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Very Long-term Frequency Stability: Estimation using a Special-purpose Statistic*

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Abstract - We introduce a statistic that can be used for a particularly difficult measurement problem, namely, determining frequency stability for frequency standards and oscillators for averaging times longer than those the traditional Allan deviation can estimate. Theoretical variance #1 ("$\widehat{\text{Theo1}}$") has statistical properties that are like "Avar" (Allan variance), with two significant enhancements: (1) it can evaluate frequency stability at longer averaging times than given by the definition of Avar, and (2) it has the highest number of equivalent degrees of freedom (edf) of any estimator of frequency stability. $\widehat{\text{Theo1}}$ is unbiased relative to Avar for white FM noise, and only moderately biased for the other noises. Given measurements of the time-error function $x(t)$ between two clocks, we have a sequence of time-error samples $\{x_n : n = 1, \ldots, N_x\}$ with a sampling period between adjacent observations given by τ_0. In integer multiples of τ_0, we can obtain an average of fractional-frequency deviates over time $\tau = m\tau_0, 1 \le m \le N_x - 1$. $\widehat{\text{Theo1}}$ is given by

$$\widehat{\text{Theo1}}(m, \tau_0, N_x) = \frac{1}{0.75(N_x - m)(m\tau_0)^2} \sum_{i=1}^{N_x - m}$$

$$\sum_{\delta=0}^{\frac{m}{2}-1} \frac{1}{(\frac{m}{2} - \delta)} \left[(x_i - x_{i-\delta+\frac{m}{2}}) + (x_{i+m} - x_{i+\delta+\frac{m}{2}})\right]^2,$$

for m even, $10 \le m \le N_x - 1$, where frequency stability is evaluated at span or stride $\tau_s = 0.75m\tau_0$. This means that the last $\tau_s = 0.75(N_x - 1)\tau_0$, or an averaging time corresponding to 3/4 of the duration of a data run, or 50 % longer than the longest τ-value of Avar.

1. Introduction

Having measured the time error between two clocks or oscillators, say, every couple of hours for one month, the maximum-overlap Allan variance estimator of frequency stability, or "Avar," cannot report frequency stability for intervals longer than half the month, or two weeks [1–3]. By definition, a zero-dead-time average frequency difference for averaging interval τ cannot possibly extend beyond 50 % of the length of the data run T, that is, beyond $\tau = \frac{T}{2}$. Furthermore, this estimate is often too low. This is because the chi-square distribution function associated with an estimate comprised of only one sample at $\tau = \frac{T}{2}$ (representing one degree of freedom) is so negatively skewed that it is twice as likely to be lower than the FM noise level's true value than above it [4,5]. In addition, if a sample estimate of frequency drift is removed, Avar is likely to respond with levels too low at longest-term compared to the expected or true underlying characteristic level [5]. This overlapping estimator for the Allan variance has sufficiently good confidence at short- and medium-term τ averaging intervals but, to be conservative in light of the reasons just stated, it is not recommended for τ beyond 10 % of a data run T [3]. In the one-month example above, this amounts to only a three-day τ-average. In this situation, the best estimator of the Allan variance, which is the Total variance, or Totvar [3, 5, 6], is recommended. Use of the Total approach yields improved confidence between 10 % and 50 % of a data run, or up to two weeks in a one-month data run. At this writing, analysts in our field are confident of Totvar's properties. Easy-to-use 32-bit Windows software is commercially distributed that implements Totvar on large data sets, computes its confidence intervals, and automatically adjusts for bias [7].

It would seem preposterous to report a reliable estimate of frequency stability at a τ of <u>three</u> <u>weeks</u>, given a one-month data run, again considering the

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reasons stated, not to mention that this is theoretically impossible with the Allan variance! In this paper, we introduce a special-purpose statistic that evaluates very long-term frequency stability at τ between $\frac{T}{2}$ and T, is less susceptible to drift removal, and has a more symmetric distribution function than that of chi-square [8]. Starting with a sequence of time-error samples $\{x_n : n = 1, \ldots, N_x\}$ with a sampling period between adjacent observations given by τ_0, $\widehat{\text{Theo1}}$ is the sample version of a theoretical variance that averages every permissible squared second-difference of time errors in a given span or stride $\tau_s = 0.75m\tau_0$ using the following definition:

$$\widehat{\text{Theo1}}(m, \tau_0, N_x) = \frac{1}{0.75(N_x - m)(m\tau_0)^2} \sum_{i=1}^{N_x-m}$$

$$\sum_{\delta=0}^{\frac{m}{2}-1} \frac{1}{(\frac{m}{2} - \delta)} \left[(x_i - x_{i-\delta+\frac{m}{2}}) + (x_{i+m} - x_{i+\delta+\frac{m}{2}})\right]^2, \tag{1}$$

for m even, $10 \le m \le N_x - 1$. At this writing, the statistic has the highest confidence in estimating long-term frequency stability. A sample calculation on a short time series is given in Appendix I.

The development of $\widehat{\text{Theo1}}$ involved the following issues. First, it is common practice to measure samples of the time-error function $x(t)$ between two oscillators and then derive frequency stability. For example, Avar is usually calculated as a normalized second difference of time-error measurements $\{x_n\}$. Measuring in this way assures Avar's statistical requirement for zero dead-time between average frequency differences [9]. Of course, one can obtain $m\tau_0$-average fractional-frequency values as

$$\overline{y}_n(m) \equiv \tfrac{1}{m} \textstyle\sum_{j=0}^{m-1} y_{n-j},$$

where $y_n = \frac{1}{\tau_0}(x_n - x_{n-1})$ with $n = t/\tau_0$ starting from a designated origin $t_0 = 0$.

Second, it is desirable to maintain Avar's half-octave frequency response with peak at reciprocal period of $f_p = \frac{1}{2\tau}$. This response efficiently extracts levels of FM power-law noise types [10–15] while retaining simple, distinct straight-line mapping (on log-log plots) to $S_y(f)$, which is the recommended characterization of frequency stability [16]. Third, we want to maximize equivalent degrees of freedom (edf) while minimizing bias relative to the conventional Allan variance. We can accomplish this by using most, if not all, of the available $\{x_n\}$ data, with small sampling interval $\tau_0 \ll T$, based on the experience gained from Totvar [5, 17, 18].

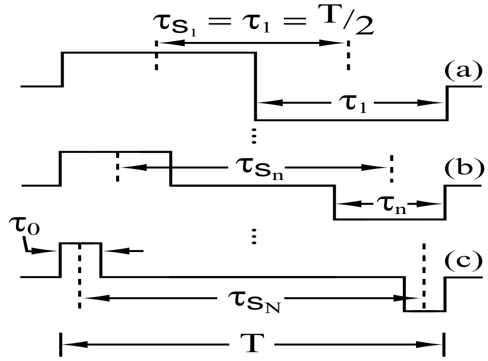

Figure 1: Sampling using $\widehat{\text{Theo1}}$ of fractional-frequency measurements $\{y_n\}$, which computes frequency differences in interval T, shows the varying *stride* $\tau_{s1,s2,etc.}$ and corresponding averaging time $\tau_{1,2,etc.}$ given by the inner summation in (1). The summation's first term ($\delta = 0$) is the sampling in (a) which is that of the classical Allan variance. In this case, stride τ_{s1} equals averaging time τ_1, and both equal $\frac{T}{2}$. For $1 < \delta \le \frac{m}{2}$, intermediate sampling functions are illustrated by (b) in which $\tau_{s(\cdot)} > \frac{T}{2}$. The summation's last sampling function is (c) in which $\tau_{s(N)} = T - \tau_0$. Therefore, the effective τ-value of the individual frequency differences averaged in $\widehat{\text{Theo1}}$ is between $\frac{T}{2}$ and $T - \tau_0$.

2. Meaning of τ

Oscillator frequency instability, the effect of random frequency-modulation (FM) noise, can be regarded as an uncertainty on that oscillator's expected or predicted average frequency [19]. At long enough intervals, a frequency error accumulates over some time span, call it τ_s. This error is due to the frequency's undesirable noise fluctuations characterized as white, flicker, or random walk, or systematic frequency drift. Given one span, an estimate of error/τ_s is simply $\frac{\Delta^\tau \overline{y}(t)}{\tau_s}$, where $\Delta^\tau \overline{y}(t)$ is a *change* or difference in fractional-frequency values, each obtained from a measurement of a pair of oscillators, τ is an averaging interval used to compute each value of fractional-frequency, and τ_s is the *stride* or span of time over which the change occured [16]. The time-domain characterization of random noise can be regarded as the rms frequency error over τ_s given by the usually-reported Allan deviation $\sigma_y(\tau)$ as $\frac{1}{\sqrt{2}} (\Delta^\tau \overline{y}(t))_{rms}$ [20, 21]. *Adjacent* values of ${}^\tau\overline{y}(t)$ must be used in the Allan definition, thus making $\tau_s = \tau$ [9]. The point is that the stride τ_s is not explicit but implied in the definition, and one could make the notation for frequency stability read $\sigma_y(\tau_s)$. With Avar, τ can mean averaging time τ or stride τ_s.

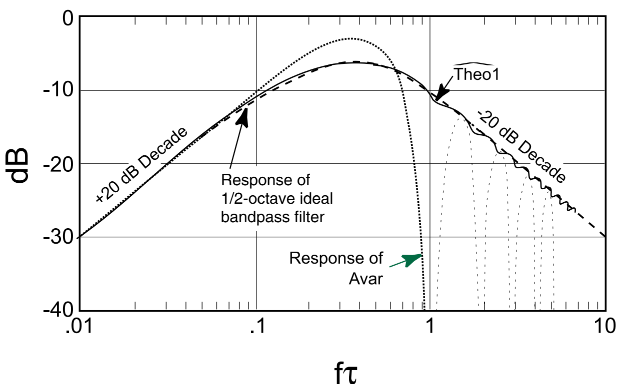

Figure 2: A comparison of frequency responses of $\widehat{\text{Theo1}}$ (bold line), Avar, and a passband variance consisting of a simple cascade of a single-pole high-pass followed by a low-pass filter with identical break points at $RC = \tau/2$ (dashed line [10]). $\widehat{\text{Theo1}}$'s frequency response with bias removed is a close approximation to the response of an ideal pass-band filter and explains why it is so efficient in extracting power-law noise levels and types.

This is illustrated in Figure 1, in which (a) is the sampling function of the classical Allan variance.

A major distinction of $\widehat{\text{Theo1}}$ is that it evaluates frequency stability at an effective stride τ_s given by an average of all strides between $\frac{T}{2}$ and $T - \tau_0$ as shown in Figure 1. "Averaging time τ" is not constant for $\widehat{\text{Theo1}}$ in the usual sense. For clarity, τ when used with $\widehat{\text{Theo1}}$ will always mean an average of stride τ_s, even if the notation "τ_s" is not explicitly used.

3. Criteria for $\tau_s = 0.75m\tau_0$

Recall that it is desirable to maintain Avar's half-octave frequency response with peak at a reciprocal period of $f_p = \frac{1}{2\tau}$. The dashed line in Figure 2 shows the response of a constant-Q, half-octave pass-band filter considered to be ideal for extracting typical power-law noise levels [10–15].

Response of a statistic is the Fourier transform of its sampling sequence that, in some cases, can be nearly impossible to interpret in the time domain but easier to understand in the frequency domain [22]. Frequency-response functions associated with $\widehat{\text{Theo1}}$ (with bias removed, see section 5) and Avar are shown in Figure 2. Prior to (1), we obtained a high-edf, low-bias prototype variance, whose frequency response peak was shifted above $f_p = \frac{1}{2\tau_s}$. We found that if $\tau_s = 0.75m\tau_0$ and the amplitude of the response is adjusted by 0.75 (in the denominator of the amplitude coefficient of definition (1)), then the frequency response could be shifted to be precisely $f_p = \frac{1}{2\tau_s}$.

4. Response to Data Periodicity

Avar has deep nulls in its response to periodic or cyclical variations in $\{x_n\}$ at frequencies $f = \frac{\text{int}}{\tau}$, int $= 1, 2, 3, \ldots$, whereas $\widehat{\text{Theo1}}$ does not (see Figure 2). This means that the response of $\widehat{\text{Theo1}}$ to a periodic term in the data with frequency near $f = \frac{\text{int}}{\tau}$ is going to be more accurate than if using Avar. In the end, $\widehat{\text{Theo1}}$'s frequency response is closer to the response of the ideal pass-band filter that Avar attempts to approximate.

5. Bias

In this writing, "bias" of $\widehat{\text{Theo1}}$ refers to the ratio of its expected value to that of the Allan variance. Bias for each noise type is listed in Table 1, and also is listed in terms of the usually reported deviation.

Table 1: Bias of $\widehat{\text{Theo1}}$.

Noise	Avar =	Adev=
White FM	$\widehat{\text{Theo1}}$	$\widehat{\text{Theo1}}$-dev
Flicker FM	1.71$\widehat{\text{Theo1}}$	1.31$\widehat{\text{Theo1}}$-dev
Rand Walk FM	2.24$\widehat{\text{Theo1}}$	1.50$\widehat{\text{Theo1}}$-dev
White PM*	0.4$\widehat{\text{Theo1}}$	0.63$\widehat{\text{Theo1}}$-dev
Flicker PM*	0.6$\widehat{\text{Theo1}}$	0.77$\widehat{\text{Theo1}}$-dev

*With PM noises, $\widehat{\text{Theo1}}$'s slope is slightly less than Avar's slope of τ^{-2} (τ^{-1} in terms of "deviation"). Therefore, the bias is approximated.

6. Equivalent Degrees of Freedom

For computing edf, empirical formulae that fit simulation can be used [1,23]. Here are the edf formulae corresponding to $\widehat{\text{Theo1}}$ for the five noises (with the condition that $\tau_0 \leq \frac{T}{10}$, as discussed in section 7 to follow):

$$\underbrace{edf}_{WHFM} = \left[\frac{4.1N_x + 0.8}{\tau_s} - \frac{3.1N_x + 6.5}{N_x}\right]\left(\frac{\tau_s^{3/2}}{\tau_s^{3/2} + 5.2}\right)$$

$$\underbrace{edf}_{FLFM} = \left(\frac{2N_x^2 - 1.3N_x\tau_s - 3.5\tau_s}{N_x\tau_s}\right)\left(\frac{\tau_s^3}{\tau_s^3 + 2.3}\right)$$

$$\underbrace{edf}_{RWFM} = \left(\frac{4.4N_x - 2}{2.9\tau_s}\right) \times$$

$$\left(\frac{(4.4N_x - 1)^2 - 8.6\tau_s(4.4N_x - 1) + 11.4\tau_s^2}{(4.4N_x - 3)^2}\right)$$

$$\underbrace{edf}_{WHPM} = \left(\frac{0.86\,(N_x+1)\left(N_x - \frac{4}{3}\cdot\tau_s\right)}{N_x - \tau_s} \right)\left(\frac{\tau_s}{\tau_s + 1.14} \right)$$

$$\underbrace{edf}_{FLPM} = \left(\frac{4.798N_x^2 - 6.374N_x\tau_s + 12.387\tau_s}{(\tau_s + 36.6)^{1/2}(N_x - \tau_s)} \right)\left(\frac{\tau_s}{\tau_s + 0.3} \right).$$

Accuracy of the fit to simulation results is ±10 %.

7. Long-term Frequency Stability Strategy

We want the inner sum in (1) to act on as many points as possible for a given data run, that is, we want m to be large for a good evaluation of $\widehat{\mathrm{Theo1}}$ at a particular desired τ_s. Therefore, it is advantageous to have a high sample rate in comparison to a desired τ_s. In other words, obtain, say, 100 to 1000 points in the data run for a desired τ_s by making τ_0 small enough to achieve this end. For example, for a sample period of $\tau_s = 1$ week $= 7$ d $= 168$ hrs, one would need 9-1/3 d $= 224$ hrs of data to get the "last point" of $\widehat{\mathrm{Theo1}}$, that is, the point at the longest possible τ_s. A good sample rate of x_i would be 1 samp / hr, thus $\tau_0 = 1$ hr, making $N_x = 224$. Because $\tau_s = 0.75m\tau_0$, $\widehat{\mathrm{Theo1}}$ will estimate frequency stability for the desired range of $\tau_s < 168$ hrs. To obtain good confidence, $N_x = 224$ yields a sufficient number of degrees of freedom in this range based on formulae in the previous section. In contrast, one would usually need over 17 days of data using Avar for the same confidence that $\widehat{\mathrm{Theo1}}$ obtains at 9-1/3 d.

8. Determination of Noise Type

Noise identification at a particular τ is needed to determine confidence intervals (by means of the edf) and bias. Presently, there exists a noise-identification algorithm that has been found effective in practice which estimates noise type *vs.* τ [24] and has been used successfully to automatically determine noise type [6,7]. It is based on the Barnes B1 function [25], which is the ratio of the N-sample (standard) variance to the two-sample (Allan) variance.

This *ad hoc* method works satisfactorily, however, computations of $\widehat{\mathrm{Theo1}}(\tau_s)$ are at stride $\tau_s = 0.75m\tau_0$, which means that its "last point" is at stride $\tau_s = 0.75(T - \tau_0)$, a point beyond which the above method works. In many cases, one can use the noise type of the point at $\tau = \frac{T}{2}$. Alternatively, the noise type at a τ-value greater than $\frac{T}{2}$ can often be narrowed down to one based on the overall surrounding frequency stability function, a technique used with Avar [19, 26–28]. A final option is to average the small, expected "range" of answers at very long term. Since the bias of $\widehat{\mathrm{Theo1}}$ is modest, one can average this range with a confidence interval that represents the peak spread of the range and still obtain a useful estimate.

9. Conclusion

- $\widehat{\mathrm{Theo1}}$ is effective to arbitrarily large τ-values, including 3/4 of the entire data run. This means that longest-term frequency stability can be obtained with only 1/3 more data-collection time. For example, the three-month stability can be obtained with four months of data, rather than the six months of data that is usually required for such a point.
- $\widehat{\mathrm{Theo1}}$ has higher edf than Avar or Totvar. In the example above, edf is about 2 to 5 times greater than Avar at τ of three months, depending on the noise type.
- $\widehat{\mathrm{Theo1}}$, like Avar and Totvar, is invariant to an overall shift in phase and frequency. $\widehat{\mathrm{Theo1}}$, like Avar and Totvar, retains simple straight-line mapping (on log-log plots) to $S_y(f)$ for easily extracting the levels of the usual five FM power-law noise types by a linear-least-squares fit [8].
- Although we have introduced a new statistic that is designed to report very long-term stability, $\widehat{\mathrm{Theo1}}$-dev can serve as an excellent alternative to the Allan deviation in medium term. It is unbiased relative to the Allan deviation for WHFM noise, and only moderately biased for other noises.

10. Acknowledgments

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11. References

[1] D. A. Howe, D. W. Allan, and J. A. Barnes, "Properties of Signal Sources and Measurement Methods," *Proc. 35th Freq. Cont. Symp.*, pp. 1–47 (1981; reprinted in [4]).

[2] International Telecommunications Union, "Measures for Random Instabilities in Frequency and Time (Phase)", Recommendation ITU-R TF.538-3, 1994.

[3] "IEEE Std 1139-1999: Standard Definitions of Physical Quantities for Fundamental Frequency and Time Metrology—Random Instabilities," *IEEE-SA Standards Board*, J.R. Vig, chairperson, March, 1999.

[4] D. B. Sullivan, D. W. Allan, D. A. Howe, and F. L. Walls (Editors), *Characterization of Clocks and Oscillators*, Natl. Inst. Stand. Technol. Technical Note 1337, 1990.

[5] D. A. Howe, "Total Deviation Approach to Long-Term Characterization of Frequency Stability," *IEEE Trans. Ultrasonics, Ferroelectrics, and Freq. Control*, **U**FFC-47 no. 5, pp. 1102–1110, Sept. 2000.

[6] D. A. Howe and T. Peppler, "Definitions of "Total" estimators of common time-domain variances," *2001 IEEE International Frequency Control Symposium*, pp. 127–132, June 2001.

[7] Stable 32, Frequency Stability Analysis for Windows and NT, Hamilton Technical Services, 195 Woodbury Street, S. Hamilton, MA 01982, (978)-468-3703, http://www.wriley.com. The latest version contains $\widehat{\text{Theo1}}$. No endorsement is implied. Products are available from other manufacturers.

[8] D. A. Howe and T. Peppler, "Characterization of Very Long-term Frequency Stability using $\widehat{\text{Theo1}}$," *Metrologia*, manuscript 1667, 2003, in process.

[9] D.W. Allan, "Statistics of Atomic Frequency Standards," *Proc. IEEE* **54**, pp. 221–230, Feb. 1966.

[10] R. G. Wiley, "A Direct Time-Domain Measure of Frequency Stability: The Modified Allan Variance," *IEEE Trans. Instrum. Meas.*, **IM-26**, pp. 38-41, Mar. 1977.

[11] D. W. Allan, M. A. Weiss, and J. L. Jespersen, "A Frequency-domain View of Time-domain Characterizations of Clocks and Time and Frequency Distribution Systems," *Proc. 45th Freq. Cont. Symp.*, pp. 667–678, 1991.

[12] D. B. Percival, "Characterization of frequency stability: frequency-domain estimation of stability measures," *Proc. IEEE*, vol. 79, pp. 961–972, 1991.

[13] D. A. Howe and D. B. Percival, "Wavelet Variance, Allan Variance, and Leakage," *IEEE Trans. Instrum. Meas.*, **IM-44**, pp. 94–97, 1995.

[14] D. Percival and A. Walden, Spectral Analysis for Physical Applications, Cambridge University Press, New York, 1993.

[15] J. Rutman, "Characterization of frequency stability: A transfer function approach and its application to measurements via filtering of phase noise," *IEEE Trans. Instrum. Meas.*, **IM-23**, pp. 40-48, Mar. 1974.

[16] J. A. Barnes, A. R. Chi, L. S. Cutler, D. J. Healy, D. B. Leeson, T. E. McGunigal, J. A. Mullen, Jr., W. L. Smith, R. L. Sydnor, R. F. C. Vessot, G. M. R. Winkler, "Characterization of frequency stability," *IEEE Trans. Instrum. Meas.*, **IM-20**, pp. 105–120, 1971.

[17] C. A. Greenhall, D. A. Howe and D. B. Percival, "Total Variance, an Estimator of Long-Term Frequency Stability," *IEEE Trans. Ultrasonics, Ferroelectrics, and Freq. Control*, **UFFC-46**, no. 5, pp. 1183–1191, Sept, 1999.

[18] D. A. Howe, "Total Variance Explained," *Proc. 13th European Frequency and Time Forum* and *1999 IEEE International Frequency Control Symposium*, pp. 1093–1099, April 1999.

[19] J. Levine, "Time and Frequency Metrology," *Review of Scientific Instruments*, vol. 70, no. 6, June 1999.

[20] M. A. Weiss and C. Hackman, "Confidence on the Three-Point Estimator of Frequency Drift," *Proc. 24th Ann. Precise Time and Time Interval Meeting*, pp. 451–460, 1992.

[21] F. Vernotte, "Relationship between Drift Coefficient Uncertainties and Noise Levels: Application to Time Error Prediction," *Proc. 29th Ann. Precise Time and Time Interval Meeting*, pp. 29–38, Long Beach, CA., Dec. 1997.

[22] D. A. Howe and C. A. Greenhall, "Total variance: a progress report on a new frequency stability characterization," *Proc. 29th Ann. Precise Time and Time Interval Meeting*, pp. 39–48, 1997.

[23] C. A. Greenhall, "Recipes for Degrees of Freedom of Frequency Stability Estimators,"*IEEE Trans. Instrum. Meas.*, **IM-40**, no. 6, pp. 994–999, Dec. 1991.

[24] D. Howe, R. Beard, C. Greenhall, F. Vernotte, and W. Riley, "A Total Estimator of the Hadamard Function Used for GPS Operations," Proc. 32nd Annual PTTI Systems and Applications Meeting, pp. 255-268, Reston, VA, Nov 28-30, 2000.

[25] J. A. Barnes, *Tables of Bias Functions, B1 and B2, for Variances Based on Finite Samples of Processes with Power Law Spectral Densities*, Natl. Inst. Stand. Technol. Technical Note NBS-375, 1969.

[26] D. A. Howe and E. E. Hagn, "Limited Live-time Measurements of Frequency Stability," *Proc. 13th European Frequency and Time Forum* and *IEEE International Frequency Control Symposium*, pp. 1113–1116, April 1999.

[27] T. E. Parker, "Hydrogen Maser Ensemble Performance and Characterization of Frequency Standards," *Proc. 13th European Frequency and Time Forum* and *IEEE International Frequency Control Symposium*, pp. 173–176, April 1999.

[28] J. Levine and T. E. Parker, "The Algorithm Used to Realize UTC(NIST)," *2002 IEEE International Frequency Control Symposium*, pp. 537–542, May 2002.

APPENDIX I

Sample Calculation

For the purpose of illustration, we calculate one value of $\widehat{\text{Theo1}}$-dev from a short sequence of numbers. The sequence can be used as a basic test of computer programs. Consider the following sequence of ten days of equispaced time error measurements $\{x_n : n = 1, \ldots, 10\}$ in units of ns:

$$\begin{array}{ll} x_1 = 1.00 & x_6 = 1.08 \\ x_2 = 2.50 & x_7 = 0.50 \\ x_3 = 0.65 & x_8 = 2.20 \\ x_4 = -3.71 & x_9 = 4.68 \\ x_5 = -3.30 & x_{10} = 3.29 \end{array}$$

Let us calculate $\widehat{\text{Theo1}}$-dev for $\tau = 6\ d$ $(5.184 \times 10^5\ s)$.

From (1), the inner summand terms for the sample variance $\widehat{\text{Theo1}}(8, 1, 10)$ for each index i are given by (for this value of τ, note that m=8):

$$\begin{array}{lll} i = 1: & \delta = 0: & \frac{1}{4}[(1 - (-3.3)) + (4.68 - (-3.3))]^2 \\ & \delta = 1: & +\frac{1}{3}[(1 - (-3.71)) + (4.68 - 1.08)]^2 \\ & \delta = 2: & +\frac{1}{2}[(1 - 0.65) + (4.68 - 0.5)]^2 \\ & \delta = 3: & +1[(1 - 2.5) + (4.68 - 2.2)]^2 \\ & & = 71.94, \\ i = 2: & \delta = 0: & \frac{1}{4}[(2.5 - 1.08) + (3.29 - 1.08)]^2 \\ & \delta = 1: & +\frac{1}{3}[(2.5 - (-3.3)) + (3.29 - 0.5)]^2 \\ & \delta = 2: & +\frac{1}{2}[(2.5 - (-3.71)) + (3.29 - 2.2)]^2 \\ & \delta = 3: & +1[(2.5 - 0.65) + (3.29 - 4.68)]^2 \\ & & = 54.75, \end{array}$$

and, for the outer summation, we obtain $\sum_{i=1}^{2}(\cdot) = 126.69$. $\widehat{\text{Theo1}}(8, 1, 10) = \frac{1}{2(8)^2 \cdot 0.75} \sum_{i=1}^{2}(\cdot) = 1.320$, and $\widehat{\text{Theo1}}\text{-dev}(8, 1, 10) = \sqrt{1.320} = 1.149$.

Since the original measurements were in units of ns/day, we obtain $1.149 \cdot \frac{1\ ns}{86400\ s}$ for a sampling period of $\tau_0 = 1\ d = 86400\ s$. Therefore, for this set, $\widehat{\text{Theo1}}\text{-dev}(8, 86400\, s, 10) = 1.330 \times 10^{-14}$.