GENERALIZATION OF THE TOTAL VARIANCE APPROACH TO THE DIFFERENT CLASSES OF STRUCTURE FUNCTIONS

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***Abstract*—The Total variance approach has been developed for increasing the confidence of the estimation of the classical Allan variance (AVAR), particularly for large integration time values. This method is based on a procedure of extension of the original data sequence called the mirror-reflection which increases the equivalent degrees of freedom of each Allan variance estimate. Recently, we applied this approach to the Modified Allan variance (MVAR) and proved that, in this case, another procedure of extension of the data sequence should be used: the reflection-only extension.**

In this paper, we propose a criterion to select the most appropriate extension procedure for a given structure function (i.e. variance). This criterion is based on the sensitivity of a structure function to the linear and quadratic drifts, or, and this is equivalent, on the convergence of this structure function for the different types of noise.

This method and this criterion will be illustrated by its application to the "pulsar variance", which is insensitive to the quadratic and linear phase drifts, and converges from f^{-4} FM to f^{+2} FM.

1. INTRODUCTION

The estimation of the long term stability of oscillators (particularly very long term) is more difficult than the estimation of the short term stability. As a consequence, the random walk and flicker FM noise levels, which are likely to dominate at long term, are always less precisely estimated than the other noise levels. Nevertheless, the low frequency noise levels contain the statistical information of the future behavior of an oscillator, which is essential for extrapolating its performances [1].

Obviously, this lack of knowledge is due to the length of the time sequence which is often insufficient for insuring the preeminence of the low frequency noises. For ultra stable quartz oscillators, a data sequence of at least one day length must be used to distinguish the flicker FM from the white FM, and 5 to 10 days may be necessary for detecting the random walk FM. Moreover, the uncertainty of the estimates increases with the integration time. For example, with the Allan variance (AVAR) and considering a T-length time sequence, the longest integration time is $\tau = T/2$. Its corresponding estimate is chi-square distributed with only one degree of freedom. This means that the standard deviation of such an estimate is $\sqrt{2}$ times greater than its expectation. Furthermore, the distribution is negatively skewed with values twice as likely to be below the actual noise level than above. AVAR is also sensitive to linear frequency drift which must be removed, thus additionally suppressing the actual low frequency random noise levels. To be safe, the $T/2$ estimate is ignored, and the longest integration time is limited in practice to $T/4$ or less. Therefore, a data run of length 20 to 40 days must be used to provide sufficient confidence for detecting the presence of, say, random walk FM, let alone estimating its level. When long enough data runs become impractical to obtain, Total variance (Totvar, or its usually reported square-root Totdev) is recommended as an improved estimator of long term stability [2].

The goal of the concepts in this paper is to increase the confidence of long term frequency stability estimates without increasing the length of a data run for other classes of variances. Additionally, it is often important to distinguish white PM from flicker PM, unlike the Allan variance. To accomplish this, we apply the Total variance approach [3] to structure functions [4]. This is motivated by two issues.

The first issue involves defining a suitable variance from a combination of order and family of structure functions such that the variance is sensitive to all expected types of noises while it is insensitive to particular types of drift. This avoids the complication of removing drifts before applying the variance and consequently avoids suppression and underestimation of actual random noise level at long term.

The second issue involves retrieving the maximum information from the data run itself by using the Total variance approach. This approach is based upon a periodic extension of data sequences [5]. Obviously, this doesn't permit an increase of the integration time beyond a variance's normal upper limit, but the uncertainty interval of the corresponding long term estimates is significantly reduced by using the Total approach.

From our experience in constructing an improved estimator of the modified Allan variance (called Mod-Totvar), several types of data extensions can be considered [6]. This paper gives criteria for the extension type selection according to the variance which is obtained from the order and family of structure functions. Section 2 illustrates how the Total variance model is implemented on the modified Allan variance to obtain the best "Modified Total" variance. Section 3 defines relevant structure functions. Section 4 gives criteria for selecting a data extension type for these structure functions and ends with an example of the Total approach applied to the pulsar variance.

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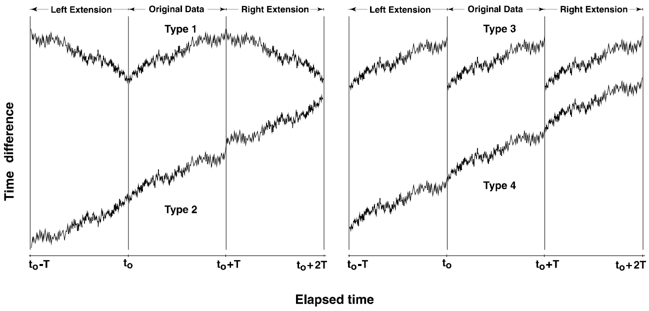

Fig. 1. The four types of data extensions.

$S_y(f)$	b	c	edf(τ_{max})	bias%
$h_{+2}f^{+2}$	1.9	2.1	3.6	-6%
$h_{+1}f^{+1}$	1.2	1.4	2.2	-17%
$h_0 f^0$	1.1	1.2	2.1	-27%
$h_{-1}f^{-1}$	0.85	0.50	2.05	-30%
$h_{-2}f^{-2}$	0.75	0.31	1.94	-31%

TABLE I

EDF MODEL DEFINED IN (2) AND BIAS FOR MOD-TOTVAR.

2. THE TOTAL VARIANCE APPROACH

2.1 Bias and equivalent degrees of freedom

The Total variance approach involves periodically extending a data sequence beyond its normal measurement duration and in such a way that a particular time statistic is expected to have the same value with extended data as without. For those statistics which estimate components of broadband noise processes, the approach can significantly reduce the spread or uncertainty in the result.

We use two quantities to check the efficiency of the method:

- *the bias* defined as the percentage of error between the classical variance estimate and the Total variance estimate;
- *the equivalent degrees of freedom (edf)* defined, assuming a χ^2 distribution of the estimates, as

$$\text{edf} = \frac{2\left[\text{Mean}(\text{estimates})\right]^2}{\text{Variance}(\text{estimates})}. \qquad (1)$$

2.2 Types of data extension

Denoting T as the length of the calculation sequence (for example, $T = 2\tau$ for AVAR, $T = 3\tau$ for MVAR, ...), four types of extension have been used (see figure 1):

1. $2T$-periodic uninverted or even mirror-reflection,
2. $2T$-periodic sign-inverted or odd mirror-reflection,
3. T-periodic straight duplication,
4. T-periodic duplication with end-to-beginning connections.

Obviously, other extension types could be used, but these types summarize essential properties of interest.

Sequence extensions were originally tested with the Allan variance. The type 2 extension was found to be optimum and is used in defining Totvar [5],[7]. Since Totvar, like AVAR, doesn't distinguish white PM and flicker PM noises, the approach was generalized to the time variance (and time deviation) [8] and to the modified Allan variance [6]. These variances are specially designed for estimating the level of the phase modulation noise types (white PM, flicker PM, and random walk PM) as well as the frequency modulation noise types (flicker FM and random walk FM). In this case, the type 1 extension should be used in order to avoid a huge bias in the presence of high frequency (or PM) noises. This is because the type 2 extension modifies the mean of a sequence at each end-to-beginning connection, inducing a step in the subestimates and causing the overall mean of the Total approach to be biased very high (see figure 2 and ref. [6]).

2.3 Modeling edf and bias

Howe and Greenhall defined an empirical model for edf of Totvar [5]:

$$\text{edf}\left[\text{Totvar}(\tau, T)\right] = b\frac{T}{\tau} - c. \qquad (2)$$

where the coefficients b and c were estimated for each type of noise from a Monte-Carlo method.

Using the same model (2) for Mod-Totvar, we obtained the coefficients b and c given in Table I, which also lists percentage bias of Mod-Totvar relative to classical MVAR for its range of noise types.

2.4 Practical implementation of Mod-Totvar

Let us consider a sequence of N time error data $\{x(t_i)\}$, with a sampling rate τ_0. Let us denote T the length of the total sequence: $T = N\tau_0$.

The integration time τ may be defined as $\tau = m\tau_0$ where m is an integer and $m \leq N$. The length of the calculation sequence of MVAR is 3τ, which will be called a *subsequence*, one of all possible consecutive sequences of 3τ length.

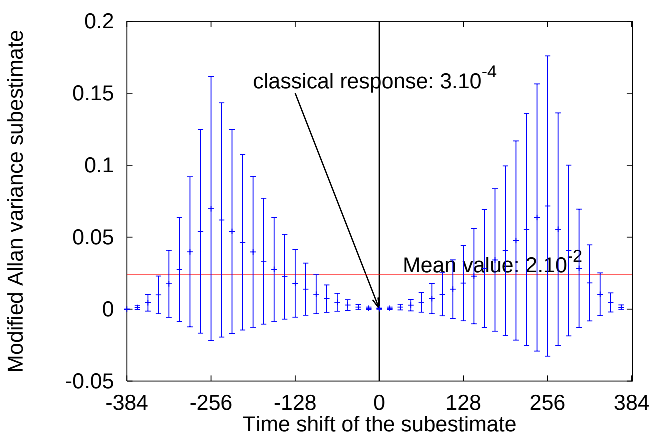

Fig. 2. Bias of the subestimates for a type 2 extension applied to a white PM noise and using MVAR. Shown are MVAR mean values and associated standard deviations (by the error bars) computed at $\pm$ time-shifts of $10\tau_0$, that is, shifts in the extended sequence of $0, \pm 10, \pm 20, \ldots \pm 384$. Each mean value is an average of 1000 estimates. The middle mean value is at a null-shift (0) which corresponds to the classical MVAR result.

In order to calculate the Mod-Totvar for one given value $\tau = m\tau_0$, one would:

1. extract all $3m$ data subsequences from the whole sequence $\{x(t_i)\}$;
2. remove their linear phase drift;
3. extend them at both ends by the type 1 even mirror-reflection to form $9m$ data subsequences;
4. calculate MVAR for each of these $9m$ subsequences;
5. average all these MVAR results.

2.4.1 2-τ and 3-τ extension

There are two cases for which MVAR can be computed. In case 1, MVAR subsequences can have a span of 2τ if three successive $\{x(t_i)\}$ values form a second-difference and consecutive second-differences spaced by τ_0 are subsequently averaged for integration time $\tau = m\tau_0$. Resulting values are then squared and averaged to compute MVAR. In case 2, MVAR subsequences can have a span of 3τ if three successive τ-averaged $\{x(t_i)\}$ values form a second-difference whose squared value is then averaged with all other possible squared second difference values. Since case 1 yields the same answer as case 2, the question arises, "Should we extend 2τ (case 1) or 3τ (case 2) subsequences?" Using simulation studies, we compared both cases, extending each subsequence to form a 2τ version and a 3τ version of Modified Total variance. The 2τ version had significantly more negative bias than the modest bias of the 3τ version shown in Table I. Moreover, the edf showed a reduction by 20% to 35% corresponding to f^{+2} FM to f^{-2} FM. These results show conclusively that the 3τ version is superior to the 2τ version, hence 3τ subsequences are used in all formulations of Modified Total variance.

2.4.2 Taking advantage of symmetries

Figure 3 shows an example of time-shifted MVAR means (and standard deviations of 1000 simulation trials) and exhibits 2 axes of symmetry located at time-shift $-384\tau_0$ and $+384\tau_0$ for a τ value equal to $256\tau_0$. This means that redundancies allow us to only calculate the subestimates for time-shifts contained between $-384\tau_0$ and $+384\tau_0$. The average of these subestimates is thus exactly the same as a complete calculation of Mod-Totvar, i.e. averaged from time-shifts $-768\tau_0$ to $+768\tau_0$.

Denoting s_k as the subestimate obtained for a time-shift equal to $k\tau_0$, it can be demonstrated that $s_{-3m+k} = s_{-k}$ and $s_{3m+k} = s_{3m-k}$.

Consequently,

- if m is even, we just have to calculate the subestimates from $s_{-3m+3m/2}$ to $s_{3m+3m/2}$ ($6m$ subestimates) ;
- if m is odd, we just have to calculate the subestimates from $s_{-3m+(3m+1)/2}$ to $s_{3m+(3m-1)/2}$ ($6m-1$ subestimates).

The periodicity of the subsequence is equal to $9m\tau_0$ (i.e. 9τ) but, thanks to the symmetries, we only have to consider $6m$ subestimates.

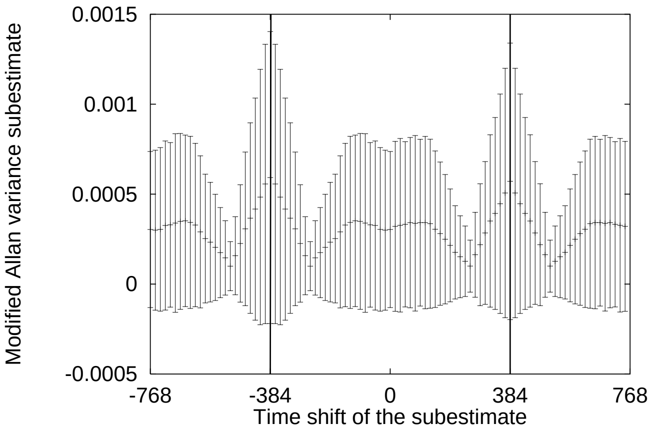

Fig. 3. Time-shifted MVAR mean and corresponding 1σ standard deviation of the mean (by the error bars) after extending a simulated subsequence by even reflection (type 1) for subsequence noise types white PM. The axis of symmetry are located at -384 and +384 for a τ value equal to $256\tau_0$.

3. The Structure Functions

The concept of structure functions is an extension of the variance approach to n^{th} difference operators [4].

The structure function $\sigma^2_{n,m}(\tau)$ is characterized by its order of difference n and its family m, expressing the "smoothness" (rectangular, linear, quadratic, ...shapes) of its sequence calculation (see figure 4).

The properties of a structure function $\sigma^2_{n,m}(\tau)$ may be summarized by:

- $\sigma^2_{n,m}(\tau)$ is insensitive to phase drifts up to t^n;
- $\sigma^2_{n,m}(\tau)$ converges for low frequency noises from f^{-2n} FM;
- $\sigma^2_{n,m}(\tau)$ converges for high frequency noises up to f^{+2m-2} FM.

It may be noticed that the convergence for low frequency noises and the insensitivity to phase drifts are linked. This property is known as the *moment condition* [9], [10]: it may be demonstrated that the insensitivity to a t^n frequency drift (t^{n+1} phase drift) yields the convergence up to f^{-2n-2} FM.

For example, we may choose the structure function $\sigma^2_{2,2}(\tau)$, called the pulsar variance [11], [12], because it converges for all types of noise and it is insensitive to the quadratic phase drifts.

4. Criteria for the Extension Type Selection

We checked

- 4 structure functions: the Allan variance ($\sigma^2_{1,1}(\tau)$), the modified Allan variance ($\sigma^2_{2,1}(\tau)$), the Picinbono variance ($\sigma^2_{1,2}(\tau)$), the Pulsar variance ($\sigma^2_{2,2}(\tau)$),
- with the 4 extension types described above,
- with or without removal of the linear or the quadratic phase drift (3 cases),

i.e. 48 different estimators.

We applied all these estimators to 100 simulated realizations (16384 data) of each of the 5 noise types (from f^{-2} FM to f^{+2} FM), i.e. 500 noise sequences.

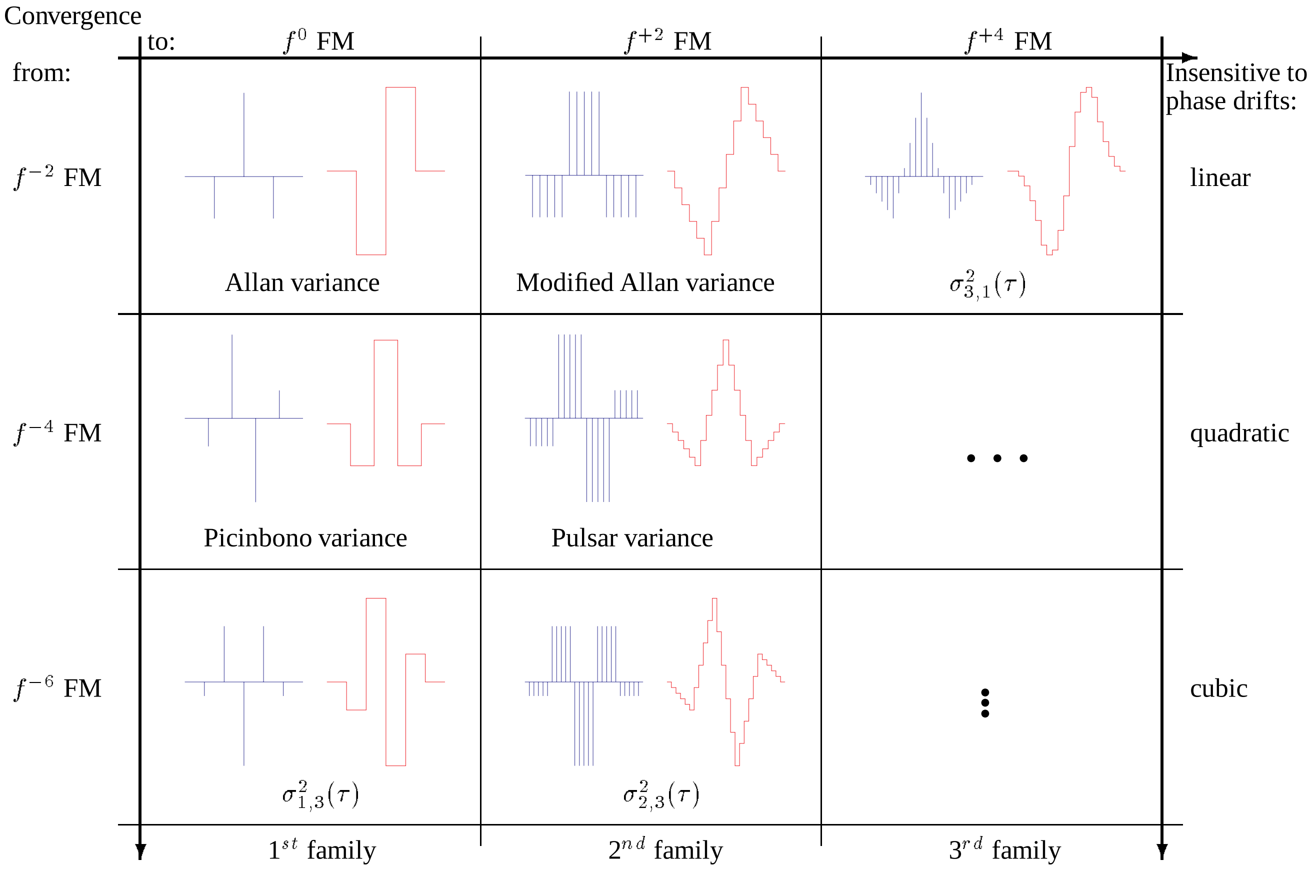

Fig. 4. The different classes of structure functions. The left figures in each box represent the calculation sequences for time error data and the right figures represent the calculation sequences for frequency deviation data. From top to bottom, the order of difference increases, yielding convergence for lower frequency noises and insensitivity to higher order polynomial drifts. From left to right, the different families yield convergence for increasingly higher frequency noises.

4.1 Results:

- extension types 1 and 2 are better than 3 and 4 (bias smaller and edf higher);
- extension type 1 is better than 2 from f^{-2} PM to white PM;
- as mentioned above, extension type 2 must be avoided for f^{+1} PM and f^{+2} PM;
- removing the quadratic phase drift decreases the edf.

4.2 Criteria:

1. since 1^{st} family structure functions don't converge for f^{+1} PM and f^{+2} PM, the type 2 extension should be used;
2. the type 1 extension must be used for the structure functions of the 2^{nd} family and higher;
3. since the 1^{st} order difference (and higher) structure functions are insensitive to linear phase drift, this drift must be removed over each subsequence of length 2τ (lower bias);
4. since the 2^{nd} order difference (and higher) structure functions are insensitive to quadratic phase drift, this drift must be removed over the whole sequence (lower bias) but not over each subsequence of length 2τ since this would decrease the edf (see last result above).

4.3 Examples:

Total variance (Totvar): extension type 2, no removal of drift;

Modified Total variance (Mod-Totvar): extension type 1, removal of the linear drift over each subsequence;

Total Picinbono variance: extension type 1, removal of the linear drift over each subsequence, removal of the quadratic drift over the whole sequence;

Total Pulsar variance: extension type 2, removal of the linear drift over each subsequence, removal of the quadratic drift over the whole sequence.

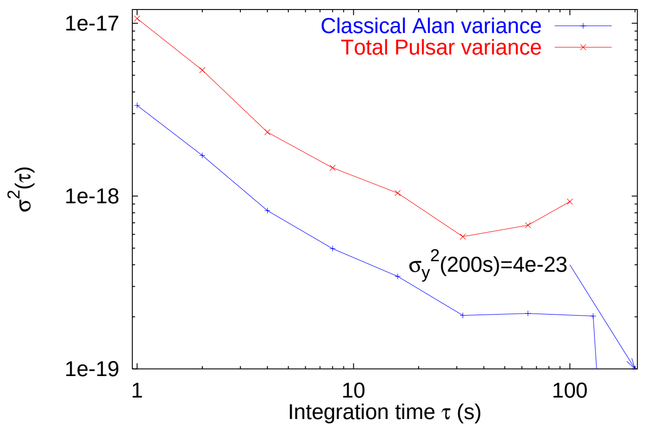

Fig. 5. Comparison of the classical Allan variance and the Total Pulsar variance. These curves were obtained from 400 frequency measurements of a quartz oscillator, with a sampling rate equal to 1s.

5. Example

Figure 5 shows the results obtained with the classical Allan variance compared to the results obtained by the Total Pulsar variance applied to the same real sequence. Despite its lower integration time limit ($\tau_{max} = T/4$ for PVAR and $\tau_{max} = T/2$ for AVAR), only the Total Pulsar variance is able to show the positive slope for large τ values, corresponding to the detection of the random walk frequency noise. It may also be noticed that for $\tau = T/2$, AVAR gives a result 5000 times too low.

6. Conclusion

By using the Total approach on different variances, the equivalent degrees of freedom of the estimates at and near the longest averaging time $\tau = T/2$ (T is the total duration of the data sequence) increases by a factor of between 2 and 4 relative to the corresponding classical variance. The confidence interval over each variance estimate using the Total approach is then reduced from 70% to 50% relative to the classical variance estimate. Thus, the noise levels are more precisely determined (and this could be crucial, see for example ref. [1]) and the use of a Total variance over a sequence of duration T may be equivalent to the use of the corresponding classical variance over a sequence of duration $2T$.

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