_{\mathrm{B}}) \right\rangle, \tag{6.4}$$

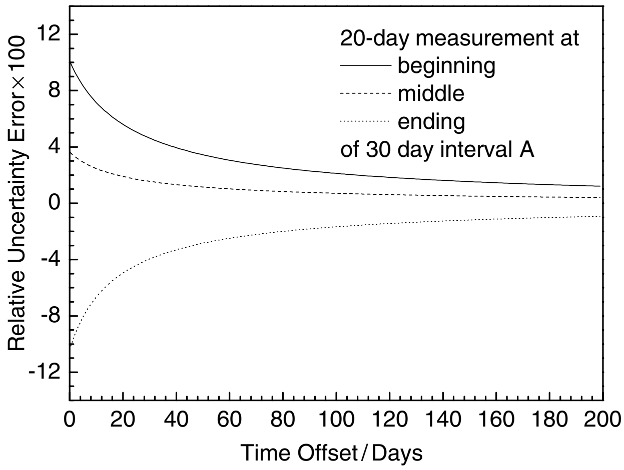

Figure 7. The percentage of the correction term with respect to the uncertainty u_1 in equations (6.1)–(6.4).

where $a_{\mathrm{A}i}$ is the weight optimized to estimate the frequency in interval A and y_{A} and y_{B} are, respectively, the 'true' frequencies of intervals A and B. u_{A} is the uncertainty to estimate the 'true' frequency y_{A} for the interval A due to dead time, u_{O} is the uncertainty to compare y_{A} to y_{B} due to measurement interval offset and u_{C} is the correlation term due to the non-white noise of the reference time scale. $a_{\mathrm{A}i}$ are the weights that minimize u_{A} in equation (6.2). As clearly seen in equations (6.1)–(6.4), we should be careful when calculating the total uncertainty, since the uncertainty is not just $\sqrt{u_{\mathrm{A}}^2 + u_{\mathrm{O}}^2}$ but has a correlation term. Sub-processes of estimating the frequency for an interval A and transferring it to the interval B are referred to the same reference time scale. If the reference time scale has non-white noise, an appropriate correction due to the correlation should also be made.

4.2. Method 2

As already noted in method 1, we can calculate a new weighted average of all the frequency measurements done in interval A by going directly to an estimate of the 'true' frequency for the interval B. We use equation (1) but with the interval B now lying outside the interval including all the real measurements. If we call the optimum weights for this case $a_{\mathrm{B}i}$, then the $a_{\mathrm{B}i}$ come from minimizing directly the total uncertainty in equation (6.1). Since the weights $a_{\mathrm{A}i}$ and $a_{\mathrm{B}i}$ are generally different, the frequency estimates and the total uncertainties also have different values.

First, we investigate the effect of the correlation term of method 1 in equations (6.1)–(6.4). Twenty days of lumped live measurements are performed at three different times for the beginning, middle and ending of the interval A. For each case, we calculate the difference between the 'true' total uncertainty u_1 and the incorrectly assumed total uncertainty $\sqrt{u_{\mathrm{A}}^2 + u_{\mathrm{O}}^2}$, divided by u_1. This can be called the 'relative uncertainty error' and is shown in figure 7. Use of the estimated frequency in interval A to compare it in interval B with the simple uncertainty of $\sqrt{u_{\mathrm{A}}^2 + u_{\mathrm{O}}^2}$ can produce errors as large as 10% of the correct uncertainty that are biased low or high depending upon when the measurement is performed. The

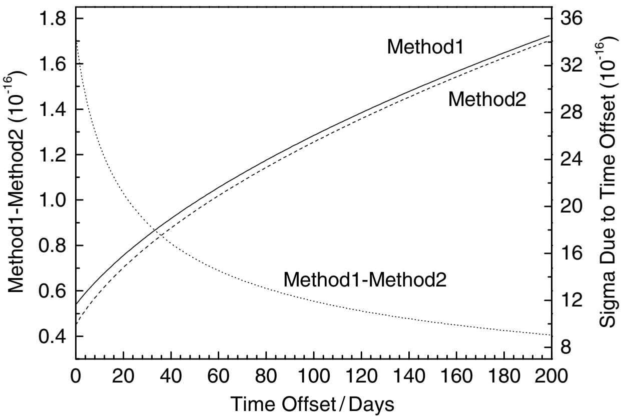

Figure 8. The uncertainties obtained by two different methods. Total accumulated run time is 20 days with 1 day consecutive measurements distributed as shown in figure 3.

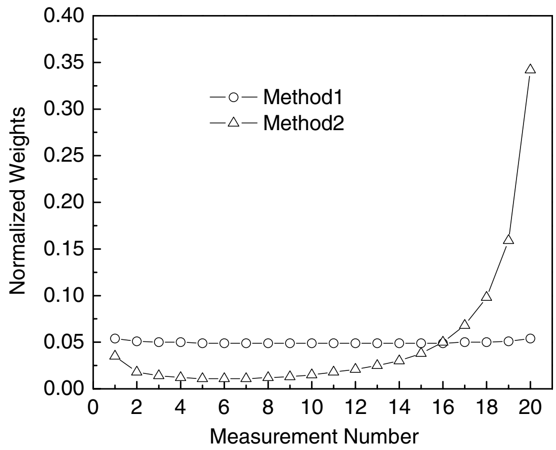

Figure 9. The optimum weights obtained by two different methods when the measurement interval offset is 200 days.

maximum difference between the beginning and the ending measurement case is more than 20% of the correct uncertainty and corresponds to 2.5×10^{-16} in magnitude for the assumed time scale noise. The relative uncertainty error becomes larger as the offset becomes smaller. Generally, the error depends on the total accumulated time, position and distribution of the live measurements in interval A and is due mostly to the random-walk FM noise. If the reference time scale has a large random-walk FM noise level, the error cannot be neglected. This result shows that detailed information may be necessary to compare frequencies with the correct uncertainty in the presence of dead time. In [2], each calibration was assumed to have no dead time in it. If the calibration were done with dead time, the uncertainty would have a contributing term other than the reported dead time uncertainty u_{A} and the measurement interval offset uncertainty u_{O}. This may lead to different weights for the presumably same calibrations but they may actually have different configurations of dead time in them.

Figure 8 shows the uncertainty obtained by the above two different methods. The total accumulated run time is 20 days with 1 day live measurements distributed in the same way as shown in figure 3. Also shown is the difference between the two uncertainties. Method 1 first calculates the optimum frequency estimate for an interval A and uses it for another interval B with measurement interval offset. As shown in figure 8, when the estimated frequency for an interval A is used for an interval B, the uncertainty is always larger than that when a frequency estimate is calculated for the interval B directly. In other words, method 1 is optimum for a frequency estimate for interval A but not for another interval B. Generally, the frequencies are different for the different methods since the weights are different. Different optimum weights for the two different methods are shown in figure 9 when the measurement interval offset is 200 days. Since method 1 uses optimum weights for the interval A, the weights for all 20 separate measurements are almost the same, as already shown in figure 5. However, the optimum weights for method 2 have a different behaviour. The measurement closest to interval B has the largest weight. The first measurement, which is furthest away from the target interval B, also has a relatively large weight compared with the measurements done around the centre. This behaviour is due largely to the random-walk FM noise of the reference time scale.

5. Conclusion

We have developed a simple method to calculate the uncertainty of a frequency comparison in the presence of arbitrary configurations of dead time and measurement interval offset. The results show that given a real mixed noise type model, the uncertainty due to dead time for a given total accumulated run time can be reduced to less than one third with distributed measurements of frequencies compared with the lumped measurement. In other words, 15 days of distributed run time is enough to reduce the uncertainty below 1.0×10^{-16} compared with 25 days needed for the lumped case.

We also investigated the effect on the frequency transfer uncertainty of a measurement interval offset. There are two different methods to estimate the frequency for the target interval with measurement interval offset. Use of the 'old' estimated frequency always has a higher uncertainty than that found by estimating the 'new' frequency with the 'old' individual measurements. We also demonstrated that the uncertainty of using a two-step estimating process, where the frequency for an 'old' interval is estimated and then transferred to the 'new' interval, is not as simple as taking the square root of the sum of the two uncertainties squared. For the considered reference time scale, the correction terms due to correlations can be more than 10% of the true values. The maximum difference between the beginning and the ending measurement case is more than 20% of the correct uncertainty.

References

[1] Douglas R J and Boulanger J-S 1997 Standard uncertainty for average frequency traceability *Proc. 11th European Frequency and Time Forum (Neuchâtel, Switzerland)* pp 345–9

[2] Azoubib J, Granveaud M and Guinot B 1977 Estimation of the scale unit duration of time scale *Metrologia* **13** 87–93

[3] Douglas R J and Boulanger J-S 1992 Local oscillator requirements for timekeeping in the $10^{-14}\tau^{-1/2}$ era *Proc. 1992 IEEE Int. Frequency Control Symp. (Hershey, PA)* pp 6–26

[4] Douglas R J, Boulanger J-S and Jacques C 1993 Accuracy metrics for judging time scale algorithms *Proc. 25th Precise Time and Time Interval Meeting (Marina Del Ray, CA)* pp 249–66

[5] Morris D, Douglas R J and Boulanger J-S 1994 The role of the hydrogen maser frequency transfer from cesium fountains *Japan. J. Appl. Phys.* **33** 1659–68

[6] Douglas R J and Boulanger J-S 1994 Frequency control of hydrogen masers using high accuracy calibration *Proc. 1994 IEEE Int. Frequency Control Symp. (Boston, MA)* pp 695–708

[7] Douglas R J and Boulanger J-S 1994 Mettafitting: weight optimization for least-square fitting of PTTI data *Proc. 26th Precise Time and Time Interval Meeting (Reston, VA)* pp 347–60

[8] Parker T E, Howe D A and Weiss M 1998 Accurate frequency comparison at the 1×10^{-15} level *Proc. 1998 IEEE Int. Frequency Control Symp. (Pasadena, CA)* pp 265–72

[9] Kasdin N J and Walter T 1992 Discrete simulation of power law noise *Proc. 1992 IEEE Int. Frequency Control Symp. (Hershey, PA)* pp 274–83

[10] Greenhall C A 2002 FFT-based method for simulating flicker FM *Proc. 34th Precise Time and Time Interval Meeting (Reston, VA)* pp 481–9