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STATISTICAL ASPECTS OF CLOCK ERRORS

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IMPORTANT NOTICE

STATISTICAL ASPECTS OF CLOCK ERRORS

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Abstract

A relationship between the variance of time indicated by an ensemble of clocks and the spectral density of the phase or frequency fluctuations of the periodic elements of the clocks is developed. Tables which apply this relationship to obtain explicit expressions for time variances corresponding to several assumed spectral densities of phase and frequency are presented. The particularly plausible case in which frequency fluctuations exhibit the same spectral behavior as white noise filtered by a single stage RC filter is examined in some detail. This spectral behavior leads to results that are shown to be closely related to the one-dimensional random walk problem. For the low-pass filter type of spectral behavior the rms relative time fluctuation is shown to decrease substantially below the rms relative frequency fluctuation as running time increases.

1. Introduction

The technology of frequency control and frequency standards has made possible the realization of oscillators having rms frequency deviations of one part in 10^{10}. In popular exposition one often hears that a clock constructed from such an oscillator could be expected to gain or lose not much more than a second in 10^{10} seconds or 300 years of running time. Attempts to find the exact relationship between oscillator stability and clock accuracy led to some interesting results. Various assumptions were made regarding the spectral density of the phase and frequency variance of an oscillator and the corresponding formulas for the rms time error of a clock constructed from the oscillator having the assumed statistical properties were obtained.

2. Theory Relating rms Time Error of a Clock to the Statistical Properties of the Oscillator

The details of clock construction vary considerably from one type of clock to another, but modern clocks in contrast to the hour glass or water clocks all have a periodic element, or oscillator, which may be mechanical or electrical. In addition clocks contain an integrating system or counter which displays a number proportional to the number of oscillations or the phase of the oscillator. The elapsed time indicated by a clock is proportional to the elapsed phase shown by the integrating device, the constant of proportionality depending on the unit of time, the frequency, and the unit of phase, i.e., complete oscillations, or radians. Thus

$$T = a\phi \tag{1}$$

where T is the elapsed time indicated by the clock,

φ is the elapsed phase in radians,

and a is the constant of proportionality.

. The concept of an rms deviation of time indicated by a clock
can be investigated by considering an ensemble of identical clocks
each of which consists of an oscillator and a counter that shows the
number of complete cycles executed by the oscillator.   The clocks
are set at the same instant which can be referred to as t = 0.   At
t = 0 all the counters read zero.   As a result of fluctuations, or
noise, the instantaneous frequencies of the oscillators in the ensemble
do not agree at any instant t; but because all the oscillators do have
the same construction, the average frequencies approach a common
limit ω as the averaging time increases.   The reciprocal of this
common limit is the a of equation 1.   At time t the instantaneous

frequency of the j-th oscillator of the ensemble is $\omega + \phi'_j(t)$, the

phase is $\omega t + \phi_j(t) - \phi_j(0)$, the cycle counter of the j-th clock

shows $N_j = \dfrac{1}{2\pi}\left[(\omega t + \phi_j(t) - \phi_j(0)\right] \pm X$ where $|X| \le 1$ and the

time $T_j(t)$ indicated by the j-th clock at time t is

$a\left[\omega t + \phi_j(t) - \phi_j(0) \pm 2\pi X\right]$.   Here φ and φ' the derivative of

φ are the random variables and the set $\phi_j(t)$ constitute a stationary
random process.   The counting error represented by X will be
neglected in order to study the rms deviation of indicated time that
results solely from the random process $\phi_j(t)$.   For the j-th clock
the indicated time T_j deviates from time t by an amount

$$T_j - t = a\left[\phi_j(t) - \phi_j(0)\right] = a\,\Delta\phi_j \qquad (2)$$

$$(T_j - t)^2 = a^2 \Delta\phi_j^{\,2} \tag{3}$$

By stationarity one may either average over j or consider various starting times to obtain the mean square deviation of indicated time. The mean square phase discrepancy $\overline{\Delta\phi^2}$ may be related to the autocorrelation function $R_\phi(t)$ of the random variable ϕ very easily.

$$\Delta\phi^2 = \text{average over } t' \text{ of } \left[\phi(t' + t) - \phi(t') \right]^2$$

$$= \overline{\phi(t' + t)^2} - \overline{2\phi(t' + t)\,\phi(t')} + \overline{\phi(t')^2}$$

$$= 2\overline{\phi^2} - 2R_\phi(t) \ . \tag{4}$$

By the Wiener-Khintchine theorem the autocorrelation function $R_\phi(t)$ can be expressed in terms of the spectral density of the variance of phase fluctuations $G_\phi(f)$ or the spectral density of the variance of frequency fluctuations $G_{\phi'}(f)$. Thus

$$R_\phi(t) = \int_{f=o}^{f=\infty} G_\phi(f) \cos(2\pi t f)\,df \tag{5}$$

or

$$R_\phi(t) = \int_{f=o}^{f=\infty} G_{\phi'}(f) \frac{\cos(2\pi t f)}{4\pi^2 f^2}\,df \tag{6}$$

since

$$4\pi^2 f^2 G_\phi(f) = G_{\phi'}(f) \ . \tag{7}$$

Relations 3, 4, and 5 can be combined to obtain

$$\overline{\left[T(t) - t \right]^2} = 2a^2 \int_{f=0}^{f=\infty} G_\phi(f)(1 - \cos 2\pi t f)\, df$$

$$= 4a^2 \int_{f=0}^{f=\infty} G_\phi(f) \sin^2 \pi t f\, df \tag{8}$$

since

$$\overline{\phi^2} = \int_{f=0}^{f=\infty} G_\phi(f)\, df \quad . \tag{9}$$

By using (7) one can write

$$\overline{\left[T(t) - t \right]^2} = \frac{a^2}{\pi^2} \int_{f=0}^{f=\infty} \frac{G_{\phi'}(f) \sin^2 \pi t f\, df}{f^2} \quad . \tag{10}$$

Equations 8 and 10 were used to calculate the mean square time error of clocks having oscillators with various assumed $G_\phi(f)$ or $G_{\phi'}(f)$. The mean square angular frequency fluctuation of an oscillator having $G_{\phi'}(f)$ as its spectral density of frequency variance is given by equation 11.

$$\overline{(\phi')^2} = \int_{f=0}^{f=\infty} G_{\phi'}(f)\, df \tag{11}$$

For certain spectral densities of phase or frequency fluctuations the integral in (10) or (8) will exist even though the integrals in (5), (6), (9), or (11) fail to converge. These difficulties arise from assumed spectral densities which do not vanish sufficiently rapidly at $f = 0$ or $f = \infty$. For instance a white spectral density of

frequency variance, admittedly an abstraction from physical reality, is a spectral density for which equation 10 is meaningful despite the lack of convergence of equations 6 and 9. To circumvent the difficulty in the derivation one can introduce a spectral density which is white between some lower frequency f_1 and some upper frequency f_2, and is zero outside of this range. For this modified white spectrum the derivation of equation 10 is valid for any $f_1 > 0$ and $f_2 < \infty$. After the integration in (10) is performed one can then let $f_1 \to 0$ and $f_2 \to \infty$ to obtain a relationship that could be compared with experimental results.

3. Results

It is convenient to analyse the behavior of a fluctuating oscillator by considering a generalized noise source, which may be electrical, mechanical, thermal, etc., followed by a filter whose output modulates an otherwise constant frequency oscillator. Tables I and II were obtained by assuming noise source characteristics resulting in the spectral density of phase variance, Table I, or frequency variance, Table II, shown in Column 4. The frequency independent factor h contains the intensity of the primary source, the transfer characteristics of the filter, and the proportionality factor relating instantaneous filter output to oscillator phase or frequency. Column 5 gives the rms frequency deviation obtained from equations 11 and 7 in Table I and from equation 11 alone in Table II, Column 6 gives an exact expression for $\overline{(T - t)^2}$, the mean square time error at any time t, Column 7 gives an approximate expression for $\overline{(T - t)^2}$ valid for small t,

TABLE I

Mean Square Time Errors for Various Spectral Densities of Phase Variance

Case	Spectral density of primary noise source	Filter characteristics	Spectral density of phase variance, $G_\phi(f)$	rms frequency deviation, $\overline{\dot{\phi}^2}$	Exact expression for mean square time deviation, $\overline{(T-t)^2}$	Approximate expression for mean square time deviation valid for small t	Approximate expression for mean square time deviation valid for large t
1	White noise	Ideal band pass	$= h, \quad F_1 \leq f \leq F_2$ $= 0, \quad f > F_2 \text{ or } f < F_1$	$\frac{4h\pi^2}{3}\left(F_2^3 - F_1^3\right)$	$2a^2hF_2\left(1 - \frac{\sin 2\pi F_2 t}{2\pi F_2 t}\right)$ $- 2a^2hF_1\left(1 - \frac{\sin 2\pi F_1 t}{2\pi F_1 t}\right)$	$\frac{4}{3}a^2h\pi^2(F_2^3 - F_1^3)t^2$	$2a^2h(F_2 - F_1)$
2	" "	Ideal low pass filter below F_2	$= h, \quad f \leq F_2$ $= 0, \quad f > F_2$	$\frac{4h\pi^2}{3}F_2^3$	$2a^2hF_2\left(1 - \frac{\sin 2\pi F_2 t}{2\pi F_2 t}\right)$	$\frac{4a^2}{3}h\pi^2 F_2^3 t^2$	$2a^2hF_2$
3	" "	Low pass RC filter, $\tau = RC$	$\frac{h}{1 + \tau^2\omega^2}$		$\frac{a^2h}{2\tau}\left(1 - e^{-t/\tau}\right)$	$\frac{a^2h}{2\tau}t - \frac{h}{4\tau^3}t^2$	$\frac{a^2h}{2\tau}$
4	" "	High pass filter $\tau = RC$	$\frac{h\tau^2\omega^2}{1 + \tau^2\omega^2}$		$\frac{2a^2h}{\pi\tau}\int_{2\pi F_1\tau}^{2\pi F_2\tau} \frac{x^2 \sin^2\frac{t}{2\tau}x\, dx}{1 + x^2}$		
5	" "	Parallel resonant circuit	$\frac{h\left(\frac{\omega_0 R}{Q}\right)^2\omega^2}{(\omega_0^2 - \omega^2)^2 + \left(\frac{\omega_0}{Q}\right)^2\omega^2}$		$\frac{a^2h\omega_0 R^2}{2Q}\left(1 - e^{-\frac{\omega_0 t}{2Q}}\cos\left(1 - \frac{1}{4Q^2}\right)^{\frac{1}{2}}\omega_0 t\right.$ $\left. + \frac{e^{-\frac{\omega_0 t}{2Q}}\sin\left(1 - \frac{1}{4Q^2}\right)^{\frac{1}{2}}\omega_0 t}{2Q\left(1 - \frac{1}{4Q^2}\right)}\right)$	$\frac{a^2h\omega_0^2 R^2}{2Q^2}t$	$\frac{a^2h\omega_0 R^2}{2Q}$
6	Flicker noise	Ideal band pass filter between F_1 and F_2	$\frac{h}{f}, \quad F_1 \leq f \leq F_2$ $0, \quad f > F_2 \text{ or } f < F_1$	$h\pi(F_2^2 - F_1^2)$	$\frac{a^2h}{\pi}\left[\ln\frac{F_2}{F_1} + \text{Ci}(2\pi t F_1) - \text{Ci}(2\pi t F_2)\right]$	$a^2h\pi(F_2^2 - F_1^2)t^2$	$\frac{a^2h}{\pi}\ln\frac{F_2}{F_1}$
7	" "	Ideal low pass filter below F_2	$\frac{h}{f}, \quad f \leq F_2$ $0, \quad f > F_2$	$h\pi F_2^2$	$\frac{a^2h}{\pi}\left[\ln 2\pi F_2 t + \text{Ci}(2\pi F_2 t)\right]$	$a^2h\pi F_2^2 t^2$	$\frac{a^2h}{\pi}\ln 2\pi F_2 t$
8	Flicker noise between F_1 and F_2, zero outside	Low pass RC filter, $\tau = RC$	$\frac{h}{\omega(1 + \tau^2\omega^2)}, \quad F_1 < f < F_2$ $0, \quad f < F_1, f > F_2$	$\frac{h}{4\pi^2} \times$ $\left(\ln\frac{4\pi^2 F_2^2\tau^2 + 1}{4\pi^2 F_1^2\tau^2 + 1}\right)$	$\frac{2a^2h}{\pi}\int_{2\pi\tau F_1}^{2\pi\tau F_2} \frac{\sin^2\frac{t}{2\tau}x\, dx}{x(1 + x^2)}$		
9	Flicker noise between F_1 and F_2, zero outside	High pass RC filter, $\tau = RC$	$\frac{h\tau^2\omega^2}{1 + \tau^2\omega^2}$	$\frac{h}{4\pi}(F_2^2 - F_1^2) - \frac{h}{4\pi^2} \times$ $\ln\left(\frac{4\pi^2 F_2^2\tau^2 + 1}{4\pi^2 F_1^2\tau^2 + 1}\right)$	$\frac{2a^2h}{\pi}\int_{2\pi\tau F_1}^{2\pi\tau F_2} \frac{x\sin^2\frac{t}{2\tau}x\, dx}{(1 + x^2)}$		

TABLE II
Mean Square Time Errors for Various Spectral Densities of Frequency Variance

Case	Spectral density of primary noise source	Filter characteristics	Spectral density of frequency variance, $G_\phi(f)$	rms frequency deviation, $\overline{e'^2}$	Exact expression for mean square time deviation, $\overline{(T-t)^2}$	Approximate expression for mean square time deviation valid for small t	Approximate expression for mean square time deviation valid for large t
1	White noise	Ideal band pass filter between F_1 and F_2	$= h, \quad F_1 \leq f \leq F_2$ $= 0, \quad f < F_1, f > F_2$	$(F_2 - F_1)h$	$\frac{a^2 h}{2\pi^2}\left[\frac{1}{F_1} - \frac{1}{F_2} - \frac{a^2 h t}{\pi}\left(\frac{\cos 2\pi F_1 t}{2\pi F_1 t} - \frac{\cos 2\pi F_2 t}{2\pi F_2 t} - \int_{2\pi F_1 t}^{2\pi F_2 t}\frac{\sin x}{x}dx\right)\right]$		
2	" "	Ideal low pass filter below F_2	$= h, \quad f \leq F_2$ $= 0, \quad f > F_2$	$F_2 h$	$\frac{-a^2 h}{2\pi^2 F_2} + \frac{a^2 h}{2\pi^2 F_2}\cos 2\pi F_2 t + \frac{a^2 h t}{\pi}Si(2\pi F_2 t)$	$2a^2 h F_2 t^2$	$\frac{2a^2 h t}{2} - \frac{a^2 h}{2F_2\pi^2}$
3	" "	Ideal high pass filter above F_1	$= h, \quad f \geq F_1$ $= 0, \quad f < F_1$		$\frac{a^2 h}{2\pi^2}\left[\frac{1}{F_1} - \frac{\cos 2\pi F_1 t}{F_1} + 2\pi t\int_{2\pi F_1 t}^{\infty}\frac{\sin x}{x}dx\right]$	$\frac{a^2 h t}{2} - a^2 h F_1 t^2$	$\frac{a^2 h}{2\pi^2 F_1}$
4	" "	Without filter	h		$\frac{a^2 h}{2}t$	$\frac{a^2 h}{2}t$	$\frac{a^2 h}{2}t$
5	" "	Low pass RC filter $\tau = RC$	$\frac{h}{1 + \tau^2\omega^2}$	$\frac{h}{4\tau}$	$\frac{a^2 h}{2}\tau(e^{-t/\tau} + \frac{t}{\tau} - 1)$	$\frac{a^2 h}{4\tau}t^2$	$\frac{a^2 h t}{2} - \frac{a^2 h\tau}{2}$
6	" "	High pass RC filter, $\tau = RC$	$\frac{h\tau^2\omega^2}{1 + \tau^2\omega^2}$		$\frac{a^2 h\tau}{2}(1 - e^{-t/\tau})$	$\frac{a^2 h t}{2}$	$\frac{a^2 h\tau}{2}$
7	" "	Parallel resonant Circuit	$\frac{h\left(\frac{\omega_o R}{Q}\right)^2\omega^2}{(\omega_o^2 - \omega^2)^2 + \left(\frac{\omega_o}{Q}\right)^2\omega^2}$	$\frac{h\omega_o R^2}{4Q}$	$\frac{a^2 h R^2}{2Q\omega_o}\left[1 - e^{-\frac{\omega_o t}{2Q}}\cos\left(1 - \frac{1}{4Q^2}\right)^{\frac{1}{2}}\omega_o t - \frac{e^{-\frac{\omega_o t}{2Q}}\sin\left(1 - \frac{1}{4Q^2}\right)^{\frac{1}{2}}\omega_o t}{2Q\left(1 - \frac{1}{4Q^2}\right)^{\frac{1}{2}}}\right]$	$\frac{a^2 h\omega_o R^2 t^2}{4Q}$	$\frac{hR^2}{2Q\omega_o}$
8	Flicker noise	Ideal band pass filter between F_1 and F_2	$\frac{h}{\omega}, \quad F_1 \leq f \leq F_2$ $0, \quad f < F_1, f > F_2$	$\frac{h}{2\pi}\ln\frac{F_2}{F_1}$	$\frac{a^2\pi t^2}{2\pi}\left(-Ci(2\pi t F_1) + \frac{\sin 2\pi t F_1}{2\pi t F_1} + \frac{1}{2\pi t F_1} - \frac{\cos 2\pi t F_1}{(2\pi t F_1)^2}\right)$	$\frac{a^2 h t^2}{2\pi}(1 - \ln\gamma 2\pi t F_1)$ $\ln\gamma = .577216$	
9	Flicker noise between F_1 and F_2, zero noise outside	Low pass RC filter $\tau = RC$	$\frac{h}{\omega(1 + \tau^2\omega^2)}$	$\frac{h}{4\pi}\ln\frac{(4\pi^2 F_1^2\tau^2 + 1)F_2^2}{(4\pi^2 F_2^2\tau^2 + 1)F_1^2}$	$\frac{2a^2 h}{\pi}\tau^2\int_{2\pi\tau F_1}^{2\pi\tau F_2}\frac{\sin^2\frac{t}{2\tau}x\,dx}{x^3(1 + x^2)}$		
10	Flicker noise between F_1 and F_2, zero noise outside	High pass RC filter $\tau = RC$	$\frac{h\tau^2\omega^2}{(1 + \tau^2\omega^2)}$	$\frac{h}{4\pi}\ln\frac{4\pi^2 F_2^2\tau^2 + 1}{4\pi^2 F_1^2\tau^2 + 1}$	$\frac{2a^2 h}{\pi}\tau^2\int_{2\pi\tau F_1}^{2\pi\tau F_2}\frac{\sin^2\frac{t}{2\tau}x\,dx}{x(1 + x^2)}$		

and Column 8 gives an approximate expression for the mean square
time error valid for large t .

4. Discussion and Interpretation

The simplest assumption one could make regarding the
spectral density of the frequency fluctuations, namely a white spectrum,
leads to a mean square time error of the indicated time T that in-
creases in direct proportion to the time that has elapsed since the
clock was set.   Figure 1 illustrates this behavior for an ensemble of
clocks.   The abcissa T is indicated time, the ordinate y(T) is the
distribution function defined so that y(T)dT is the number of clocks
that indicate a time within a dT neighborhood of T, and the param-
eter t is the physical time that has elapsed since the clocks were set
at t = 0.   The area under each curve is constant and equals the total
number of clocks N in the ensemble.   The variance of each distribu-
tion is proportional to t.   At t = 0 the distribution is $N\delta(t)$,
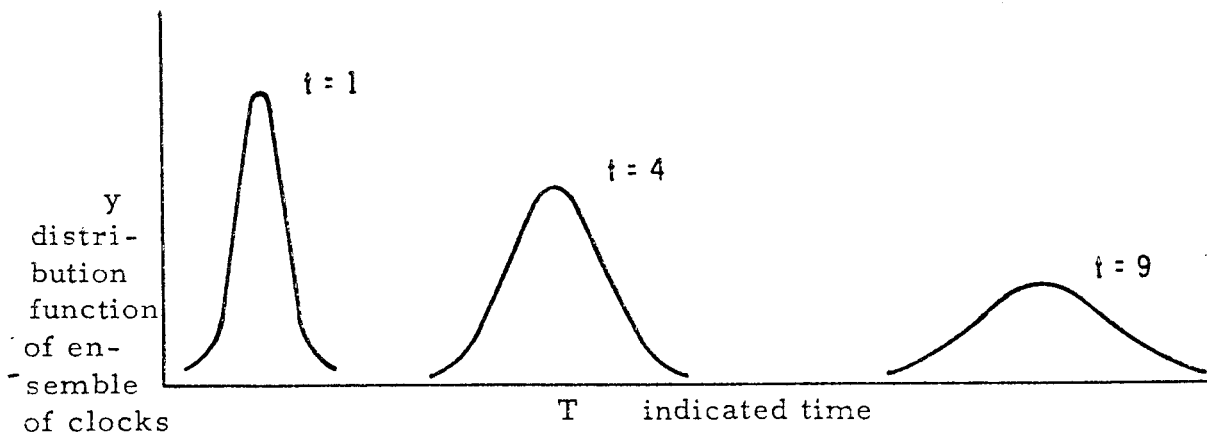where $\delta(t)$ is the delta function.

Figure 1

Diffusion of Indicated Time shown by an Ensemble of Identical Clocks.

A direct increase in variance of the indicated time with physical time can be given an alternate explanation which is somewhat less formal than the explanation given by equation 10. This alternate explanation tells us that the distribution function y of Figure 1 is Gaussian. The starting point is the assumption that the increase in indicated time shown by a clock during a time interval t/n is completely independent of the increase in indicated time shown by the clock during any previous time interval of duration t/n. In this discussion n is an integer which is made to increase without limit. The assumption implies that the autocorrelation function of the frequency fluctuations is zero for any argument greater than zero; consequently the spectrum of frequency fluctuations is white so that the starting assumption is equivalent to the assumption used in the 4th case considered in Table II. The time T indicated after counting for a time t, which for argumentative purposes is divided into subintervals of duration t/n, is the sum of the increases in indicated time during each of the n subintervals into which t is divided. That is

$$T(t) = T_1(\frac{t}{n}) + T_2(\frac{t}{n}) + T_3(\frac{t}{n}) + T_4(\frac{t}{n}) \ldots T_n(\frac{t}{n}) \qquad (12)$$

where T is the indicated time t units of time after the clock was set.

$T_j(\frac{t}{n})$ is the increase in indicated time during the j-th interval of duration t/n into which t was decomposed.

By the central limit theorem of probability the variance of T is n times the variance of $T_j(\frac{t}{n})$ and the T's shown by an ensemble of clocks must be distributed in a Gaussian fashion. Thus the mean square error of indicated time would be directly proportional to the time that has elapsed since the clock was set.

In attempting to explain the behavior of actual clocks it would seem that case 5 of Table II would be more realistic than case 4 which involves a white spectrum. Case 5 would arise if Johnson noise or shot noise passed through a low pass filter and then linearly modulated the frequency of the oscillator. A second mechanism for the assumed spectral density would require that the frequency determining element of the oscillator be sensitive to temperature, that it be in thermal contact with a thermal capacitance (i.e., a body with non zero mass and non zero specific heat) and that the thermal capacitance be connected by a finite thermal conductance to an ambient which undergoes temperature fluctuations that show a white spectrum. Still another mechanism would involve a microphonic frequency determining element excited by mechanically-filtered white acoustic waves. In any actual oscillator frequency fluctuations can not occur at infinite rates, so that the introduction of a filtering time constant τ is very plausible. For the mechanism involving modulation of the frequency, by a voltage obtained by filtering white electrical noise with a low pass filter τ is given by RC. Irrespective of the mechanism that produces the spectral density of case 5 Table II whether it be electrical, thermal, or acoustic, the parameter τ can be characterized by the following statement:

    a. $G_{\phi'}(\tau^{-1}) = \frac{1}{2} G_{\phi'}(0)$

    b. $R_{\phi'}(\tau) = e^{-1} R_{\phi'}(0)$

        here $R_{\phi'}$ is the autocorrelation function of the angular frequency fluctuations.

    c. The frequency deviation of the oscillator at any instant is almost independent of the frequency deviation at earlier

epochs separated from the instant by times greater than 10τ.

d. The mean square change in frequency during intervals less than $.1\tau$ is small in comparison to the mean square frequency deviation.

It is interesting to compare the tabulated mean square time error $\dfrac{a^2 h \tau}{2}\left(e^{-t/\tau} + \dfrac{t}{\tau} - 1\right)$ with the mean square time error of a clock whose frequency is determined by the flip of a coin at the beginning of successive intervals of duration τ. The frequency is to be one of the two values ω_1, or $\omega_2 > \omega_1$. During the intervals between flips the clock will either gain $a\left(\omega_2 - \dfrac{\omega_2 - \omega_1}{2}\right)\tau$ units of time or lose $a\left(\dfrac{\omega_2 + \omega_1}{2} - \omega_1\right)\tau$ units, that is, the clock will gain or lose $\dfrac{a\tau}{2}\left(\omega_2 - \omega_1\right)$ seconds during each interval τ. After t/τ flips or t seconds, the probability W of accumulating a time error ΔT may be obtained by using the results of random walk theory.

$$W(\Delta T, t) = \left[C^{t/\tau}_{\frac{t}{2\tau} + \frac{\Delta T}{a(\omega_2 - \omega_1)\tau}} \right] (\tfrac{1}{2})^{t/\tau} \quad (13)$$

here C is the binomial coefficient.

$$\overline{(\Delta T)}^2 = \frac{a^2 t \tau}{4}\left(\omega_2 - \omega_1\right)^2 \quad (14)$$

or

$$\overline{(\Delta T)^2} = a^2 t \tau \quad \text{(mean square frequency deviation)}. \qquad (15)$$

For large t case 5 of Table II gives

$$\overline{(\Delta T)^2} = 2 a^2 t \tau \left(\frac{h}{4\tau}\right) = 2 a^2 t \tau \quad \text{(mean square frequency deviation)}.$$

The expression given in Table II for the mean square time error can be put into the following form by using the rms relative time deviation and the rms relative frequency deviation.

$$\left[\overline{\left(\frac{\Delta T}{t}\right)^2}\right]^{\frac{1}{2}} = \left[\overline{\left(\frac{\phi'}{\omega}\right)^2}\right]^{\frac{1}{2}} X(t/\tau) \qquad (16)$$

where the left hand term is the rms relative time deviation,

the first factor on the right is the rms relative frequency deviation, and $X(t/\tau)$ is given by equation 17.

$$X(t/\tau) = 2^{\frac{1}{2}} (t/\tau)^{-1} (e^{-t/\tau} + t/\tau - 1)^{\frac{1}{2}} \qquad (17)$$

$$X = \left(\frac{2\tau}{t}\right)^{\frac{1}{2}} \quad \text{for} \quad t \gg \tau$$

$$X = 1 - \frac{1}{2} t/\tau \quad \text{for} \quad t \ll \tau$$

Values of X are given in Table III. The smoothing factor X, which is always less than 1, tells the effectiveness of the integrating process in balancing out periods during which the clock runs too rapidly with periods during which the clock runs too slowly. According to

equation 16 a clock with $\tau = 1$ sec and an rms relative frequency deviation of 10^{-10} will 10^{10} sec ($\simeq 300$ years) after being set have an rms relative time error of 1.414 parts in 10^{15} or an rms time error of 14.14 microseconds. This calculation neglects all sources of error other than those which result from oscillator instability and supposes that the spectral density of the frequency fluctuations is of the form given in case 5 in Table II and that the a of equation 1 is known precisely.

TABLE III

Values of the smoothing function X for various values of t/τ

Ratio of elapsed time to life time of deviation t/τ	Smoothing function X
0.0	1.00
.2	.97
1.0	.86
2.0	.75
4.0	.61
10.0	.42
100.	.14
1000.	.04
10000.	.01