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The relativistic redshift with 3×10^{-17} uncertainty at NIST, Boulder, Colorado, USA

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Abstract
We have estimated the relativistic redshift correction due to gravity, necessary to reference to the geoid the measurements of the new frequency standards at the National Institute of Standards and Technology (NIST) in Boulder, Colorado, USA, using a new local survey and various methods and models. We referenced the frequency offsets computed from different methods to the same geoid surface, one defined with respect to the current best estimate of an ideal mean-Earth ellipsoid. The new fractional frequency results are (1) -1797.61×10^{-16}, based on the global gravitational model EGM96; (2a) -1798.72×10^{-16}, based on the regional, high-resolution geoid model G96SSS; (2b) -1798.49×10^{-16}, based on the regional, high-resolution geoid model G99SSS; and (3) -1798.91×10^{-16}, based on the value for the geopotential number provided in the National Geodetic Survey's data sheet for the NIST reference marker. The minus sign implies that clocks run faster in the laboratory in Boulder than a standard clock located on the geoid. The values from (2b) and (3) are expected to be the most accurate and are also independent. Based on these results, we estimate the frequency shift at the reference point at NIST to be -1798.7×10^{-16}, with an estimated standard uncertainty of $\pm 0.3 \times 10^{-16}$.

1. Theoretical background

This is a continuation of earlier work [1, 2], where we used coordinates accurate to about 1 m. Now, using a recent GPS survey, we have coordinates that should be accurate to 20 cm or better. In addition, we also used a new regional model of the geoid available for the continental US in our computations.

With the advent of new primary frequency standards whose uncertainties approach 1 part in 10^{15}, and the potential for clocks with smaller uncertainties or greater stabilities, there is a need for improved estimates of the relativistic redshift. This is an effect predicted by the theory of relativity as the sum of a special and a general relativistic effect. The theory of general relativity predicts that a clock farther away from the Earth runs faster relative to a clock closer to the Earth. The effect is proportional to the gravitational potential due to the Earth, the geopotential. In relativity, the geopotential is defined by the convention giving the potential a negative value, approaching zero as a particle moves towards infinity away from an attracting body. Geodesy uses the sign convention for geopotentials opposite to that used in relativity theory. In geodesy, all potentials are positive, so that a higher potential would generally be closer to the Earth. In this paper we will use the geodetic convention, in which all geopotentials are positive.

A second effect in relativity enters, the so-called second-order Doppler shift of special relativity, in which a standard clock runs slower as it moves faster, relative to a clock at rest with the observer. The rotation of the Earth, therefore, gives rise to a centripetal potential that also changes the clock's frequency. We differentiate between the potential due to gravitation and that due to gravity: the former arises from the presence of attracting masses only, the latter contains in addition the centripetal potential due to the Earth's rotation [3, section 2-1]. It is the gravity potential that we need to consider here, therefore the term 'gravitational redshift' is somewhat misleading and has been avoided herein.

A primary frequency standard that contributes to international atomic time (TAI) must be corrected to run at the frequency at which clocks would run on the Earth's geoid, a surface of constant gravity potential that approximates mean sea level in a well-defined way. It is therefore necessary to determine the difference in gravity potential ($W_0 - W_P$), between the geoid (subscript 0) and the location of a primary frequency standard (subscript P), in order to correct for this frequency offset, according to [4]

$$\frac{f_0 - f_P}{f_0} = \frac{\Delta f}{f_0} = \frac{W_P - W_0}{c^2}, \tag{1}$$

where c denotes the speed of light. Note that if the point P is above the geoid, we generally have $W_P < W_0$, using the convention in which potentials are positive. Hence, f is negative in this case, since this clock correction would make the clock in Boulder run slower, to match the rate of a standard clock on the geoid.

The geopotential number $C = W_0 - W_P$ [3, p 56] is given by

$$C = W_0 - W_P = \int_{H=0}^{H=H_P} \boldsymbol{g} \cdot \mathrm{d}\boldsymbol{H}, \tag{2}$$

where g is the magnitude of the gravity acceleration vector, and $\mathrm{d}H$ is the length increment along the positive upward plumb line. The path-independent line integral in equation (2) starts from a reference equipotential surface whose gravity potential is W_0 (on which every point has orthometric height equal to zero) and ends at the station location where $W = W_P$ and $H = H_P$. Although the reference equipotential surface can be defined unambiguously through a prescribed value of W_0, such a definition has limited practical value for the physical realization of this surface, since absolute potentials cannot be measured. In theory, any equipotential surface of the gravity field is a suitable reference surface for orthometric heights worldwide. However, the human conception of 'heights' and historic practices make it convenient for such a reference surface (vertical datum) to be 'close' to the mean sea surface (MSS). Historically, the vertical datum of a country or a set of countries has been realized by prescribing a certain value to the orthometric height or the geopotential number of some tide gauge station(s). The geopotential numbers and orthometric heights of other points could then be determined using spirit levelling and gravity measurements, through the evaluation of a discrete counterpart (summation) of equation (2) [3, chapter 4].

The presence of a quasi-stationary (i.e. non-vanishing through averaging over long time periods) component within the dynamic ocean topography (DOT) results in departures of the MSS from an equipotential surface ranging geographically between -2.1 m and $+1.3$ m, approximately. Due to these departures (and in some cases due to additional considerations related to mapping applications), different vertical datums refer to different equipotential surfaces. Therefore, given a datum-dependent value of C, the determination of $\Delta f/f$ with respect to a unique equipotential surface requires the estimation of that datum's offset from that unique equipotential surface. A unique equipotential surface—the geoid—that closely approximates (in some prescribed fashion) the MSS has to be defined and realized through the operational development of models [5, 6]. There exist global geoid models, developed through a combination of satellite tracking data, surface gravimetry, and satellite altimetry. Such a state-of-the-art model, complete to degree and order 360, corresponding to a half-wavelength resolution of $\sim$55 km at the equator, is EGM96 [7]. The resolution of such global models is limited primarily by the available surface gravimetric data used in their development. Detailed (i.e. higher-resolution) local or regional geoid models are developed by incorporating the information contained within dense gravity and topography data into a global geoid model. This adds high spatial frequency details to the broader geoid features represented within a global model. G96SSS [8] and the updated G99SSS [9] are such regional geoid models for the US. Global and regional geoid models can also be used to estimate the geopotential number C, given the geocentric coordinates of the point P.

We distinguish, therefore, two general approaches for the computation of C and hence $\Delta f/f$: one based on spirit levelling and gravity observations, and another based on the use of geoid models, either global or regional/local. Each approach has its own advantages and disadvantages, and its own error characteristics. It is important, however, to recognize that each computational method and/or model used may yield a result that refers to a different equipotential surface. Since the various reference surfaces may be offset by several decimetres, estimation of their relative offsets becomes important if one desires to compare the various results at the level of a decimetre or less.

It is useful to recall the correspondence between the approximate magnitude changes of H, C, and $\Delta f/f$. Near the Earth's surface $g \approx 9.8\,\mathrm{m\,s^{-2}}$, and since $c = 299\,792\,458\,\mathrm{m\,s^{-1}}$, a change in H by 1 m implies roughly a $9.8\,\mathrm{m^2\,s^{-2}}$ change in C, and therefore a change in $\Delta f/f$ of -1.1×10^{-16}. To support present frequency standards with uncertainties near $\pm 1 \times 10^{-15}$ it is sufficient that $\Delta f/f$ be computed with an error not exceeding $\pm 1 \times 10^{-16}$. Therefore, the total error in an absolute determination of the geopotential number C, consisting of the error in W_0 (absolute) and the error in $W_0 - W_P$ (relative), should not exceed $\sim\pm 9.8\,\mathrm{m^2\,s^{-2}}$, or equivalently, the absolute orthometric height H_P of our station should be determined to better than ± 1 m.

2. Computational aspects

In the following paragraphs we discuss the specific computations involved in the estimation of $\Delta f/f$, according to three methods. The first two methods are based on geoid model information, global and regional respectively, while the third method is based on spirit levelling and gravity observations. The first two methods share some long-wavelength errors, but the third method is independent of the other two.

2.1. Mean-Earth ellipsoid

The concept of a mean-Earth ellipsoid [3, section 2-21] is of central importance in gravimetric geodesy and in our specific application. This purely mathematical construct is a rotating ellipsoid of revolution (i.e. biaxial), whose surface is also an equipotential surface of its gravity field. The gravity potential on its surface is presupposed to equal the gravity potential on the geoid. Four parameters are necessary and sufficient to define uniquely its size, shape, rotation, and gravity field. One may presuppose that these parameters are numerically equal to the corresponding parameters of the real Earth. Then, the departures of the geoid from such an 'ideal' ellipsoid, called geoid undulations and denoted by N, will have a vanishing zero-degree term, i.e. their average over the whole Earth equals zero. Therefore, by suppressing the zero-degree term in the spherical harmonic expansion of N, one obtains 'automatically' geoid undulations that refer to this 'ideal' mean-Earth ellipsoid, without the need to know the specific scale (semi-major axis) of this ellipsoid. Specification of the scale and the gravity field of this 'ideal' ellipsoid require numerical specification of its defining parameters. These values can be determined only from analyses of various geodetic observations and, therefore, contain random and possibly systematic errors. Here, we will define this 'ideal' mean-Earth ellipsoid, in a tide-free system [10], by adopting the current best estimates for the values of the following parameters [11]:

$$\begin{aligned}
&\text{Equatorial radius: } a = 6\,378\,136.46\,\text{m},\\
&\text{Flattening: } f = \frac{1}{298.257\,65},\\
&\text{Geocentric gravitational constant:}\\
&\quad GM = 3.986\,004\,418 \times 10^{14}\,\text{m}^3\,\text{s}^{-2},\\
&\text{Mean rotational speed: } \omega = 7\,292\,115 \times 10^{-11}\,\text{rad}\,\text{s}^{-1}.
\end{aligned} \tag{3}$$

We should emphasize here that the mean-Earth ellipsoid defined by the above four values is only as 'ideal' as the current accuracy of these values allows. These values are constantly being refined through improved geodetic determinations. GM is currently determined most accurately from analyses of laser ranging data acquired on high altitude geodynamic satellites, and so is f (more precisely the second-degree zonal gravitational coefficient, J_2, from which f can be derived). ω is deduced most accurately from very long baseline interferometry. The equatorial radius a is currently determined best from analyses of satellite radar altimeter data over the oceans. Jekeli discusses the fundamental concepts involved in the determination of the best estimate of the equatorial radius [6]. The adopted defining values of equation (3) imply a value of the gravity potential on the geoid equal to

$$W_0 = 62\,636\,856.88\,\text{m}^2\,\text{s}^{-2}, \tag{4}$$

with an estimated uncertainty of $\pm 1.0\,\text{m}^2\,\text{s}^{-2}$ [11].

We note that this uncertainty implies a limitation on the realization of the second. With this uncertainty on the geoid, the second can be realized to no better than $\pm 1 \times 10^{-17}$. Standards can be compared to better than this, however. The limitation for comparison due to the relativistic redshift is the uncertainty in determining the offsets for the standards of the gravity potential at their locations, from a fixed surface of gravity potential. We will find in this paper an uncertainty of 3×10^{-17} for standards at NIST, Boulder, due to the offset of the gravity potential from the current best estimate of the geoid.

2.2. Reference point

This paper compares gravity potential based on models with gravity potential based on spirit levelling and gravity measurements. To accomplish this we used two different markers at NIST. The US National Geodetic Survey surveyed three points on the NIST campus in September 2000 [12]. We used one of these, identified as DMA (it was first surveyed by the Defense Mapping Agency in 1977 using the Transit satellite system), to obtain geocentric Cartesian coordinates for use in evaluating models. This point, DMA, is located on the flat roof above the fourth floor at NIST, Boulder. There is also a point on the side of the second floor of the building designated Q407. This point is part of the North American Vertical Datum 1988 (NAVD88) network of spirit levelling and gravity measurements. Since most of the frequency standards at NIST are on the second or third floors, it is more convenient to evaluate the relativistic redshift at an elevation equal to that of point Q407 rather than that of the DMA point.

Q407 is approximately 18.6 m distant, horizontally, from a point directly below the DMA point. We measured the Q407 point as 9.903 m vertically below the DMA point. For evaluating the global and regional models, we used the coordinates of a point 9.903 m below the DMA point. We refer to this, rather fictitious, point as P in what follows. The change in gravity potential from point P to Q407 should be small due to a horizontal shift of about 18.6 m. It would be exactly zero, of course, if we moved along an equipotential line. Since we are simply shifting by maintaining the change in coordinate height as zero, we can at least be certain that the relativistic redshift at P should agree with the value at the Q407 point to better than 10^{-18}, i.e. 1 cm in terms of orthometric height.

2.3. Method 1

For method 1 we evaluated the EGM96 global model for the gravitational potential, i.e. the potential due to the Earth's attracting mass. To this we added the centripetal potential. This sum yields the gravity potential W_P given by

$$W_P = V(r_P, \theta_P, \lambda_P) + \Phi(r_P, \theta_P), \tag{5}$$

where $V(r_P, \theta_P, \lambda_P)$ is the gravitational potential, $\Phi(r_P, \theta_P)$ the centripetal potential, and $(r_P, \theta_P, \lambda_P)$ are geocentric radius, geocentric co-latitude (90° minus latitude), and longitude, respectively, at point P. One has [3, chapter 2]:

$$V(r_P, \theta_P, \lambda_P) = \frac{GM}{r_P}\left[1 + \sum_{n=2}^{\infty}\left(\frac{a}{r_P}\right)^n \sum_{m=-n}^{n} \bar{C}_{nm}\bar{Y}_{nm}(\theta_P, \lambda_P)\right], \tag{6}$$

with

$$\bar{Y}_{nm}(\theta_P, \lambda_P) = \bar{P}_{n|m|}(\cos\theta_P) \times \begin{cases}\cos m\lambda_P, & \text{if } m \geqslant 0,\\ \sin|m|\lambda_P, & \text{if } m < 0,\end{cases} \tag{7}$$

and

$$\Phi(r_P, \theta_P) = \tfrac{1}{2}\omega^2 r_P^2 \sin^2\theta_P. \tag{8}$$

$\bar{P}_{nm}(\cos\theta_P)$ is the fully-normalized associated Legendre function of the first kind [3, sections 1-11, 1-14], of degree n and order m, and $\bar{C}_{nm}$ are the (unitless) fully-normalized potential coefficients. The numerical values of $\bar{P}_{nm}(\cos\theta_P)$

were evaluated here using (a modification of) the routine LEGFDN, originally written by Colombo [13, p 131]. EGM96 [7] provides, currently, the most accurate estimate of a set of $\bar{C}_{nm}$, complete to degree and order 360.

The geocentric Cartesian coordinates of our reference point, P, at 9.903 m below the DMA marker for NIST Boulder in ITRF94 are

$$\begin{aligned} X_P &= -1\,288\,394.075\,\mathrm{m}, \\ Y_P &= -4\,721\,673.869\,\mathrm{m}, \\ Z_P &= 4\,078\,630.782\,\mathrm{m}. \end{aligned} \tag{9}$$

These coordinates are expected to be accurate to 20 cm or better.

We converted these coordinates to $(r_P, \theta_P, \lambda_P)$ and evaluated equation (6) truncated, of course, to maximum degree and order 360, and equations (8) and (5). We obtained

$$W_P = 62\,620\,700.75\,\mathrm{m^2\,s^{-2}}, \tag{10}$$

which implies, due to equations (1) and (4),

$$\frac{\Delta f}{f} = -1797.61 \times 10^{-16}. \tag{11}$$

There are two types of errors associated with the use of EGM96: (a) errors of commission due to the fact that the coefficients $\bar{C}_{nm}$ are imperfectly known; and (b) errors of omission due to the truncation to degree 360, of the infinite series in (6). The commission error of EGM96 has two components. The first one (long-wavelength component) can be computed rigorously from the error covariance matrix that accompanies the part of the model up to degree and order 70. The second component, corresponding to degrees 71–360, is only available in terms of a global root mean square (RMS) estimate that does not account for the specific geographic location of our station. This estimate can be computed from the standard deviations of the EGM96 coefficients above degree 70.

The omission error of EGM96 can be estimated based on some theoretical model describing the decay of the gravitational spectrum of the Earth globally. Such a computation also yields a global RMS value, without geographic specificity.

Details on the EGM96 geoid error assessment can be found in [7, sections 7.3.3.1 and 10.3.2]. Based on that assessment, we estimate the total (commission plus omission) geoid undulation error of EGM96 to be approximately ± 0.6 m, in an RMS sense over the conterminous USA. To obtain this estimate we proceeded as follows. Over the conterminous USA the RMS geoid error obtained from rigorous error covariance propagation up to degree 70 is ± 0.26 m, while the quadratic summation of the EGM96 coefficient standard deviations (i.e. neglecting coefficient error correlation) gives ± 0.19 m. We used the ratio of these two quantities (~ 1.4) to scale the higher-degree (71–360) commission error, as well as the omission error beyond degree 360, thereby ‘converting’ these two global RMS estimates to corresponding values that are more representative of our specific area. We fully recognize here that this approach is only approximate. Considering the mountainous terrain of the region around our station (which is expected to increase, primarily, the omission, but also the commission error of the model), it is not unreasonable to estimate the EGM96 undulation error at our location, P, to be between ± 0.6 m and ± 1.0 m. Pavlis *et al* [14] reported an evaluation of EGM96 using independent data. Over 5168 benchmarks distributed over the conterminous USA, the standard deviation of the differences between the EGM96 geoid undulation estimates and independent estimates obtained from GPS positioning and spirit levelling is approximately ± 0.40 m, which verifies that our present error assessment is not unreasonable (it may actually be slightly pessimistic). This EGM96 geoid error estimate for our site implies that the error of the $\Delta f/f$ value given in (11) is not expected to exceed $\pm 1 \times 10^{-16}$.

2.4. Method 2

A significant reduction of the omission error encountered with EGM96 can be effected through the use of a detailed regional geoid model. We have used the coordinates obtained from the new survey both with the $2' \times 2'$ gravimetric geoid model G96SSS [8] (which we also used in [2] along with the old survey coordinates), and the updated G99SSS [9] model. We proceeded as follows.

1. We first computed the geoid undulations implied by EGM96 (to degree 360) at the $2' \times 2'$ grid nodes of G96SSS, using [15]

$$N_{\mathrm{EGM96}}(\theta_P, \lambda_P) = \frac{GM}{r_P \gamma_P} \sum_{n=2}^{360} \left(\frac{a}{r_P}\right)^n \times \sum_{m=-n}^{n} \bar{C}^*_{nm} \bar{Y}_{nm}(\theta_P, \lambda_P) + \frac{\Delta g_B}{\bar{\gamma}} H. \tag{12}$$

In equation (12), γ_P is normal gravity at P, $\bar{\gamma}$ is an average value of normal gravity between the projections of P on the ellipsoid and telluroid, Δg_B is the Bouguer gravity anomaly, and $\bar{C}^*_{nm}$ denote the potential coefficient remainders after the even zonal reference coefficients of the ellipsoidal (normal) field are subtracted from $\bar{C}_{nm}$. A detailed discussion of the underlying theory and the details of the numerical implementation of equation (12) can be found in [15]. Notice that the geoid undulations from (12) refer to our current best estimate of the ideal mean-Earth ellipsoid.

2. Let us denote as xx either 96 or 99, for the two detailed regional geoid models. We subtracted the GxxSSS geoid undulations, $N_{\mathrm{G}xx\mathrm{SSS}}$, exactly as these are given on the distributed CD, from the undulations computed from equation (12). The average value of these differences over the domain of GxxSSS provides an estimate of the shift that is required in order to reference the $N_{\mathrm{G}xx\mathrm{SSS}}$ values to an ideal mean-Earth ellipsoid. For both G96SSS and G99SSS, we found

$$N_{\mathrm{G}xx\mathrm{SSS}}(\text{ideal}) = N_{\mathrm{G}xx\mathrm{SSS}} + 0.40\,\mathrm{m}. \tag{13}$$

3. From the geocentric Cartesian coordinates of P, we computed its geodetic coordinates with respect to an ellipsoid defined by the (a, f) values given in equation (3). This yields

$$\begin{aligned} \varphi_P &= 39°59'42.861'', \\ \lambda_P &= 254°44'14.541'', \\ h_P &= 1634.421\,\mathrm{m}. \end{aligned} \tag{14}$$

Therefore the orthometric height H_P, as implied by the geocentric positioning data (h_P) and the N_{GxxSSS}(ideal) value, is

$$H_P(\text{ideal}) = h_P - N_{GxxSSS}(\text{ideal}). \tag{15}$$

The evaluation of equation (15) requires N_{GxxSSS}(ideal) at the point P. This value was obtained using bicubic spline interpolation of the grids on which the G*xx*SSS values are given, $2' \times 2'$ for G96SSS and $1' \times 1'$ for G99SSS.

4. From H_P(ideal) we computed the geopotential number of P using Helmert's equation [3, equation (4-26)]:

$$C = H_P(\text{ideal})[g_P + 0.0424 H_P(\text{ideal})], \tag{16}$$

where C is in g.p.u. (1 g.p.u. $= 10\,\text{m}^2\,\text{s}^{-2}$), H_P(ideal) in km, and g_P (the value of gravity acceleration at P) is in Gal (1 Gal $= 10^{-2}\,\text{m}\,\text{s}^{-2}$). $g_P = 9.796\,022\,\text{m}\,\text{s}^{-2}$, a value obtained from the National Geodetic Survey's (NGS) data sheet for the NIST reference marker. The approximations involved in equation (16) introduce errors of only a few centimetres [3, p 169]. More accurate formulations in place of equation (16) (which also take into account the terrain correction) are possible [3, p 169]; however, such formulations were not implemented here.

2.4.1. Method 2a. Using G96SSS, the model we used previously in [2], we now obtained with the new coordinates

$$C = 16\,166.08\,\text{m}^2\,\text{s}^{-2}, \tag{17a}$$

which yields, from equation (1),

$$\frac{\Delta f}{f} = -1798.72 \times 10^{-16}. \tag{18a}$$

Unlike EGM96, the G*xx*SSS regional geoid models are not accompanied by propagated error estimates. Their accuracy has been assessed only through comparisons with independent geoid undulation estimates obtained from GPS positioning and levelling observations [8, 9]. Based on this uncertainty assessment, we estimate the error in N_{G96SSS} to be approximately 0.20 m. Considering also a 0.20 m error in the ellipsoidal height h_P; this implies an error for the $\Delta f/f$ value given in (18*a*) of 0.31×10^{-16}.

2.4.2. Method 2b. Using the new geoid model G99SSS, we now obtained with the new coordinates:

$$C = 16\,164.01\,\text{m}^2\,\text{s}^{-2}, \tag{17b}$$

which, from equation (1), yields

$$\frac{\Delta f}{f} = -1798.49 \times 10^{-16}. \tag{18b}$$

From comparisons with independent geoid undulation estimates obtained from GPS positioning and levelling observations [9] we estimate the error in N_{G99SSS} to be approximately 0.18 m. Considering the 0.20 m error in the ellipsoidal height h_P, this implies an error for the $\Delta f/f$ value given in (18*b*) of 0.30×10^{-16}.

2.5. Method 3

The $\Delta f/f$ values given in equations (11), (18*a*) and (18*b*) were computed based on a global model and the two regional geoid models, respectively. We turn now to the $\Delta f/f$ computation from spirit levelling and gravity measurements, as shown in equation (2). We performed this computation as follows.

1. From the NGS data sheet for our reference marker we obtained its dynamic height [3, p 163] value $H_P^{\text{dyn}} = 1649.034$ m. This value is related to the geopotential number of P by [3, equation (4-9)]:

$$H_P^{\text{dyn}} = \frac{C}{\gamma_0}, \tag{19}$$

where $\gamma_0 = 9.806\,199\,\text{m}\,\text{s}^{-2}$ is the value of normal gravity on the GRS80 reference ellipsoid, at $\varphi = 45°$ (this value was taken from the NGS data sheet, exact to the digits given there). From equation (19) we therefore computed

$$C_{\text{NAVD88}} = 16\,170.76\,\text{m}^2\,\text{s}^{-2}, \tag{20}$$

where the subscript 'NAVD88' emphasizes the fact that this value refers to the equipotential surface that passes through the origin point (Father Point/Rimouski, located in Quebec) of the North American Vertical Datum 1988 [16].

To estimate the offset between the NAVD88 reference equipotential surface and the 'ideal' geoid surface, we proceeded, in principle, as described in [17]. From NGS [Milbert, Private communication, 1998] we have available a set of 5168 GPS/levelling points distributed (not evenly) over the conterminous USA. The geodetic coordinates and the NAVD88 orthometric heights of these points are given. The Cartesian coordinates were obtained from GPS positioning and are given with respect to the ITRF94(1996.0) reference frame. From these Cartesian coordinates, the geodetic coordinates (φ, λ, h) have been computed with respect to the GRS80 reference ellipsoid and were provided to us. For our computation, however, we need h to be defined with respect to the 'ideal' values of (a, f) given in equation (3). We performed this conversion using [18, equation (64)]:

$$h(\text{ideal}) = h(\text{GRS80}) - w\Delta a + \frac{a(1-f)}{w}\sin^2\varphi\Delta f, \qquad w = (1 - e^2\sin^2\varphi)^{1/2}, \tag{21}$$

where

$$\Delta a = 6\,378\,136.46\,\text{m} - 6\,378\,137\,\text{m}, \qquad \Delta f = \frac{1}{298.257\,65} - \frac{1}{298.257\,222\,101}. \tag{22}$$

Using equation (12) we computed the EGM96-implied geoid undulations, N_{EGM96}, at the locations of these 5168 points, with respect to an ideal mean-Earth ellipsoid, in the tide-free system. We then formed the differences

$$d = h(\text{ideal}) - H_{\text{NAVD88}} - N_{\text{EGM96}}. \tag{23}$$

2. The mean value of d over the 5168 points provides an estimate of the offset between the NAVD88 reference surface

and the 'ideal' geoid surface. For reasons explained in [15], we computed this mean value in three ways:

using all 5168 points,

$$\Rightarrow \bar{d}_1 = -0.448\,\mathrm{m};$$

using 2067 points whose distance is not less than 25 km (thinned set),

$$\Rightarrow \bar{d}_2 = -0.477\,\mathrm{m};$$

using 438 points of the thinned set where $H < 100\,\mathrm{m}$

$$\Rightarrow \bar{d}_3 = -0.235\,\mathrm{m}.$$

Rapp [15, p 286] discusses the reason for the value $\bar{d}_3$ to be considered more reliable than the other two. On the other hand, $\bar{d}_3$ is computed on the basis of only a subset of the total number of available points. Estimating the 'best' value of $\bar{d}$ depends on the 'relative weights' that one is willing to assign to these three estimates. We adopted the value $\bar{d} = -0.300\,\mathrm{m}$ as our current 'best' estimate of the offset between the NAVD88 reference equipotential surface and the geoid surface. The minus sign implies that the equipotential surface passing through the origin of NAVD88 is below the geoid surface that is realized through the EGM96 model, when the latter is referenced to our current best estimate of a mean-Earth ellipsoid. Smith and Milbert report a value of $\bar{d} = -0.314\,\mathrm{m}$ with an uncertainty of $\pm 0.156\,\mathrm{m}$ [8]. Their value is based on comparisons over 2951 GPS/levelling points over the conterminous USA and agrees very well with our estimate. We should also mention that in the above analysis the permanent tide effect was consistently accounted for. All three quantities in equation (23) were expressed in the tide-free system.

3. The offset $\bar{d}$ can now be input to equation (16), in the place of $H_{\mathrm{P}}(\text{ideal})$, to estimate the correction $\mathrm{d}C$ necessary to convert C_{NAVD88} to $C(\text{ideal})$. We find

$$\mathrm{d}C = -2.94\,\mathrm{m^2\,s^{-2}} \quad \Rightarrow \quad C(\text{ideal}) = 16\,167.82\,\mathrm{m^2\,s^{-2}}, \tag{24}$$

which implies

$$\frac{\Delta f}{f} = -1798.91 \times 10^{-16}. \tag{25}$$

Errors in the estimate of $\Delta f/f$ given in (25) arise from errors in the NAVD88 dynamic height value provided in the NGS data sheet for our reference marker, and errors in our estimation of the NAVD88 datum offset. The NGS data sheet for our reference marker contained no error estimates, other than the designation that 'first order, class II' levelling was performed to determine our station's height. Zilkoski *et al* [16] discuss a comparison of NAVD88 heights with corresponding independent estimates from Canadian levelling observations over the USA–Canada border. Over 14 points the maximum difference found was 0.11 m. This value does not necessarily apply to our station; nevertheless (and in lieu of more precise information) a reasonable estimate of our station's dynamic height error may be about $\pm 0.15\,\mathrm{m}$. Considering an error of 0.20 m in our estimate of the NAVD88 datum's offset, we conclude that the $\Delta f/f$ value given in (25) is probably accurate to 0.28×10^{-16}.

Table 1.

Method	Redshift parts in 10^{-16}	Uncertainty parts in 10^{-16}
(1) EGM96	−1797.61	0.70
(2a) G96SSS	−1798.72	0.31
(2b) G99SSS	−1798.49	0.30
(3) Levelling/gravity	−1798.91	0.28

3. Combined redshift estimate

The results from the three methods are summarized in table 1.

Method (3) is independent of the others. We accept G99SSS as an update to G96SSS and use only the results from methods (2b) and (3) to determine our final result. These differ by 0.42×10^{-16}, while our estimated errors from table 1 imply a 0.41×10^{-16} standard uncertainty for this difference. Averaging methods (2b) and (3) to evaluate $\Delta f/f$, we estimate its value and uncertainty for our reference marker to be

$$(-1798.70 \pm 0.3) \times 10^{-16}. \tag{26}$$

We note that the uncertainty here is larger than our previous estimate of 0.2×10^{-16}. We consider this uncertainty estimate to be perhaps more realistic.

We should mention that we have not accounted here for luni-solar tidal effects. At this level of accuracy the effects of (at least) the semi-diurnal lunar tide M_2 (and possibly of other constituents) must be considered. One should, therefore, interpret our $\Delta f/f$ result as an average value over multiples of the periods of the main tidal constituents. In practice, however, there may be a number of different periods with components at this level. Water table variations and barometric pressure may also be significant.

Loading effects involve the deformation of the crust and the shifting of equipotential surfaces due to time-varying loads (e.g. due to ocean tides, atmospheric mass redistribution, etc). Ocean loading effects for inland stations away from the coastline (such as the NIST reference point) are not expected to exceed a few millimetres in magnitude.

4. Summary and future prospects

Based on our work, it appears that the existing measurements and models of the Earth's gravity field may not support estimates of the relativistic redshift correction to better than the 10^{-17} level for frequency standards on the Earth. Since this number contributes to the error budget of a primary frequency standard in an RMS sense, this implies that a primary frequency standard in an Earth-bound laboratory will have difficulty contributing to TAI at better than the 10^{-16} level. In the next decade it seems reasonable to expect frequency standards to reach accuracies challenging our current accuracy in the determination of the redshift correction. It may be important to establish standard ways of computing the relativistic redshift, so that standards are effectively compared using the same geoid.

There is some discussion in the timing community about putting primary frequency standards in space. The advantage is that it would be easier to compare frequency among terrestrial labs. One disadvantage is that it would be difficult if not impossible to service the standard. Most laboratory standards, to date, have required, or at least benefited from, having researchers available to work with the standards. There are experiments in progress that will launch cold-caesium standards with state-of-the-art accuracies on the International Space Station (ISS). Two of these are the European project, Atomic Clock Ensembles in Space (ACES) [19], and the US project, Primary Atomic Reference Clocks in Space (PARCS) [20]. We expect it to be easier to determine the gravity potential on the ISS or other Earth orbiters, because of the distance from the Earth. The higher spatial frequencies in the geopotential attenuate as one moves away from the Earth.

Currently, two geopotential mapping missions are expected to support a significant advance in the knowledge of the Earth's geopotential: NASA's Gravity Recovery And Climate Experiment (GRACE) [21], and ESA's Gravity Field and Steady-State Ocean Circulation (GOCE) missions [22, 23]. GRACE was launched in March 2002, and promises to deliver centimetre-level geoid undulation accuracy with a half-wavelength resolution of 200 km to 300 km. GOCE (scheduled for launch in about 2006) is expected to improve the resolution even further, allowing centimetre-level geoid undulation accuracy down to $\sim$80 km resolution. The global geopotential models expected from these missions, in combination with locally available detailed surface gravity and topography data, may permit point geoid undulation determination approaching centimetre-level accuracy. In addition, radar altimeter data from satellites such as TOPEX/Poseidon and its follow-on Jason-1 [24], in combination with the global geopotential models from GRACE and GOCE, should permit improvements in the determination of the equatorial radius of the mean-Earth ellipsoid, which directly affects the accuracy of W_0. These advances may permit determination of $\Delta f/f$ accurate to a few parts in 10^{18}.

The GRACE and GOCE missions will also probably improve the accuracy with which TAI can be transferred to astronomical timescales. Since the IAU resolution B1.9 (2000), the rate of terrestrial time (TT), the astronomical timescale on the geoid, is a defined offset from the Earth-centred astronomical timescale, TCG. This offset defines a geoid relative to the gravity potential at the Earth's centre. Operationally, one removes estimated frequency offsets from TAI then identifies that corrected rate with TT. This process equates whatever *de facto* geoid TAI is referenced to with the geoid defined by the IAU resolution. Now, TAI is an operational timescale. Also, the value of any potential cannot be measured, but is measurable only as a difference from one place or time to another. Hence, there is some error in identifying the geoid to which TAI operationally refers, with the defined equipotential surface that references TT. Any inaccuracy in this identification must pass into the transfer from TAI to astronomical measurements.

The GRACE, and later the GOCE missions, in conjunction with precise satellite altimetry data, are expected to have a direct impact on this problem, since

(a) they will allow a better determination and realization of the geoid, through better estimation of DOT (the separation between the marine geoid and the sea surface). This would permit a more accurate determination of the rate offset between TT and TCG.

(b) GRACE will permit certain temporal geoid variations to be monitored, which appears to be one of the main considerations relevant to the definition and realization of TT.

On the other side, development of frequency standards accurate to 10^{-17} or better may provide one possibility for the verification and error calibration of geopotential differences estimated from data acquired (in part) from the GRACE and GOCE missions. This could be attempted following ideas such as those proposed originally by Bjerhammar [4]. In addition, frequency standards of such high accuracy, located on different continents, provide an alternative technique well recognized among geodesists for connecting different vertical datum points. While there is promise for standards of such accuracies, methods for transferring such time and frequency measurements appear to be lacking. The current best time transfer methods appear to be at the level of 100 ps to 200 ps stability, or about (1–2) $\times$ 10^{-15} frequency transfers at 1 day [25–28]. In conclusion, it appears that technology advances in the development of frequency standards and advances in gravity field determination over the next few years are expected to benefit both disciplines in complementary ways.

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