AN APPROACH TO THE PREDICTION OF COORDINATED UNIVERSAL TIME

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By computing the root mean square of the third difference of the fluctuations of Universal Time for various sample times, a spectral classification of the fluctuations can be made. For periods from 0.02 years to 20 years, the data indicate a power spectral density for the fluctuations which varies as $|\omega|^{-\alpha}$, where α lies in the range $3 \leq \alpha \leq 5$.

Based on the spectral classification, some simple and practical methods of predicting frequency offsets for coordinated Universal Time are compared to theoretically optimum (linear) prediction. Three methods are considered and are applied to past UT 2 data for comparison. A conclusion from the paper is that the frequency offset for the coordinated time scales (following present regulations) may be predicted such that no resets of epoch are required for roughly 60 per cent of the years for which the prediction is made.

KEY WORDS: Universal Time, Prediction, Coordinated Universal Time.

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INTRODUCTION

Prior to the year 1956, the definition of the second was the fraction 1/86,400 of the mean solar day. Since the late 1930's, however, astronomers have been aware of fluctuations in the Earth's rate of rotation.[1,2] In 1952 a statistical classification of these fluctuations was presented by Brouwer.[3] His conclusion was that the *rate* of rotation behaved as an "accumulation of random digits," i.e., a random walk.

Because of the fluctuations in the Earth's rate of rotation, a new definition of the unit of time (the second) was adopted in 1956. This definition is based upon the Earth's orbital motion around the sun and is called Ephemeris time (ET). In 1964 an alternate standard for measuring the second received international acceptance. This standard is the cesium beam atomic clock and generates a second of time which does not differ from the Ephemeris second within the present limits of measurement (limitations in the measurement stem from difficulties in the precise determination of ET).

Late in the 1950's, several laboratories began to maintain an epoch of atomic time based upon the cesium atom's frequency. Of significance for this paper are two atomic time scales, the A.1 scale of the U. S. Naval Observatory[4] and the A.3 scale of the International Time Bureau (BIH)[5] in Paris. Regular comparisons between these time scales and Universal time (UT, based upon the rotation of the Earth on its axis) are available covering the interval from 1956 to the present. As is considered below, this allows one to significantly augment the ET-UT data by virtue of the high precision of the atomic measurements, which allow more frequent determination of the UT epoch.

Universal time (UT) is significant because many forms of navigation depend upon fairly accurate knowledge of the Earth's angular position relative to the sun and stars. For this reason all time signals broadcast by stations coordinating with the International Time Bureau (BIH) in Paris, in accordance with international regulations, maintain close agreement with UT 2 (mean solar time with corrections for some known perturbations).

Present CCIR regulations require some[6] coordinated "time" signals (UTC) to be offset in rate relative to the atomic second by integral multiples of 50 parts in

10^{10}. The offset in frequency is announced by the BIH in the fall of the year preceding the year in which it is to be used. When UT 2 and coordinated "time" signals differ in excess of 0.1 second, the coordinated "time" signals are reset exactly one tenth of a second as announced by BIH. Thus, each year various observatories and laboratories are interested in predicting the Earth's rotational rate for the next year.

CLASSIFICATION OF UT 2 BEHAVIOR

The methods of analysis used here may be found in the Special Issue of the Proceedings of the IEEE, Vol. 54, No. 2, February, 1966. In particular, Cutler and Searle,[7] Vessot, et al.,[8] Allan,[9] and Barnes[10] use time domain analysis to infer power spectral density. The present authors considered it worthwhile to apply some of these techniques to Universal time.

By correcting Universal time (UT) for the migration of the Earth's poles one obtains UT 1. Then by removing the remaining yearly periodic fluctuations one obtains UT 2. Numerous observatories determine UT 2. Some of the most extensive data may be found in the publications of the BIH. In particular, data of UT 2 relative to an atomic time scale may be found in reference 5 covering the years 1956.0 to 1965.0 at 10-day intervals. The rms third difference[10] of these data is plotted in Figure 1 as a function of the delay time, τ. The third difference is given by

$$\Delta^3\varphi = \varphi(t+3\tau) - 3\varphi(t+2\tau) + 3\varphi(t+\tau) - \varphi(t)\,. \qquad (1)$$

Also plotted in Figure 1 is the rms third difference of UT-ET as obtained from Brouwer[3] for the years 1820.5 to 1900.5 and from reference 11 for the years 1901.5 to 1962.5. While UT differs from UT 2, the corrections from UT to UT 1 are too small to be resolved in the data for periods longer than one year and the UT 1 to UT 2 corrections are periodic at one year, which is exactly the sampling rate. While one may question the reliability of the ET measurements, there is as yet no significant indication that ET differs from Atomic time (AT). The precision of determination of ET is not nearly as high as for AT[12], but for periods of two years or longer the measurement errors of ET are not significant for Figure 1. Thus Figure 1 may be considered to reflect the fluctuations in UT 2 relative to a uniform clock.

The confidence limits indicated by Figure 1 were computed according to the number of independent, non-overlapping third differences used to calculate the rms third difference according to reference 13. The limits indicated on Figure 1 are calculated for a 90 per cent confidence. The single "best" straight line is not, in fact, contained within 90 per cent of the confidence intervals shown. This single "best" line has a slope of about 1.36.

The same data of Figure 1 are plotted on Figure 2 with two straight lines intersecting at about one year.

It is interesting to note that Brouwer,[8] using a bit more extensive data than the ET-UT data used here, concludes that: "The resulting value for the secular increase in the length of the day is $+0.^{s}00135 \pm 0.^{s}00038$ (s = seconds) per century. The principal uncertainty in these evaluations is due to the random process which causes the amplitude of the fluctuations to increase proportionally to the power 3/2 of the time . . ." (We note that if the rms amplitude of fluctuations increase in proportion to the 3/2 power of time, then the rms third difference does also.) Although Brouwer's remarks were based on data with a spacing of one year and although a line of slope 3/2 cannot be contained within very many confidence intervals of Figure 1, one cannot completely rule out the possibility of a 3/2 power law for the amplitude of the fluctuations for

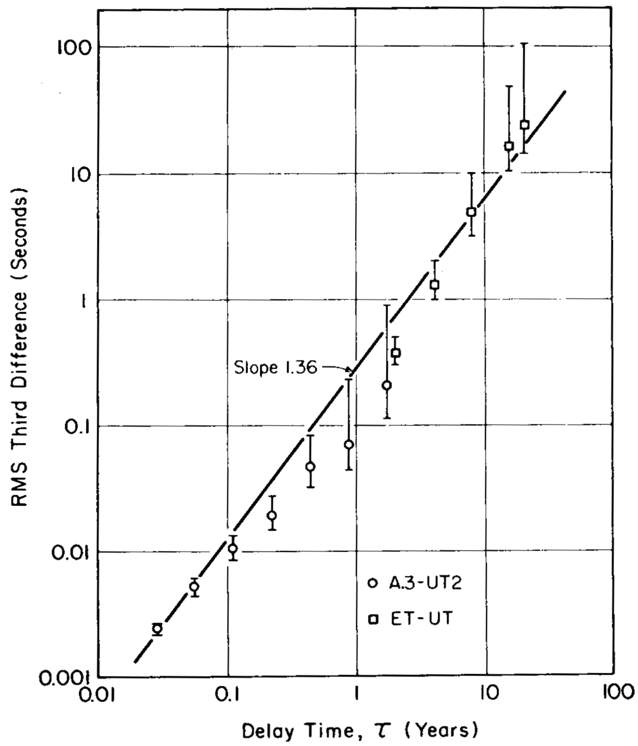

Figure 1 — RMS third difference of UT data.

times smaller than one year. One is tempted however, to draw the lines as indicated in Figure 2. A line of slope 1 fits the data very well for times less than one year. There are additional reasons, which will be covered later, to suspect that for times longer than one year a slope of 2 as shown in Figure 2 is not real.

One may construct a table relating the τ dependence of the rms third difference of a time series to the frequency dependence of the power spectral density of that series. The first four values of Table I were obtained from Vessot, et al.,[8] Allan,[9] and Barnes.[10] The remaining values were obtained by extending their mode of analysis.

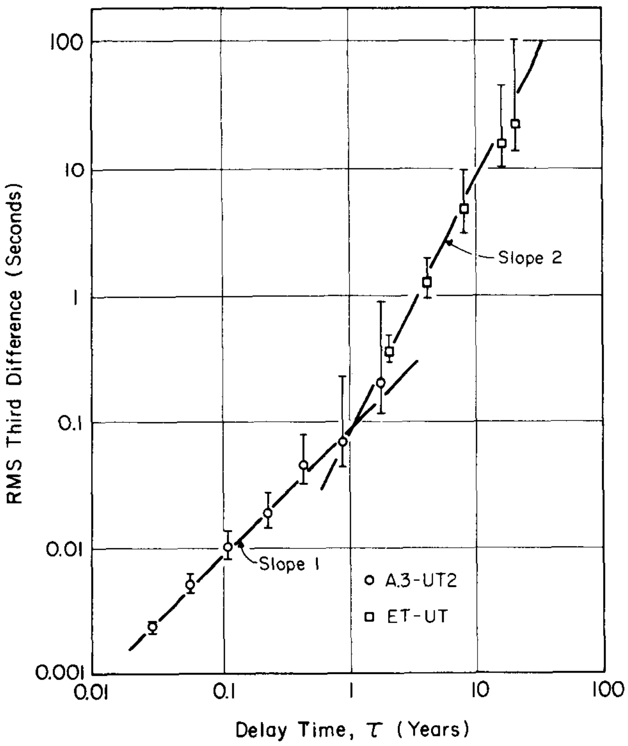

Figure 2 — RMS third difference of UT data.

Also, it should be noted that Brouwer[3] states that "the observed fluctuations in the moon's mean longitude are compatible with the hypothesis that the rate of rotation of the earth is affected by cumulative random changes." Thus, Brouwer's model is completely consistent with the fifth line of Table 1.

Table I

τ-dependence of the third difference for power law spectra

Noise Spectral Type	α	τ-Dependence of rms Third Difference	Name of Noise
$S(\omega) = k$	0	$\sim \tau^0$	White
$= \frac{k}{\lvert\omega\rvert}$	1	$\sim \ln(\tau\omega_h)$	Flicker
$= \frac{k}{\lvert\omega\rvert^2}$	2	$\sim \tau^{1/2}$	Random Walk
$= \frac{k}{\lvert\omega\rvert^3}$	3	$\sim \tau$	Flicker Rate
$= \frac{k}{\lvert\omega\rvert^4}$	4	$\sim \tau^{3/2}$	Random Walk Rate
$= \frac{k}{\lvert\omega\rvert^5}$	5	$\sim \tau^2$	Flicker Acceleration

where ω_h is a high frequency cutoff.

A Noise Model

It is possible to construct a filter[14-16] whose transfer function, $K(\omega)$, is approximated by $\left(\frac{1}{j\omega}\right)^{\frac{\alpha}{2}}$. Assume that $\varphi(t)$ is the output of such a filter when white noise is the input to the filter. Davenport and Root[17] show [their Eq. (11-57)] that the minimum mean square error of prediction for $\varphi(t+\tau)$ is given by

$$\varepsilon_{\min} = \int_0^\tau |g(\eta)|^2 \, d\eta$$

where $g(\eta)$ is the impulse response function of the filter which is processing the white noise (of unit power). The impulse response function, $g(\eta)$, is defined as the Fourier transform of the transfer function, $K(\omega)$; i.e.,

$$g(\eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} K(\omega) e^{i\omega\eta} \, d\omega.$$

For the case of a filter with transfer function $K(\omega) = \left(\frac{1}{j\omega}\right)^{\frac{\alpha}{2}}$, the impulse response function is[18]

$$g(\eta) = \frac{1}{\Gamma\left(\frac{\alpha}{2}\right)} \eta^{\frac{\alpha}{2}-1} \quad . \tag{2}$$

The minimum mean square error of prediction then becomes

$$\varepsilon_{\min} = \left(\frac{1}{\Gamma\left(\frac{\alpha}{2}\right)}\right)^2 \int_0^\tau \eta^{\alpha-2} \, d\eta,$$

Thus, for a signal, $\varphi(t)$, whose power spectral density (relative to an angular frequency to be consistent with references 9 and 10) is given by

$$S_\varphi(\omega) = h|k(\omega)|^2 = h|\omega|^{-\alpha}, \; \alpha > 1, \tag{3}$$

where h is the input power spectral density (relative to ω), the minimum mean square error of prediction is expected to be

$$\varepsilon_{\min}{}^2 = \frac{2\pi h}{\Gamma^2\left(\frac{\alpha}{2}\right)} \left(\frac{\tau^{\alpha-1}}{\alpha-1}\right), \; \alpha > 1. \tag{4}$$

The factor of 2π occurs because (3) is a density relative to an angular frequency rather than a cycle frequency.

The Prediction Problem

By virtue of the fact that α is not precisely determined by the data, it was not possible to determine *the* best method of prediction. Instead, it was considered simplest by the authors to predict on the basis of an equation of the form

$$\hat{\varphi}(t+\tau) = \sum_{n=1}^{M} a_n \varphi(t - b_n\tau) \tag{5}$$

where the coefficients $\{a_n, b_n\}$ are to be determined (1) on the basis of available data and (2) "near" optimum prediction for that range of α which is consistent with the previous analysis. (By "Optimum Prediction" one means to predict with a minimum resulting mean square error.)

The first sets of $\{a_n\ b_n\}$ investigated by the authors were constructed with the assumption that the actual predicted value, $\hat{\varphi}(t + \tau)$, would satisfy an equation in which the m-th finite difference is zero. As an example, the third difference of the variable χ_n is given by

$$\Delta^3\chi_n = \chi_{n+3} - 3\chi_{n+2} + 3\chi_{n+1} - \chi_n,$$

and thus the predicted value would be given by

$$\hat{\varphi}(t+\tau) = 3\varphi(t) - 3\varphi(t-\tau) + \varphi(t-2\tau). \qquad (6)$$

In general the errors based on the finite difference method of prediction are significantly worse than optimum prediction. Figure 3 shows a plot of the ratio of finite difference prediction to optimum prediction for various orders of differencing and as a function of α. Figure 3 was obtained from (4) and (6) and references 9 and 10 for $1 < \alpha \leq 3$. The method of analysis cited in the previous references was extended for the range $3 < \alpha < 7$.

If one accepts the possibility that, out to periods of one year, the Earth's behavior is dominated by an $|\omega|^{-3}$ behavior of $S_\varphi(\omega)$ (slope 1 in Figure 2), then a second difference method is only 20 per cent worse than optimum (see Figure 3).

If, however, one accepts Brouwer's $\tau 3/2$ dependence, one can show that optimum prediction is accomplished by a simple continuation of the instantaneous rate. (This is exactly analogous to noting that, for a gambler playing at even odds, the gambler's total funds are as likely to increase as decrease during the next few plays.) The instantaneous rate, however, is not exactly measurable, and one logically retreats to a prediction based upon an equation of the form

$$\hat{\varphi}(t+\tau) = \varphi(t) + \frac{\tau}{T}(\varphi(t) - \varphi(t-T)). \qquad (7)$$

Assuming that the errors of measurement of $\varphi(t)$ and $\varphi(t-T)$ are each five milliseconds[12], the error in $\hat{\varphi}(t + \tau)$ arising from the measurement of $\varphi(t)$ and $\varphi(t - T)$ can be shown to become near to the error of optimum prediction at a T value of about 2 weeks (15×10^5 sec.) for $\tau = 1$ year.

If one accepts an $|\omega|^{-5}$ dependence for the spectral density of the fluctuations in the Earth's angular position, then one finds the variance of the second difference of the position does not exist (Figure 3). This is equivalent to saying that the secular (mean) increase in the length of the day is not a measurable quantity. Based on the more extensive data in Brouwer's[3] paper and some recent results of coral growth,[19-23] however, the secular increase in the length of the day seems well understood. If an $|\omega|^{-5}$ dependence is real, it probably does not extend to frequencies much below one cycle per century. Munk and MacDonald[24] (their Figure 8.1) indicate this behavior stops at about one cycle per 30 years. Dicke[25] has also discussed the long term fluctuations in the rotation of the Earth.

Because of the rather steep slope to the date of Figure 2 for periods longer than one year, a prediction method which is a compromise between a pure third difference and a pure second difference was also investigated. The form of the prediction studied is given by

$$\hat{\varphi}(t+\tau) = 2\varphi(t) - \varphi(t-\tau) + \frac{\tau^2}{T^2}[\varphi(t) - 2\varphi(t-T) + \varphi(t-2T)]. \qquad (8)$$

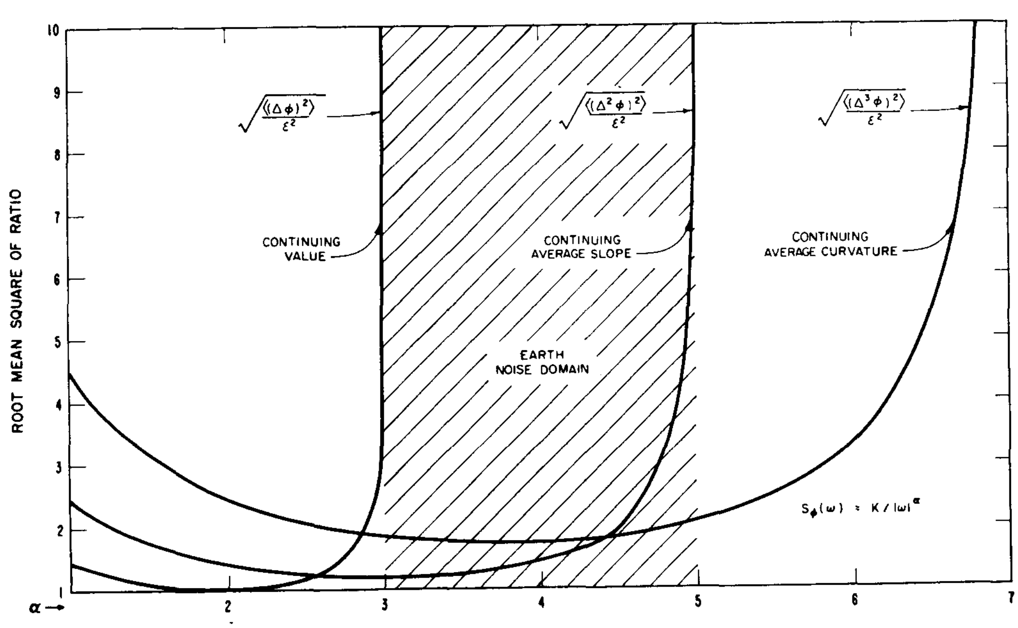

Figure 3 — Ratio of prediction based on finite differences to optimum.

Prediction Results

Because the U. S. Naval Observatory regularly supplies the National Bureau of Standards with current values of the time difference between A.1 (the U. S. Naval Observatory atomic time) and UT 2, "predictions" for the past years were computed on the basis of a second difference and also on the bases of (7) and (8) ($T = 3\tau$) using the A.1 - UT 2 data. In actual practice the frequency offset (relative to atomic time scales) is announced in the fall of the year preceding the year it is used.

The authors chose to predict the year-end value of UT 2 - UTC for the Nth year from early September (Nth year) values using one of the prediction schemes. Based on this year-end value and the desire to reduce this error to zero during the (N + 1)th year, frequency offsets, $\frac{\Delta f}{f}$, computed from each of the simple methods were predicted. The results of such calculations for each of the three prediction methods are presented in Table II for the past years and for 1967. The frequency offsets were held to integral multiples of 50 parts in 10^{10}, and if the UT 2-UTC value exceeded 0.1 second a "reset" is assumed to have occurred. While these are the current conditions which UTC satisfies, it should be remarked that these conditions were *not* actually used during the earlier part of the period covered. Thus this table should not be compared naively to actual past performance although it may be considered as an indication of what these prediction methods might yield in the future.

On the basis of the data presented in Table II, it appears that no one technique is obviously more reliable than the others. Certainly the second-difference method and the method based on (7) are simpler than (8), and may be preferred for this reason.

The predictions for 1968 are -250×10^{-10} for both the second difference method and on the basis of (7). Based on (8) one would predict -300×10^{-10} for 1968.

Table II

Prediction Results

Year	Second Difference		Equation 7		Equation 8 ($T = 3\tau$)	
	$(\Delta f/f) \times 10^{10}$	number of resets	$(\Delta f/f) \times 10^{10}$	number of resets	$(\Delta f/f) \times 10^{10}$	number of resets
1958	−150		−150			
1959	−150		−150			
1960	−150		−200	1		
1961	−150		−150			
1962	−100	1	−100			
1963	−150		−200		−150	1
1964	−250	1	−200		−200	1
1965	−250		−350	2	−250	
1966	−250	1	−300		−300	
resets/years		3/8		3/8		2/4
1967	−250		−300		−300	

CONCLUSIONS

An analysis of recent UT 2 data has shown that Brouwer's 3/2 power law is reasonable for periods shorter than one year. That is, even for short periods of time, "the observed fluctuations . . . are compatible with the hypothesis that the rate of rotation of the earth is affected by cumulative random changes" — as Brouwer[3] commented in regard to times longer than a year. Based only on these recent UT 2 data, however, a better fit to the data is a flicker noise (1/f) law affecting the rate of rotation of the earth for periods shorter than one year.

Three methods of predicting the offset frequencies for Coordinated Universal time (UTC) have been considered which are not "far" from an optimum prediction scheme. These schemes of prediction have been applied to past UT 2 data to test their reliability (Table II). Because all three schemes seem roughly equivalent, one is tempted to choose the simplest, i.e., a second-difference method of prediction.

REFERENCES

1. Jones, Sir Harold Spencer, Ann. Cape Obs., 13, Part 3, 1932.
2. Jones, Sir Harold Spencer, *M. N.*, 99, 541, 1939.
3. Brouwer, D., "A Study of the Changes in the Rate of Rotation of the Earth," *Astronomical Journal*, Vol. 57, No. 5, pp. 125-146, September 1952.
4. Markowitz, W., "The System of Atomic Time, A. 1," *Proc. of the 13th Annual Symposium on Frequency Control*, New Jersey: U. S. Army Signal Research and Development Laboratory, p. 316, May 1959.
5. Guinot, B., "Temps Atomique," *Bulletin Horaire du Bureau International de l'Heure*, Series J., No. 7, January-February 1965.
6. Hudson, G. E., "Some Characteristics of Commonly Used Time Scales," *Proc. IEEE*, Vol. 55, No. 6, June 1967.
7. Cutler, L. S., and C. L. Searle, "Some Aspects of the Theory and Measurement of Frequency Fluctuations in Frequency Standards," *Proc. IEEE*, Vol. 54, No. 2, pp. 136-154, February 1966.
8. Vessot, R. F. C., L. Mueller, and J. Vanier, "The Specification of Oscillator Characteristics From Measurements Made in the Frequency Domain," *Proc. IEEE*, Vol. 54, No. 2, pp. 199-207, February 1966.
9. Allan, D. W., "Statistics of Atomic Frequency Standards," *Proc. IEEE*, Vol. 54, No. 2, pp. 221-230, February 1966.
10. Barnes, J. A., "Atomic Timekeeping and the Statistics of Precision Signal Generators," *Proc. IEEE*, Vol. 54, No. 2, pp. 207-220, February 1966.
11. U. S. Naval Observatory and Royal Greenwich Observatory, 1967, *American Ephemeris and Nautical Almanac*, Her Majesty's Stationery Office, London, p. vii. U. S. Naval Observatory and Royal Greenwich Observatory (1961), Explanatory Supplement to the Astronomical Ephemeris and Nautical Almanac (Her Majesty's Stationery Office, London), pp. 443-453.
12. Kovalevsky, J., "Astronomical Time," *Metrologia*, Vol. 1, No. 4, October 1965.
13. Crow, E. L., F. A. Davis, and M. W. Maxfield, *Statistics Manual, with Examples Taken From Ordnance Development*, China Lake, Calif., (or Dover, 1960), U. S. Naval Ordnance Test Station, p. 242, 1955.
14. Heaviside, O., *Electromagnetic Theory*, Dover Publication, New York, pp. 128-129, 1950.
15. Carlson, G. E. and C. A. Halijak, "Approximations of Fractional Capacitors $(1/s)^{1/n}$ by a Regular Newton Process," *IEEE Trans. on Circuit Theory*, pp. 210-213, June 1964.
16. Roy, S. C. D. and B. A. Shenoi, "Distributed and Lumped RC Realization of a Constant Argument Impedance," *Journal of the Franklin Institute*, Vol. 282, No. 5, pp. 318-329, November 1966.
17. Davenport, W. B., Jr., and W. L. Root, *Random Signals and Noise*, McGraw-Hill, New York, Chap. 11, 1958.
18. Campbell, G. A., and R. M. Foster, *Fourier Integrals*, Van Nostrand, New York: p. 50, pair 521, 1948.
19. Wells, J. W., "Coral Growth and Geochronology," *Nature*, *197*, p. 948, (1963).
20. Scrutton, C. T., "Periodicity in Devonian Coral Growth," *Palaeontology*, *7*, p. 552 (1964).
21. Runcorn, S. K., "Changes in the Earth's Moment of Inertia," *Nature*, *204*, p. 823 (1964).
22. Lamar, D. L. and P. M. Marfield, "Length of Devonian Day From Scrutton's Coral Data," *J. Geophys. Res. 71*, 4429 (1966).
23. Runcorn, S. K., "Corals As Paleontological Clocks," *Sci. Am.*, Vol. 215, No. 4, pp. 26-33, October 1966.
24. Munk, W. H., and G. J. F. MacDonald, *The Rotation of the Earth*, Cambridge, 1960, Chapters 8 and 11.
25. Dicke, R. H., *The Earth-Moon System*, Plenum Press, New York: 1966, pp. 98-164.