

contributions

Propagation of a Ground Wave Pulse Around a Finitely Conducting Spherical Earth from a Damped Sinusoidal Source Current*

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Summary—The form of the transient electromagnetic ground wave which has been propagated over a finitely conducting spherical earth from a source current dipole can be calculated by a direct quadrature evaluation of the Fourier integral. The method is illustrated in this paper by a calculation of the transient field radiated by the particular case of the damped sinusoidal source current dipole. At short distances from the source, the earth was assumed to be a plane and the displacement currents in the earth were neglected. The pulse was then calculated by a direct evaluation of the Fourier integral and the integration was verified by special operational methods (inverse Laplace transformation). The form of this pulse was then predicted at great distance from the source by a direct evaluation of the Fourier integral in which the displacement currents in the earth and the earth's curvature were introduced into the Fourier transform. The form of the transient signal was found to be dispersed by the propagation medium. The most noteworthy attribute of this dispersion is a stretching of the period of the wave so that the form of the source is somewhat obscured by the filtering action of the medium.

INTRODUCTION

IN RECENT YEARS, the interest in the change with distance in the form or shape of various propagated radio-frequency transients, especially at low frequencies, can be attributed to the use of pulse tech-

niques in radio navigation systems and the interest in sferics which radiate from thunderstorms. The prediction of these propagated transients at various distances, employing idealized source models, constitutes the theoretical problem.

This paper introduces a vertically polarized point source current or Hertz¹ dipole, the amplitude of which varies in time as a damped sinusoid. The step function and the impulse function are limiting cases of this general type of source. More complicated sources can be simulated by superposition of this basic type.

THEORY

The propagated pulse, at a distance, d , and a time, t , which is described in this paper as a space-time function, $E(t, d)$, is built upon a source, $F_s(t)$, for which the transform, $f_s(\omega)$, may be written:

$$f_s(\omega) = \int_{-\infty}^{\infty} \exp(-i\omega t) F_s(t) dt, \quad (1)$$

where $\omega = 2\pi f$, f = frequency, cycles.

The transfer function which characterizes the propagation medium, $E(\omega, d)$, and the source integral (1) together describe a Fourier transform, $f(\omega, d)$,

$$f(\omega, d) = f_s(\omega) E(\omega, d). \quad (2)$$

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¹ H. Hertz, "Die Kräfte Elektrischer Schwingungen Behandelt Nach der Maxwell'schen Theorie," *Ann. Phys. Chem. (Leipzig)*, vol. 36, pp. 1-22; 1889.

The function, $E(\omega, d)$, can be interpreted as the amplitude, $|E(\omega, d)|$, and the phase lag, $\text{Arg } E(\omega, d)$, of a "continuous wave" signal, *i.e.*, a sinusoidal wave of constant amplitude, uninterrupted in time, and propagated around the surface of the earth from a Hertzian dipole source current. The product of such a function and the transform, $f_s(\omega)$, of the particular source under consideration, $F_s(t)$, (1), therefore describes the Fourier transform of a pulse. The propagated transient, $E(t, d)$, can then be formally represented in space and time as a Fourier integral,

$$E(t, d) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(i\omega t) f_s(\omega) E(\omega, d) d\omega. \quad (3)$$

Primary interest is concerned with the real part of the space-time function, $\text{Re } E(t, d)$, since this represents the instantaneous signal observed in an "infinite bandwidth" receiver. The amplitude envelope, $|E(t, d)|$, and the phase envelope, $\phi(t, d)$, where

$$E(t, d) = |E(t, d)| \exp\{i[\phi(t, d) - \omega t]\}, \quad (4)$$

are only of secondary interest since these waveforms would ordinarily be recovered at the output of the receiver with some sort of envelope detection.

The source currents, $F_s(t)$, employed in this paper may be represented in complex form for the cosine source,

$$\begin{aligned} \text{Re } F_s(t) &= \text{Re exp } (-\nu t) \\ &= \exp(-c_1 t) \cos \omega_c t, \quad (0 < t < \infty) \\ &= 0, \quad (t < 0) \end{aligned} \quad (5a)$$

or for the sine source,

$$\begin{aligned} -\text{Im } F_s(t) &= \text{Re } i \exp(-\nu t) \\ &= \exp(-c_1 t) \sin \omega_c t, \quad (0 < t < \infty) \\ &= 0, \quad (t < 0) \end{aligned} \quad (5b)$$

where

$$\nu = c_1 + i\omega_c. \quad (5c)$$

The transform, $E(\omega, d)$, which represents the propagation medium, can be described as an amplitude and phase transfer characteristic for a pulse as follows:

$$E(\omega, d) = |E(\omega, d)| \exp\left\{-i\left[\phi_c(\omega, d) - \frac{\pi}{2} + a\omega\right]\right\} \quad (6)$$

where

$$a = \frac{\eta_1 d}{c}, \quad (6a)$$

and where η_1 is the index of refraction of air at the surface of the earth ($\eta_1 \sim 1.000338$) and c is the speed of light [$c \sim 2.997925 (10^8)$ meters/second].

It is convenient to write

$$\begin{aligned} E(\omega, d) \exp(i\omega t) \\ = |E(\omega, d)| \exp\left\{i\left[\omega t' - \phi_c(\omega, d) + \frac{\pi}{2}\right]\right\} \end{aligned} \quad (7)$$

where the local time, t' , is

$$t' = t - a. \quad (8)$$

The primary propagation time, (6a), is thus included in the local time and $\phi_c(\omega, d)$ is the secondary phase or phase correction² resulting from the influence of the earth on the propagation mechanism. The time, $t' = 0$, thus describes the earliest time at which the pulse signal could be observed.

The neglect of the earth's curvature and the displacement currents in the earth at short distances make possible the employment of the operational calculus to give an immediate and somewhat useful solution to the problem. Employing the symbol, s , as used in the operational method,

$$s = i\omega, \quad (9)$$

the Laplace transform of the Norton³ surface wave⁴ is as follows:⁵

$$E(s, d) = C \left\{ f_0(s) + \frac{1}{a(s + \nu)} + \frac{1}{a^2 s(s + \nu)} \right\} \quad (10)$$

$$f_0(s, d) = \frac{s}{s + \nu} - \frac{s^2}{s + \nu} \sqrt{\pi\alpha} \exp(s^2\alpha) \text{erfc}(s\sqrt{\alpha}) \quad (11)$$

where

$$C = 2 \frac{I_0 l a^2}{4\pi\kappa d^3} = \frac{2(10^{-7})}{d} \quad (I_0 l = 1 \text{ ampere-meter, the dipole momentum}). \quad (12)$$

$$\kappa = \frac{\eta_1^2}{c^2 \mu_0}, \text{ a constant,}$$

² J. R. Johler, W. J. Kellar, and L. C. Walters, "Phase of the Low Radio-Frequency Ground Wave," NBS Circular 573, pp. 1-6; June 27, 1956.

³ K. A. Norton, "The propagation of radio waves over the surface of the earth and in the upper atmosphere," PROC. IRE, vol. 25, pp. 1203-1236; September, 1937. See especially (57), p. 1212. See also K. A. Norton, "The propagation of radio waves over the surface of the earth in the upper atmosphere," PROC. IRE, vol. 24, pp. 1367-1387; October, 1936. Also, K. A. Norton, "The physical reality of space and surface waves in the radiation field of radio antennas," PROC. IRE, vol. 25, pp. 1192-1202; September, 1937.

⁴ A. Sommerfeld, "Über Die Ausbreitung der Wellen in der Drahtlosen Telegraphie," Ann. Phys., vol. 28, pp. 665-736; March, 1909.

⁵ The complementary error function, $\text{erfc}(z)$ is defined,

$$\text{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} \exp(-u^2) du.$$

where, μ_0 is the permeability of space, $\mu_0 = 4\pi(10^{-7})$, and⁶

$$\alpha = \frac{\eta_1^3 d}{2\sigma\mu_0 c^3}, \quad (13)$$

in which σ is the conductivity of the earth ($\sigma = 0.005$ mho/meter for typical land).

The inverse Laplace transform of the Norton surface wave may then be written:⁷

$$E(t', d) = \nu C \exp(-\nu t') \left\{ -1 - \nu\sqrt{\pi\alpha} \exp(\nu^2\alpha) \cdot \left[\operatorname{erfc}(-\nu\sqrt{\alpha}) - \operatorname{erfc}\left(\frac{t'}{2\alpha} - \nu\sqrt{\alpha}\right) \right] + \frac{1}{\alpha\nu} - \frac{1}{a^2\nu^2} \right\} + \left[\frac{t'}{2\alpha} + \nu \right] C \exp\left[\frac{-t'^2}{4\alpha}\right] + \nu C \left[\frac{1}{a^2\nu^2} \right]. \quad (14)$$

The introduction of the displacement currents ($\epsilon_2 \neq 0$) and the earth's curvature, Fig. 1, complicates the func-

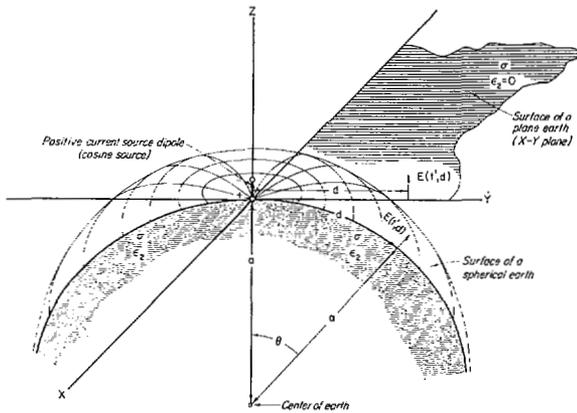


Fig. 1—Orientation of the source dipole and the earth with respect to coordinate systems.

tion $f_0(s)$. The transformation may be formally written as follows:

$$E(t', d) = F_0(t', d) + F_{i,e}(t', d) \quad (15)$$

$$F_0(t', d) = C\mathcal{L}^{-1}f_0(s). \quad (16)$$

The function, $F_0(t', d)$, is of primary importance in this paper since the contributions from the induction and electrostatic fields, $F_{i,e}(t', d)$, are quite minute at

⁶ The index of refraction, η_1 , of the earth's atmosphere is constant at the surface value with respect to altitude for purposes of the surface wave transformation at short distances. The effect of the vertical lapse of the index of refraction at great distances is introduced as the factor, β , (29), ($\beta \sim 0.75$).

⁷ J. R. Johler, "Transient radiofrequency ground waves over the surface of a finitely conducting plane earth," *J. Res. NBS*, vol. 60, pp. 281-285; April, 1958. See also, J. R. Johler, "Propagation of the radiofrequency ground wave transient over finitely conducting plane earth," *Geofis. pura e appl. (Milan)*, vol. 37, pp. 116-126; February, 1957.

great distances. The function, $F_0(t', d)$, may be re-written in the Fourier integral, (3), notation employing the variable ω instead of the operational symbol s ,

$$F_0(t', d) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(i\omega t') E(\omega, d) \cdot \int_{-\infty}^{\infty} F_s(t) \exp(-i\omega t) dt d\omega \quad (17)$$

or, with the aid of (5a) and (5b), and evaluating the inner integral of (17),

$$E(t', d) = \frac{1}{2\pi} \int_0^{\infty} |E(\omega, d)| \cdot \left\{ \frac{\cos \left[\omega t' - \phi_c + \tan^{-1} \frac{-(\omega_c + \omega)}{c_1} \right]}{\sqrt{c_1^2 + (\omega_c + \omega)^2}} + \frac{\cos \left[-\omega t' + \phi_c' + \tan^{-1} \frac{-(\omega_c - \omega)}{c_1} \right]}{\sqrt{c_1^2 + (\omega_c - \omega)^2}} + i \left[\frac{\sin \left[\omega t' - \phi_c' + \tan^{-1} \frac{-(\omega_c + \omega)}{c_1} \right]}{\sqrt{c_1^2 + (\omega_c + \omega)^2}} + \frac{\sin \left[-\omega t' + \phi_c' + \tan^{-1} \frac{-(\omega_c - \omega)}{c_1} \right]}{\sqrt{c_1^2 + (\omega_c - \omega)^2}} \right] \right\} d\omega, \quad (18)$$

where

$$\phi_c' = \phi_c - \frac{\pi}{2}. \quad (18a)$$

It immediately becomes obvious that the real part of the amplitude time function (18), $\operatorname{Re} E(t', d)$, is the desired function for the cosine source (5a), and the imaginary part, $-\operatorname{Im} E(t', d) = \operatorname{Re} [iE(t', d)]$, is the desired function for the sine source (5b). Thus the problem has been reduced to the evaluation of a real integral,

$$E(t', d) = \int_0^{\infty} F_i(\omega, d) d\omega = \operatorname{Re} \int_{-\infty}^{\infty} f(\omega, d) \exp(i\omega t) d\omega. \quad (19)$$

The transform of the pulse is asymmetrical as a result of the source function. The Fourier spectrum, $f_z(\omega, d)$, Fig. 2, can be determined for the transform, $f(\omega, d)$, the amplitude spectrum,

$$|f_z(\omega, d)| = |f(\omega, d) + f(-\omega, d)| \quad (\omega \geq 0), \quad (20)$$

and the phase spectrum,

$$\phi_z(\omega, d) = \operatorname{Arg} [f(\omega, d) + f(-\omega, d)], \quad (\omega \geq 0). \quad (20a)$$

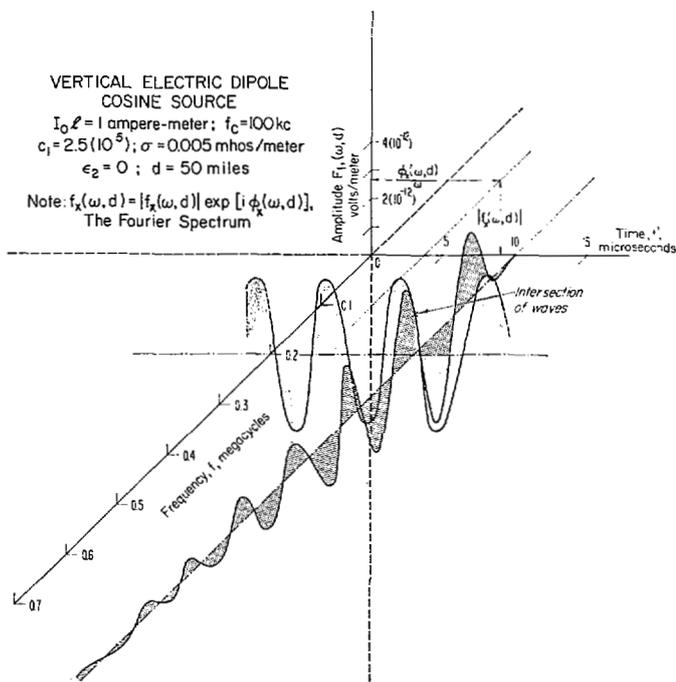


Fig. 2—Integrand of Fourier integral or image of pulse in the frequency-amplitude plane at small distance from the source, illustrating the relation of the integrand to the Fourier spectrum waves.

The integrand, $F_t(\omega, d)$, Figs. 2, 3, in the frequency-amplitude plane, is rather severely mutilated at great distances by the conduction and displacement currents in the earth and the effect of the earth's curvature. This is quite advantageous from a computational point of view, since the convergence of the integral at higher frequencies is rapid, and, as a matter of fact, analog methods such as a planimeter integration could be employed to evaluate the amplitude-time function, $E(t', d)$. However, numerical mastery of the problem is achieved by a reduction of the computation to a digital process.

The integral, $E(t', d)$, which involves the limits zero and infinity is rewritten as the sum of integrals with finite limits as follows:

$$\int_0^{\infty} F_t(\omega, d) d\omega = \int_0^{b_1} F_t(\omega, d) d\omega + \int_{b_1}^{b_2} F_t(\omega, d) d\omega + \int_{b_2}^{b_3} F_t(\omega, d) d\omega + \cdots, \quad (21)$$

where enough terms are taken so that any remainder error is small. Each finite integral is evaluated by Gaussian quadrature.^{8,9} The limits of each finite integral are somewhat arbitrary, but are chosen consistent with the required accuracy and the availability of Gaussian quadrature weights and abscissas.⁹ If the range of inte-

⁸ Z. Kopal, "Numerical Analysis," John Wiley and Sons, Inc., New York, N. Y., p. 367; 1955.

⁹ P. Davis and P. Rabinowitz, "Abcissas and weights for Gaussian quadratures of high order," *J. Res. NBS*, vol. 56, pp. 35-37; January, 1956.

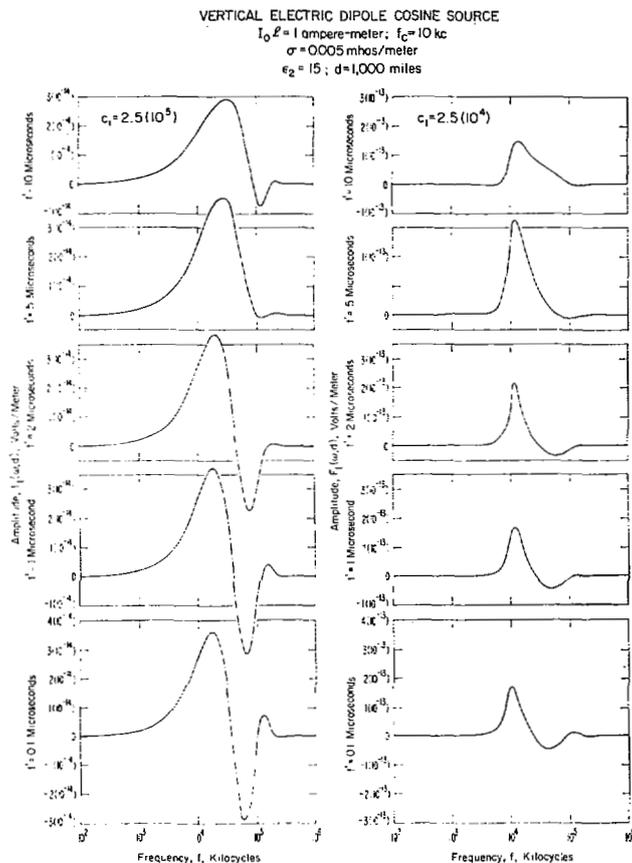


Fig. 3—Integrand of Fourier integral or image of pulse in the frequency-amplitude plane at great distance from the source.

gration is increased for a given integrand and accuracy specification, either more intervals or more Gaussian weights and abscissas are required. Thus, it is possible to express each finite integral as a sum:⁸

$$\int_{b_n}^{b_{n+1}} F_t(\omega, d) d\omega = \sum_{m=1}^M W_m F_t(\omega_m, d) + \epsilon(M), \quad (22)$$

where $\epsilon(M)$ is an error term which can, in general, be made arbitrarily small by increasing M , and where

$$m = 1, 2, 3, \cdots M. \quad (22a)$$

$$\omega_m = \frac{1}{2}[b_{n+1} - b_n]x_m + \frac{1}{2}[b_{n+1} + b_n]. \quad (22b)$$

The x_m 's are the Gaussian abscissas and M determines the number of values of $F_s(\omega, d)$ to be used in the quadrature. The Gaussian weights and abscissas can be determined from the following considerations:

$$\int_{-1}^1 f(x) dx = \sum_{m=1}^M H_m f(x_m) \quad (23)$$

$$W_m = \frac{1}{2}[b_{n+1} - b_n]H_m. \quad (24)$$

The x_m 's are the roots of the Legendre polynomial defined by

$$\frac{d^m}{dx^m} (x^2 - 1)^m = 2^m n! P_m(x) \quad (25)$$

$$\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= \frac{3}{2}x^2 - \frac{1}{2} \\
P_3(x) &= \frac{5}{2}x^3 - \frac{3}{2}x \\
P_4(x) &= \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8} \\
&\dots
\end{aligned} \tag{26}$$

Polynomials of higher degree are determined by use of the recursion formula.

$$(m+1)P_{m+1}(x) + mP_{m-1}(x) = (2m+1)xP_m(x). \tag{27}$$

Upon determination of the roots, the weight coefficients, H_m , of the corresponding quadrature formula are evaluated as follows:

$$H_m = \frac{2}{(1-x_m^2)[P_m'(x_m)]^2}. \tag{28}$$

The results of Davis and Rabinowitz⁹ (forty-eight Gaussian weights and abscissas) were employed in the quadrature.

The amplitude $|E(\omega, d)|$ and the phase correction, $\phi_c(\omega, d)$, for the ground wave were calculated by means of the convergent residue series (s series) of Watson,¹⁰ Bremmer,^{11,12} van der Pol¹¹ and Norton,¹³ employing the conventional time function, $\exp(-i\omega t)$ as follows:^{2,14}

$$\begin{aligned}
E^*(\omega, d) &= -i\omega C \left[2\pi\beta^{2/3}(k_1\alpha')^{1/3} \frac{d}{\alpha'} \right]^{1/2} \\
&\sum_{s=0}^{\infty} \frac{\exp \left\{ i \left[(k_1\alpha')^{1/3}\tau_s\beta^{2/3} \frac{d}{\alpha'} + \frac{\beta d}{2\alpha'} + \frac{\pi}{4} \right] \right\}}{\left[2\tau_s - \frac{1}{\delta_e^2} \right]} \tag{29}
\end{aligned}$$

in which

$$s = 1, 2, 3, \dots,$$

$$\alpha' = \text{the radius of the earth } [\alpha' \sim 6.36739(10^6) \text{ meters},$$

¹⁰ G. N. Watson, "The diffraction of electric waves by the earth and the transmission of electric waves round the earth," *Proc. Roy. Soc. (London)*, vol. 95, p. 83; October, 1918; p. 546; July, 1919.

¹¹ B. van der Pol and H. Bremmer, "The diffraction of electromagnetic waves from an electrical point source round a finitely conducting sphere, with applications to radiotelegraphy and the theory of the rainbow," *Phil. Mag.*, vol. 24, pt. I, p. 141, July, 1937; pt. II, p. 825, November, 1937; *Supp. J. Science*, vol. 25, p. 817; June, 1938.

¹² H. Bremmer, "Terrestrial Radio Waves; Theory of Propagation," Elsevier Publishing Co., New York, N. Y., pp. 11-50; 1949.

¹³ K. A. Norton, "The calculation of the ground wave field in intensity over a finitely conducting spherical earth," *Proc. IRE*, vol. 29, pp. 623-639; December, 1941.

¹⁴ J. R. Wait and H. H. Howe, "Amplitude and Phase Curves for Ground Wave Propagation in the Band 200 Cycles per Second to 500 Kilocycles," NBS Circular 574, pp. 1-17; May, 1956.

$$E^*(\omega, d) = |E(\omega, d)| \exp i \left[\phi_c - \frac{\pi}{2} \right], \tag{30}$$

$$k_1 = \frac{\omega}{c} \eta_1, \tag{31}$$

$$k_2^2 = \frac{\omega^2}{c^2} \left[\epsilon_2 + i \frac{\sigma\mu_0 c^2}{\omega} \right], \tag{32}$$

$$\delta_e = \frac{i \frac{k_2^2}{k_1^2} \beta^{1/3}}{(k_1\alpha')^{1/3} \left[\frac{k_2^2}{k_1^2} - 1 \right]^{1/2}}, \tag{33}$$

and τ_s comprises the roots (see Appendix) of Riccati's differential equation:¹⁵

$$\frac{d\delta_e}{d\tau_s} - 2\delta_e^2\tau_s + 1 = 0. \tag{34}$$

The amplitude and phase transfer characteristic of the ground wave is illustrated (Figs. 4, 5) in this paper for a conductivity, $\sigma = 0.005$ mho per meter, and a dielectric constant, $\epsilon_2 = 15$. It should be noted that at short distances, the induction and electrostatic fields enhance the amplitude and phase at low frequencies. The earth's curvature and the conduction and displacement currents in the earth enhance the phase and attenuate the signal rather severely at high frequencies and great distances. The convergence of the infinite integral (21) is primarily a result of the exponential high frequency attenuation (29),

$$\exp \left\{ -\text{Im} \left[(k_1\alpha')^{1/3}\tau_s\beta^{2/3} \frac{d}{\alpha'} \right] \right\}.$$

THE SOURCE

The cosine source used in this paper employs an abrupt initial current (Figs. 7, 9, 11). The radiated field therefore propagates an impulse type of function which is superposed upon the sinusoid. This may be explained quite simply for the case of an infinite conducting earth:

$$E(s) = 1 \quad (\nu = 0, \alpha = 0) \tag{35}$$

then

$$E(t', d) = \mathcal{L}^{-1}E(s) = \delta(t'). \tag{35a}$$

$\delta(t')$ is the Dirac impulse function which can be defined in relation to the step function, $u(t)$, as follows:

$$\int_0^t \delta(t) dt = u(t), \tag{36}$$

where

$$u(t) = 1, \quad t > 0 \tag{37}$$

$$u(t) = 0, \quad t < 0. \tag{37a}$$

¹⁵ Bremmer, *op. cit.*, p. 45.

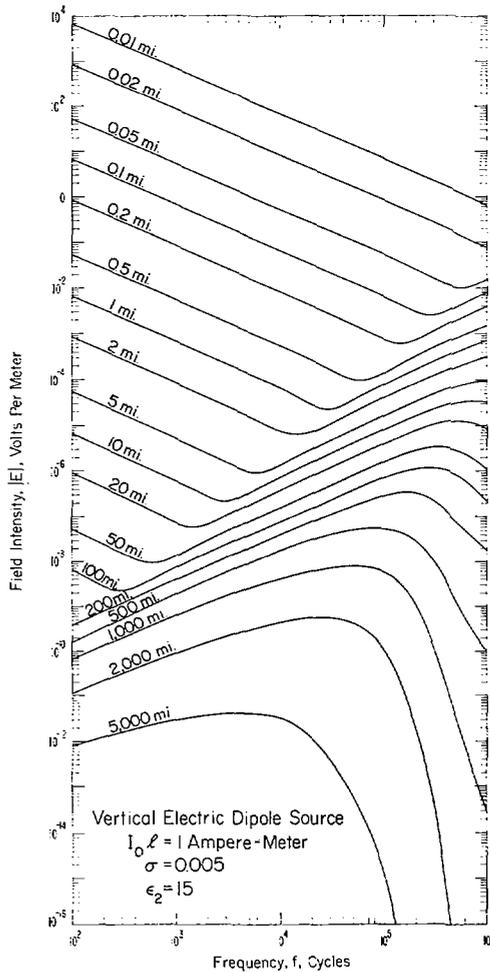


Fig. 4—Amplitude transfer characteristic of the ground wave.

The step function response of the ground wave at considerable distance from the source is a modified impulse function. Indeed, this is merely a limiting case of (14), $c_1 = 0$, $\nu = 0$, or,

$$E(t', d) = \frac{t'}{2\alpha} C \exp\left[\frac{-t'^2}{2\alpha}\right]. \quad (38)$$

This is consistent with the step function response derived by Wait.¹⁶ It should be noted that the amplitude and duration of the impulse becomes finite as a result of the introduction of the finite conductivity of the earth into the propagation mechanism.

The step function response resulting from the impulsive radiated field of the cosine current source is implicit in the calculations of this paper. The sine current source on the other hand does not imply such an additional radiation field.

¹⁶ J. R. Wait, "Transient fields of a vertical dipole over a homogeneous curved ground," *Can. J. Res.*, vol. 34, pp. 27-35; January, 1956. See also J. R. Wait, "The transient behaviour of the electromagnetic ground wave over a spherical earth," *IRE TRANS. ON ANTENNAS AND PROPAGATION*, vol. AP-5, pp. 198-202; April, 1957. Also J. R. Wait, "A note on the propagation of the transient ground wave," *Can. J. Phys.*, vol. 35, pp. 1146-1151; September, 1957.

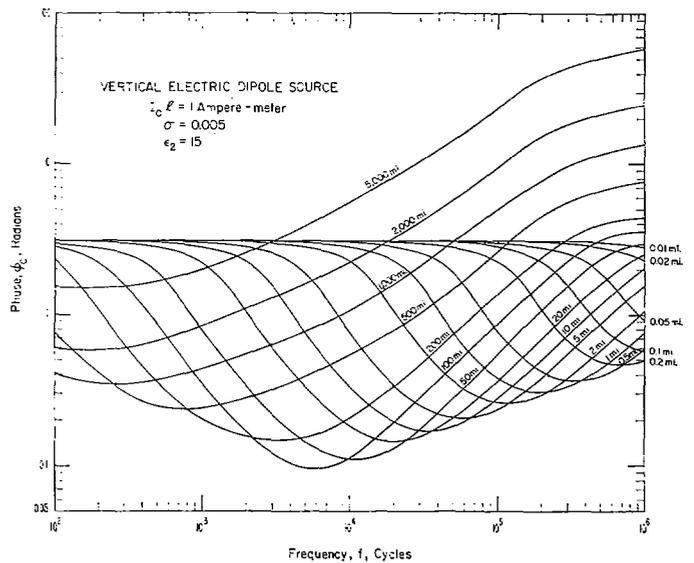


Fig. 5—Phase transfer characteristic of the ground wave.

It is quite possible to introduce the cosine current source less abruptly by redefining the source function, $F_s(t)$, as follows:

$$F_s(t) = \exp(-\nu t) - \exp(-\xi t) \quad (39)$$

where

$$\xi = c_2 + i\omega\epsilon. \quad (39a)$$

ν has been described previously (5c) and c_2 is assigned a large positive value ($c_2 \gg c_1$). In accord with the superposition principle, this merely involves the sum of two waves calculated as previously described (14), (18). The space-time function, $E'(t', d)$, may be written as follows (Figs. 9, 11):

$$E' = E'(t', d) = E_\eta(t', d) - E_\xi(t', d). \quad (40)$$

RESULTS OF THE COMPUTATION

The detailed structure of the transient for various combinations of characteristic frequency, f_c , and damping, c_1 , for both sine and cosine source functions was determined at great distance (Figs. 6-11, pp. 7-9). The transient waves with characteristic period, f_c , of 100 kc, and a damping, c_1 , of $2.5(10^5)$ (Figs. 6, 7), and at a distance, d (Fig. 1), of 50 miles were calculated by both the operational formula (14) and the direct evaluation of the Fourier integral (18). In both cases, the earth was assumed to be a plane, and the displacement currents in the earth were neglected. Close agreement was found between the two methods, in Figs. 6, 7.

The displacement currents in the earth and the effect of the earth's curvature were then introduced by the calculation of the theory (29) of Watson, Bremmer, van der Pol, and Norton (29) at great distances (Figs. 4-11). The dispersion of the pulse at great distance was evident from the calculation. The most noteworthy attribute of

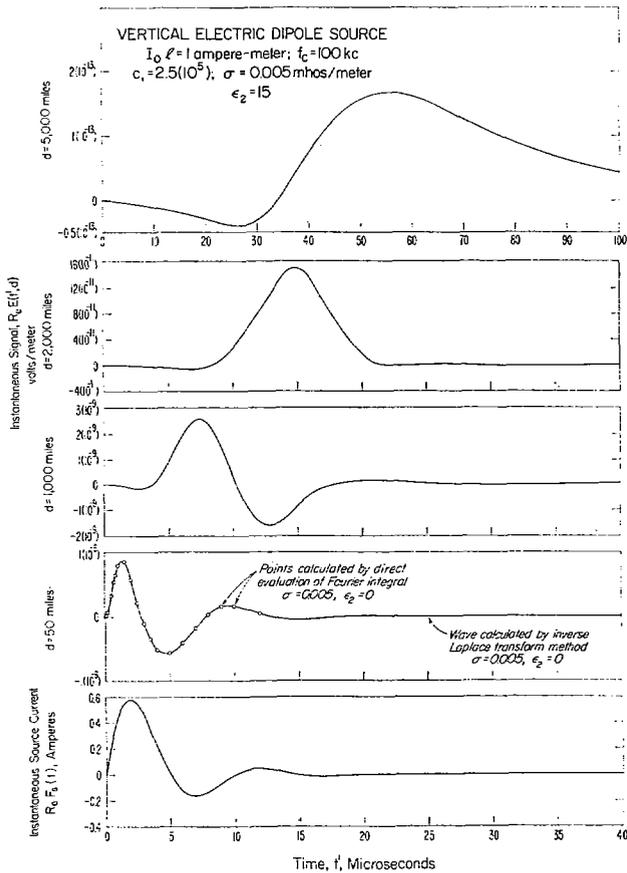


Fig. 6—Instantaneous signal and corresponding damped sine wave source current of a propagated ground wave pulse, illustrating close agreement at short distance between the form of the wave predicted by the operational formulas (inverse Laplace transform) and the form of the wave predicted by the direct evaluation of the Fourier integral.

this dispersion was an increase in the period of the pulse. The form of the source was somewhat obscured by the filtering action of the medium.

THE STEP, DELTA, AND GENERALIZED SOURCE FUNCTIONS

The special case of the step function response of the ground wave has already been discussed (38) in connection with the damped cosine source. This formulation (38) can be readily extended to the spherical earth theory with the following result:

$$E(t', d) = \frac{1}{\pi} \int_0^\infty |E(\omega, d)| \left\{ \frac{1}{\omega} \sin [\omega t' - \phi_c'] \right\} d\omega \quad (41)$$

The special case of an impulse source function or "delta function,"

$$F_s(t) = \delta(t) \quad (42)$$

reduces to the following simple formula:

$$E(t', d) = \frac{1}{\pi} \int_0^\infty |E(\omega, d)| \{ \cos [\omega t' - \phi_c'] \} d\omega \quad (43)$$

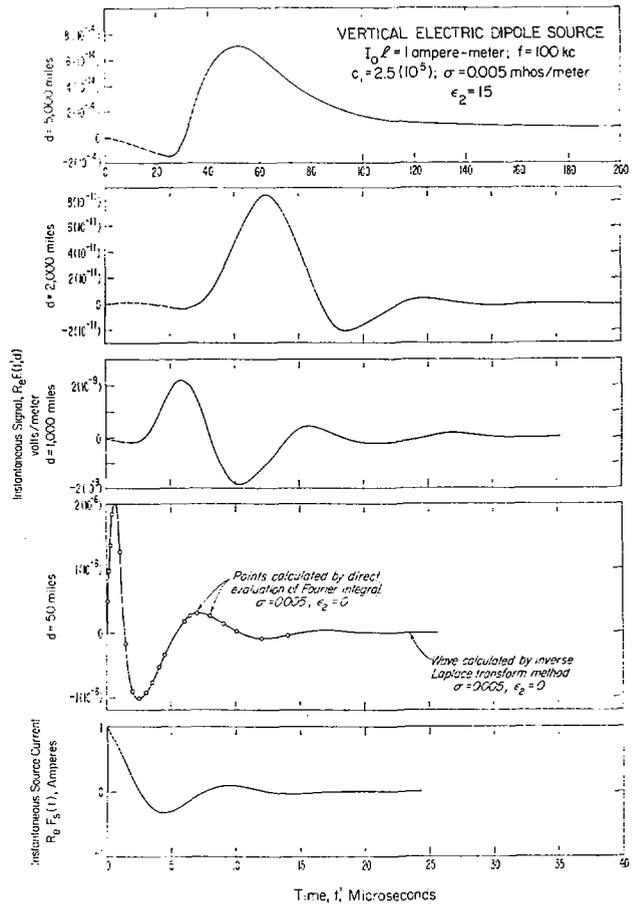


Fig. 7—Instantaneous signal and corresponding damped cosine wave source current of a propagated ground wave pulse, illustrating agreement at short distances between operational methods and the direct evaluation of the Fourier integral.

where each of the above formulas is readily evaluated by the previously described quadrature (22). The results can be checked against the operational methods based on the plane earth theory (14), which yields the following result for the impulse source function or delta function.¹⁶

$$E(t', d) = \left\{ \frac{1}{2\alpha} - \left[\frac{t'}{2\alpha} \right]^2 \right\} C \exp \left[\frac{-t'^2}{4\alpha} \right]. \quad (44)$$

During the past several years some very interesting papers have appeared which describe the propagation of a "delta" or impulse source function over the surface of the earth. Whereas it is questionable whether such a source function represents anything in nature, this is an important theoretical problem nonetheless, and presumably the more complicated source functions could be synthesized by application of the superposition principle.

The results of Levy and Keller¹⁷ account for the earth's curvature but neglect the displacement currents

¹⁷ B. R. Levy and J. B. Keller, "Propagation of electromagnetic pulses around the earth," IRE TRANS. ON ANTENNAS AND PROPAGATION, vol. AP-6, pp. 56-61; January, 1958.

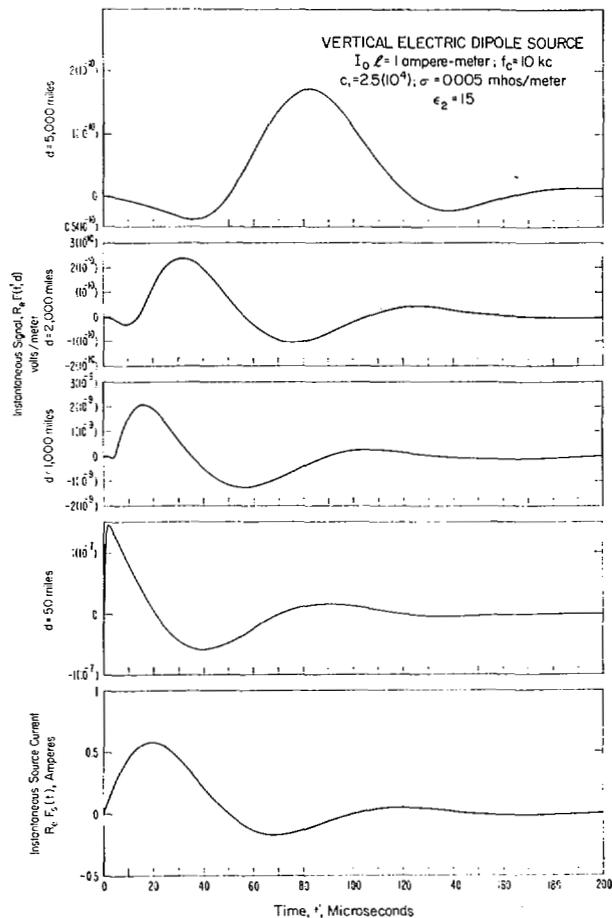


Fig. 8—Instantaneous signal and corresponding damped sine wave source current of a propagated ground wave pulse.

in the earth by the assumption of an infinite conductivity. These authors then approximate the "lossy" case by taking only the first term in the infinite series which represents the roots of Riccati's differential equation (34) (see Appendix). In contrast, the results of Pekeris and Alterman¹⁸ completely neglect the conduction currents in the earth and consider only the displacement currents, and further assume that the earth is a plane, which is reasonable only at short distances. On the other hand, Wait,¹⁶ has developed a method for the solution of this problem which not only considers displacement currents and conduction currents, but also the earth's curvature.

It seems to be quite possible to get an exact solution to this problem by the method described in this paper (43), especially since the delta or impulse source function is a much simpler one than the sinusoid with which this paper is primarily concerned.

The source functions which represent natural phenomena will not in general have the precise mathematical form of a damped sinusoid. Since the application of the superposition principle is not always practical,

¹⁸ C. L. Pekeris and Z. Alterman, "Radiation from an impulsive current in a vertical antenna placed on a dielectric ground," *J. Appl. Phys.*, vol. 28, pp. 1317-1323; November, 1957.

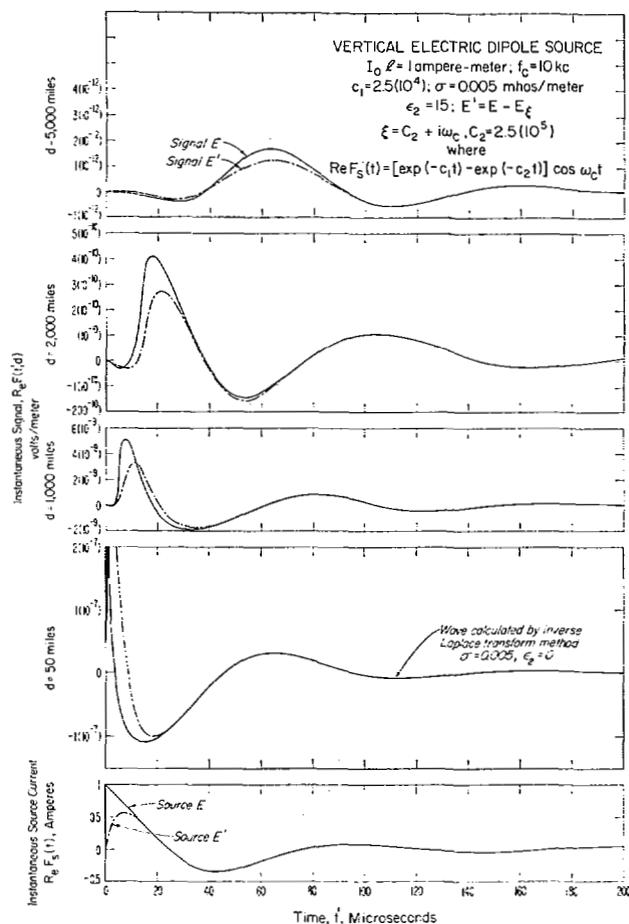


Fig. 9—Instantaneous signal and corresponding damped cosine wave source current of a propagated ground wave pulse, illustrating the introduction of a less abrupt initial source current.

another approach to the general problem is appropriate. Indeed, it is quite possible to extend the theory to complicated waveforms. Consider an experiment in which the signal, $\text{Re } E(t', d)$ is observed and recorded at some distance, d_1 , from the source. The theory is then required to predict the form of the signal recorded at some other distance, d_2 . The theory is also required to determine the form of the source, $F_s(t)$.

The spectrum (18), (20), (20a) can be determined directly from the observed signal, $\text{Re } E(t', d)$:

$$f_x(\omega, d) = \int_0^{\infty} \exp(-i\omega t') \text{Re } E(t', d) dt'. \quad (45)$$

The infinite integral can, as before, be split into the sum of finite integrals subject to the previously described (21) conditions of convergence,

$$f_x(\omega, d) = \int_0^{t_1'} F(\omega, t') dt' + \int_{t_1'}^{t_2'} F(\omega, t') dt' + \dots \\ + \int_{t_n'}^{t_{n+1}'} F(\omega, t') dt' + \dots, \quad (46)$$

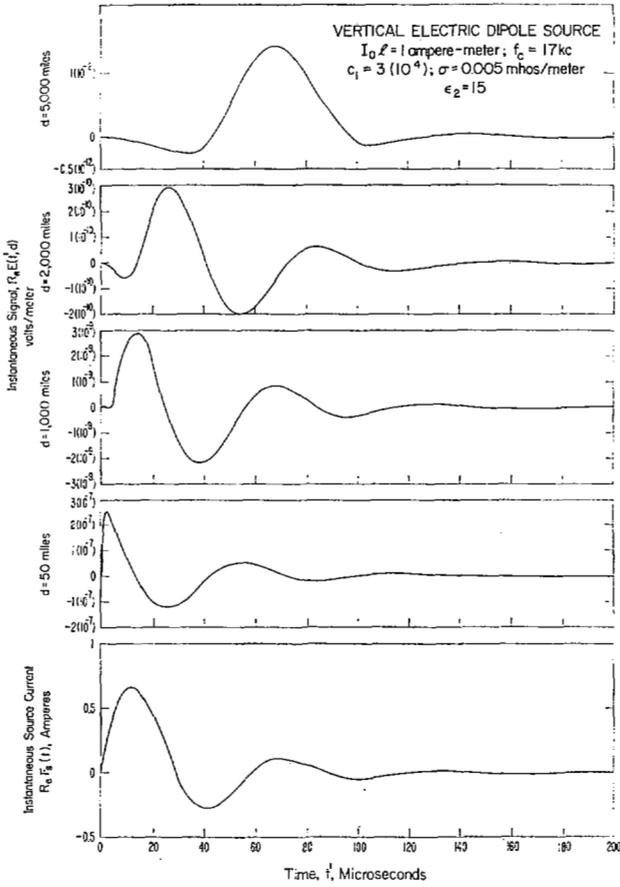


Fig. 10—Instantaneous signal and corresponding damped sine wave source current of a propagated ground wave pulse.

$$\int_{t'_n}^{t'_{n+1}} F(\omega, t') dt' = \sum_{m=1}^M W_m F(\omega, t'_m), \quad (47)$$

$$m = 1, 2, 3, \dots, M, \quad (47a)$$

$$t'_m = \frac{1}{2}(t'_{n+1} - t'_n)x_m + \frac{1}{2}(t'_{n+1} + t'_n). \quad (47b)$$

The spectrum of the source, $f_{x,s}(\omega)$, can then be determined,

$$f_{x,s}(\omega) = \frac{f_x(\omega, d)}{E(\omega, d)}. \quad (48)$$

Since the real part of the signal, $\text{Re } E(t', d)$, was employed in the analysis (45), the source function, $\text{Re } F_s(t)$, can be described as an integral with a symmetrical integrand,

$$F_s(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(i\omega t) f_{x,s}(\omega) d\omega, \quad (49)$$

or

$$F_s(t) = \frac{1}{\pi} \int_0^{\infty} |f_{x,s}(\omega)| \{ \cos [\omega t + \phi_{x,s}(\omega)] \} d\omega. \quad (50)$$

Also

$$f_x(\omega, d_2) = \frac{f_x(\omega, d_1)}{E(\omega, d_1)} E(\omega, d_2), \quad (51)$$

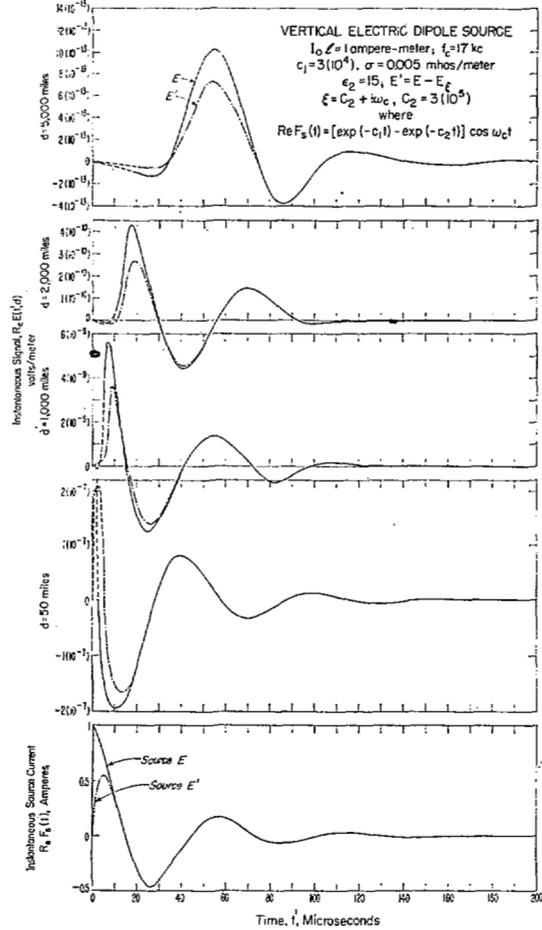


Fig. 11—Instantaneous signal and corresponding damped cosine wave source current of a propagated ground wave pulse, illustrating the introduction of a less abrupt initial source current.

and

$$E(t', d_2) = \frac{1}{\pi} \int_0^{\infty} |f_x(\omega, d_2)| \cos [\omega t' + \phi_x(\omega, d_2)] d\omega. \quad (52)$$

The integrals (50), (52) can be evaluated by previously described quadrature (22).

The recovery of the form of the source (50) is not always practical at great distance, since the ground wave is rather severely attenuated at high frequencies (Fig. 4). The limit of resolution for which the source can be recovered at a given distance is experimental, *i.e.*, the resolution is dependent upon the accuracy with which the signal, $\text{Re } E(t', d)$, can be observed and the accuracy with which the constants which characterize the propagation medium can be determined. But this is merely another way of stating that the form of the source is somewhat obscured by the propagation medium at great distances.

CONCLUSIONS

The form of the transient ground wave signal which has been propagated to great distance from a current source has been theoretically determined. More complicated signals can be formed by superposition and the

simple signals such as the step function can be formed as limiting cases of the more general sinusoid. Complicated waveforms at a given distance from the source can be

APPENDIX

The roots of Riccati's differential equation (34) can be found as follows:

$$\tau_s = \tau_{s,0} - \delta - \frac{2}{3} \tau_{s,0} \delta^3 + \frac{1}{2} \delta^4 - \frac{4}{5} \tau_{s,0}^2 \delta^5 + \frac{14}{9} \tau_{s,0} \delta^6 - \frac{1}{7} (5 + 8\tau_{s,0}^3) \delta^7 + \frac{58}{15} \tau_{s,0}^2 \delta^8 - \left(\frac{328}{81} \tau_{s,0} + \frac{16}{9} \tau_{s,0}^4 \right) \delta^9 + \left(\frac{423}{315} + \frac{1552}{175} \tau_{s,0}^3 \right) \delta^{10} - \left(\frac{7576}{495} \tau_{s,0}^2 + \frac{32}{11} \tau_{s,0}^5 \right) \delta^{11} + \dots \quad |\delta^2 \tau| < \frac{1}{2}$$

$$\tau_s = \tau_{s,\infty} - \left[\frac{1}{2\tau_{s,\infty}} \right] \frac{1}{\delta} - \left[\frac{1}{8\tau_{s,\infty}^3} \right] \frac{1}{\delta^2} - \left[\frac{1}{12\tau_{s,\infty}^2} + \frac{1}{16\tau_{s,\infty}^5} \right] \frac{1}{\delta^3} - \left[\frac{7}{96\tau_{s,\infty}^4} + \frac{5}{128\tau_{s,\infty}^7} \right] \frac{1}{\delta^4} - \left[\frac{1}{40\tau_{s,\infty}^3} + \frac{21}{320\tau_{s,\infty}^6} + \frac{7}{256\tau_{s,\infty}^9} \right] \frac{1}{\delta^5} - \left[\frac{29}{720\tau_{s,\infty}^5} + \frac{77}{1280\tau_{s,\infty}^8} + \frac{21}{1024\tau_{s,\infty}^{11}} \right] \frac{1}{\delta^6} - \left[\frac{1}{112\tau_{s,\infty}^4} + \frac{19}{360\tau_{s,\infty}^7} + \frac{143}{2560\tau_{s,\infty}^{10}} + \frac{33}{2048\tau_{s,\infty}^{13}} \right] \frac{1}{\delta^7} - \left[\frac{97}{4480\tau_{s,\infty}^6} + \frac{163}{2560\tau_{s,\infty}^9} + \frac{429}{8192\tau_{s,\infty}^{12}} + \frac{429}{32768\tau_{s,\infty}^{15}} \right] \frac{1}{\delta^8} - \left[\frac{1}{288\tau_{s,\infty}^5} + \frac{13661}{362880\tau_{s,\infty}^8} + \frac{6769}{92160\tau_{s,\infty}^{11}} + \frac{2431}{49152\tau_{s,\infty}^{14}} + \frac{715}{65536\tau_{s,\infty}^{17}} \right] \frac{1}{\delta^9} - \left[\frac{2309}{201600\tau_{s,\infty}^7} + \frac{820573}{14515200\tau_{s,\infty}^{10}} + \frac{37961}{460800\tau_{s,\infty}^{13}} + \frac{46189}{983040\tau_{s,\infty}^{16}} + \frac{2431}{262144\tau_{s,\infty}^{19}} \right] \frac{1}{\delta^{10}} + \dots \quad |\delta^2 \tau| > \frac{1}{2}$$

evaluated in practical experiments by a Fourier transformation employing numerical integration of the observed or real part of the signal. The signal can then be predicted at some other distance from the source by an inverse Fourier transformation employing similar numerical integration. The source can in principle be recovered from the observed signal, but at great distance it is somewhat obscured by the propagation medium.

The technique for the direct evaluation of the Fourier integral for the case of the ground wave suggests applications to other problems, such as the propagation of a transient through a filter or the propagation of the transient sky wave.

The poles and zeros $|\tau_{s,\infty}|$ and $|\tau_{s,0}|$ have been tabulated,¹⁹ where

$$\tau_{s,\infty} = |\tau_{s,\infty}| \exp \left[i \frac{\pi}{3} \right] \text{ and } \tau_{s,0} = |\tau_{s,0}| \exp \left[i \frac{\pi}{3} \right].$$

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¹⁹ Jöhler, *et al.*, *op. cit.*, p. 33, table 44.