Dynamics of a Simple Maser Model

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A simple maser model consisting of a single-mode field coupled to N identical two-level atoms is presented. The assumption of negligible statistical correlation between the atoms and the field, permits the Heisenberg equations of motion to be replaced by a self-consistent set of ordinary nonlinear differential equations. Relaxation terms and an energy source are introduced phenomenologically. The resulting equations exhibit a threshold, stable and unstable steady states, relaxation oscillations and, sometimes, spiking. The dynamic equations for the hydrogen maser and the rate equations of laser theory are derived as special cases. Also discussed are the maser amplifier, locking of a maser to an external signal, and the effects of cavity thermal noise.

INTRODUCTION

The purpose of this article is to present a theory which describes many of the physical phenomena associated with masers or lasers, yet is simple enough to be worked with by students. It can be used as a foundation for building more complex theories to describe actual devices. Alternatively, it can be used to introduce students to the behavior of a simple nonlinear system by working with a physically interesting example.

The work presented here is an outgrowth of a dynamical theory for the hydrogen maser. Although developed more or less independently, the ideas and most of the results of this paper exist in the literature somewhere. However, to obtain the coverage given here several papers must be studied, most of them encumbered with complications. We mention some of this earlier work in passing, but our development is entirely self-contained.

The article divides roughly into two parts: the derivation of the equations of motion, and the solution of those equations. In Sec. I, we delineate our model in terms of a fully quantum-mechanical Hamiltonian. We then develop our equations so as to bring out the physical origin of the various terms and the underlying assumptions.

The remaining sections describe several easily obtained solutions which illustrate various features of maser behavior. The emphasis is toward simple examples of the mathematical techniques, not generality. Section II describes the character of the solutions. Section III describes two limiting cases and relates them respectively to the hydrogen maser and the rate equations of laser theory. The behavior of a maser under the influence of an external signal is discussed in Sec. IV, and a bit of noise theory is given in Sec. V.

I. EQUATIONS OF MOTION

The Hamiltonian for our maser model is

$$\mathcal{H} = \hbar\left[\omega_c a^\dagger a + \sum_j \tfrac{1}{2}\omega_a \sigma_j^z + \sum_j b(\sigma_j^- a + \sigma_j^- a^\dagger)\right].$$

$$(1)$$

The first term represents the Hamiltonian for the radiation field, which we assume is confined to a resonant cavity. We assume further that only one mode of the cavity is of concern to us. Hence, we can represent the radiation field as a single quantum-mechanical harmonic oscillator of frequency ω_c with creation and annihilation operators $a^\dagger$ and a obeying $[a, a^\dagger] = 1$. The second term represents the Hamiltonian of N identical atoms indexed by j. We assume that only two stationary states of each atom are involved in maser action. Hence, we can represent an atom by Pauli spin matrices:

$$\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

each atom having resonant frequency ω_a. The remainder of the Hamiltonian represents the interaction between the atoms and the field. An expression for the coupling constant b (assumed real) is given in Eq. (A8) of the Appendix. We have assumed the atoms are so confined that they

all see the same amplitude of the field. Thus, no spatial coordinates need be introduced. Similar Hamiltonians have been used by Haken,[1] Buley and Cummings,[2] and Louisell,[3] among others.

Some comment is in order on the form of the interaction term. The usual interaction between an atom and the electromagnetic field is $\mathbf{p}\cdot\mathbf{A}$, or an atomic moment times a field strength. Such an interaction for a two-level atom would reduce to some constants times σ^x times the coordinate operator for the field. Expressing the operators in terms of σ^+, σ^-, a, and a^+, we have a constant times

$$(\sigma^+ + \sigma^-)(a + a^\dagger) = (\sigma^+ a + \sigma^- a^\dagger) + (\sigma^+ a^\dagger + \sigma^- a).$$

The term $\sigma^+ a$ represents the absorption of a photon together with the excitation of an atom, while $\sigma^- a^\dagger$ represents the inverse process. Thus, the first pair of terms represents an exchange of energy between an atom and the field. The term $\sigma^+ a^\dagger$ represents the emission of a photon together with the excitation of an atom, both processes requiring energy. Since b is usually many orders of magnitude smaller than ω_0, the interaction term can be of importance only if it nearly conserves energy. This occurs for $\omega_c \approx \omega_a$ in the first pair of terms, but not at all for the second pair. Hence, we have dropped the second pair from our Hamiltonian.

The use of only the first pair for the interaction is often referred to as the rotating field approximation. In magnetic resonance, it corresponds to keeping that component of an oscillating magnetic field which rotates in the same sense as the precessing spins, and discarding the "antirotating" component. The validity of this approximation has been extensively studied[4] and found to be quite good for $b^2\langle a^\dagger a\rangle \ll \omega_a^2$. The principal correction is the so-called Bloch–Siegert shift[5] in the resonance frequency of the atom.

Many workers have used the density-matrix formalism in maser theory.[6-8] In principle, the density matrix contains all the information about the system that can be learned within the limitations of quantum mechanics. Since no one has come up with an exact solution for the time development of the density matrix with the Hamiltonian (1) (except when only one atom is included[9]), we have decided to be less ambitious and find a minimum of information about the system, say, only the expectation values of the main observables. Expectation values are classical variables and should obey classical-type equations. Accordingly, we proceed to get rid of the quantum mechanics of the problem as fast as possible. From the Hamiltonian (1), we obtain the Heisenberg equations of motion:

$$i\dot{a} = \omega_c a + b \sum_j \sigma_j^-,$$

$$i\dot{\sigma}_j^- = \omega_a \sigma_j^- - b\sigma_j^z a,$$

and

$$i\dot{\sigma}_j^z = 2b(\sigma_j^+ a - \sigma_j^- a^\dagger). \tag{2}$$

We then form the expectation value of each of these equations:

$$(d/dt)\langle a\rangle = -i\omega_c\langle a\rangle - ib \sum_j \langle \sigma_j^-\rangle,$$

$$(d/dt)\langle \sigma_j^-\rangle = -i\omega_a\langle \sigma_j^-\rangle + ib\langle \sigma_j^z a\rangle,$$

and

$$(d/dt)\langle \sigma_j^z\rangle = -2ib(\langle \sigma_j^+ a\rangle - \langle \sigma_j^- a^\dagger\rangle). \tag{3}$$

We now have a set of ordinary differential equations whose solutions are scalar functions of time.

If $\langle \sigma a\rangle$ could be factored into $\langle \sigma\rangle\langle a\rangle$, then Eqs. (3) would be a complete system of coupled equations. That the expectation value of a product cannot be factored in general is a point drilled into all students of quantum mechanics. Nevertheless, for the purpose of our theory we contend that this factoring is *approximately* valid. Such an approximation has been used by many others under a variety of names ("neoclassical,"[9] self-consistent-field approximation,[6] "dielectric approxima-

[1] H. Haken, *Dynamical Processes in Solid State Optics*, R. Kubo and H. Kamimura, Eds. (W. A. Benjamin, Inc., New York, 1967), pp. 168–194.

[2] E. R. Buley and F. W. Cummings, Phys. Rev. **134**, A1454 (1964).

[3] W. H. Louisell, *Radiation and Noise in Quantum Electronics* (McGraw-Hill Book Co., New York, 1964).

[4] J. H. Shirley, Phys. Rev. **138**, B979 (1965).

[5] F. Bloch and A. Siegert, Phys. Rev. **57**, 522 (1940).

[6] C. R. Willis, J. Math. Phys. **5**, 1241 (1964).

[7] W. Weidlich and F. Haake, Z. Physik **185**, 30 (1965).

[8] J. P. Gordon, Phys. Rev. **161**, 367 (1967).

[9] E. T. Jaynes and F. W. Cummings, Proc. IEEE **51**, 89 (1963).

tion["10]) or no name at all,[1] but is equivalent to having a semiclassical theory. For if the field is treated as a nonfluctuating classical quantity and only the atoms treated quantum mechanically, the field coordinate a becomes just a number and factors out of the expectation value. Spontaneous emission is lost in this approximation, but the stimulated emission responsible for maser action is still there.

Intuitively, the semiclassical approximation can be justified in two ways. First, the radiation-field mode becomes strongly excited (a large number of photons present); hence, a classical description of the field should be adequate. Second, an atom is statistically correlated only with that part of the field it itself has generated. For a large number of atoms present, most of the field has been generated by the other atoms. We assume an atom is not correlated with others except through their mutual interaction with the field. Then each atom is uncorrelated with most of the field, i.e., the expectation value of the product factors. The validity of the factorization has been investigated by Willis,[6] Weidlich, and Haake.[11] A more complete justification by the author will be published elsewhere.

We further simplify Eqs. (3) by introducing dimensionless macroscopic variables:

$$A = \langle a \rangle e^{i\omega t},$$

$$M = \sum_j \langle \sigma_j^- \rangle e^{i\omega t},$$

and

$$W = \sum_j \langle \sigma_j^z \rangle. \tag{4}$$

By defining the sum over all atoms as a single variable, we eliminate the sums and any mention of individual atoms from our equations. We also factor out the gross time dependence $e^{i\omega t}$. Here ω is the frequency at which the maser actually oscillates, which may be different from the resonant frequency of either the cavity or the atoms. Physically, A can be interpreted as the complex amplitude of the field, so scaled that A^*A is the mean photon number; M is proportional to the complex amplitude of the magnetization or electric polarization acquired by the

atoms; and W is the population difference between the two levels. When W is positive, the majority of the atoms are in the upper state (inverted population). Note that $\langle a \rangle$ and $\langle \sigma^- \rangle$ nonzero implies that the field and polarization have a reasonably well defined phase. Our theory assumes coherent oscillation.

The equations of motion for the macroscopic variables (4) are obtained from Eqs. (3) with the semiclassical approximation:

$$\dot{A} = i(\omega - \omega_c)A - ibM, \tag{5a}$$

$$\dot{M} = i(\omega - \omega_a)M + ibWA, \tag{5b}$$

and

$$\dot{W} = -2ib(M^*A - MA^*). \tag{5c}$$

Since ω will be close to ω_a and ω_c, the basic rapid oscillation of the maser has totally disappeared from our equations. This is a consequence of the rotating field approximation and the nature of the nonlinearity. Without the rotating field approximation, Eqs. (5) would contain terms in $e^{2i\omega t}$ [compare (A7) in the Appendix]. The maser then generates harmonics, just like other self-oscillators (such as the Van der Pol[12]) without a rotating field.

One familiar solution of Eqs. (5) can be obtained by considering one atom and a field strong enough that we can neglect the effect of the atom upon it. In Eqs. (5b) and (5c), we then treat A as a constant. Eliminating M, we find

$$(d/dt)^3 W + 4p^2(d/dt)W = 0.$$

where $p^2 = c^2 + \frac{1}{4}(\omega - \omega_a)^2$, and $c = b|A|$. For the atom initially in the upper state, we have the solution $W = 1 - 2(c^2/p^2)\sin^2 pt$. The probability that it will be in the lower state is $\frac{1}{2}(1 - W) = (c^2/p^2)\sin^2 pt$. This is identical with the Rabi transition probability used in the theory of molecular beam spectrometers.[13]

Equations (5) have the following integrals of the motion:

$$A^*A + \frac{1}{2}W = \text{const} = n + \frac{1}{2}N,$$

$$M^*M + \frac{1}{4}W^2 = \text{const} = \frac{1}{4}N^2,$$

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[10] D. E. McCumber, Phys. Rev. **130**, 675 (1963).

[11] W. Weidlich and F. Haake, Z. Physik **186**, 203 (1965).

[12] A. Blaquiere, *Nonlinear System Analysis* (Academic Press Inc., New York, 1966).

[13] N. F. Ramsey, *Molecular Beams* (Oxford University Press, London, England, 1956).

and

$$M^*A + MA^* + W(\omega_a - \omega_c)/2b$$

$$= \text{const} = N(\omega_a - \omega_c)/2b.$$

The constants have been evaluated by assuming there is a point in the motion for which $M = 0$. The value of W there is called N and the value of A^*A is called n. Using these integrals, we find that W obeys the following differential equation:

$$(d/dt)^2 W + [4b^2 n + 2b^2 N + (\omega_a - \omega_c)^2]W - 3b^2 W^2$$

$$= (\omega_a - \omega_c)^2 N - b^2 N^2.$$

This can be solved in terms of Jacobian elliptic functions:

$$W = N - \lambda cn^2(qt, k).$$

Here, the zero of time is chosen when W is a minimum, and the parameters are given by:

$$\lambda = d - a,$$

$$k^2 = (d-a)/2d,$$

$$q^2 = b^2 d,$$

$$d^2 = a^2 + 4nN,$$

and

$$a = n - N + (\omega_a - \omega_c)^2/4b^2.$$

The other variables can be found from W via the integrals of the motion. A special case of this solution was found by Jaynes and Cummings.[9]

The elliptic functions are periodic, so W oscillates. From the energy integral $A^*A + \frac{1}{2}W = \text{const}$, we see that energy is periodically exchanged between the atoms and the field. This will not go on forever in a physical system, since the atoms and field will eventually interact with external influences not included in our Hamiltonian. More specifically, the radiation-field energy is absorbed by the cavity walls and coupled out of the cavity for observation. The atoms collide with the walls of their container, with each other, or with other atoms present. Many efforts have been made to develop models for such dissipating interactions.[3,7,11,14-16] We do not go into these models here,

but merely adopt their conclusions, which are equivalent to the introduction of phenomenological damping terms into our equations.

All three variables A, M, and W are damped, in general at different rates. For the field A, we introduce the damping constant $\beta = \omega_c/2Q$, where Q is the (loaded) quality factor of the cavity. For the atoms we introduce the damping constants γ_1 and γ_2 for W and M, respectively. These gammas correspond to the reciprocals of the T_1 and T_2 relaxation times in magnetic resonance theory.[17] We make no assumption about the size of the relaxation rates except that they should be small as compared to ω.

Now A and M relax to zero, but the population difference W may relax to an unequal population W_0, as in thermal equilibrium. We should include a constant term $\gamma_1 W_0$ in the W equation to account for this. Also, to get anything interesting out of a maser, one must furnish a source of energy (pumping) in the form of excited atoms which can then radiate energy to make up for the losses. This source also appears in the W equation. For mathematical convenience, we lump this term together with $\gamma_1 W_0$ and call the result I. In the absence of interaction $(b=0)$, W will then relax to I/γ_1. Maser models with more than two levels may have a different form of relaxation.

We should mention here that theories of dissipation give not only a damping term, but also a driving term of a stochastic nature. This term provides a source for the noise which invariably accompanies any dissipative process. Since the mean of such a term is zero and we are primarily concerned with mean values, we omit these terms from our equations. The effects of including one are considered in Sec. V.

We now present the final equations for our maser model with dissipation:

$$\dot{A} = -\beta A + i(\omega - \omega_c)A - ibM, \quad (6a)$$

$$\dot{M} = -\gamma_2 M + i(\omega - \omega_a)M + ibWA, \quad (6b)$$

and

$$\dot{W} = I - \gamma_1 W - 2ib(M^*A - MA^*). \quad (6c)$$

Equation (6a), describing the evolution of the

[14] R. K. Wangsness and F. Bloch, Phys. Rev. **89**, 728 (1953).

[15] I. R. Senitzky, Phys. Rev. **119**, 670 (1960); **131**, 2827 (1963).

[16] M. Lax, Phys. Rev. **145**, 110 (1966).

[17] A. Abragam, *The Principles of Nuclear Magnetism* (The Clarendon Press, Oxford, England, 1961).

field as driven by the atoms, is linear. The other two, describing the evolution of the atoms as driven by the field, are nonlinear, but only because they involve the product of a field and atomic variable. Considering A as an external field, we note that Eqs. (6b) and (6c) are equivalent to the Bloch equations of magnetic-resonance theory.[17] They can be developed from a classical model of precessing spins. Equation (6a) can be obtained from classical electromagnetic-field theory when the rotating-field approximation is made. In short, Eqs. (6) are essentially classical. The approximation $\langle \sigma a \rangle \approx \langle \sigma \rangle \langle a \rangle$ is all that is necessary to convert the quantum equations (2) to classical ones.

Earlier work[9,15-20] leading to dynamic equations in terms of macroscopic variables had the equations for A and M second order. Although these equations are more accurate in that they do not involve the rotating field approximation, this approximation has always been invoked in their solution. In the Appendix we illustrate equivalence by reducing the equation of Davis[18] to Eqs. 6.

In Eqs. (6), A and M are complex while W is real. There are five real variables in all. These five can be made explicit by defining $A = re^{i\theta}$ and $M = i\mu e^{i(\theta - \varphi)}$. The equations of motion become:

$$\dot{r} = -\beta r + b\mu \cos\phi, \tag{7a}$$

$$\dot{\mu} = -\gamma_2\mu + bWr \cos\phi, \tag{7b}$$

$$\dot{W} = I - \gamma_1 W - 4b\mu r \cos\phi, \tag{7c}$$

$$\dot{\phi} = \omega_a - \omega_c - b(\mu/r + Wr/\mu) \sin\phi, \tag{7d}$$

and

$$\dot{\theta} = \omega - \omega_c - b(\mu/r) \sin\phi. \tag{7e}$$

Note that θ appears only in the last equation. Only the first four equations must be solved simultaneously. If the cavity is on tune ($\omega_c = \omega_a$), then ϕ can be set equal to zero and only the first three equations are required.

[18] L. W. Davis, Proc. IEEE **51**, 76 (1963).

[19] A. N. Oraevskii, Radiotekhnika i Elektronika **4**, 4, 718 (1959) [English transl. Radio Eng. Electron. **4**, 4, 228 (1959)].

[20] A. S. Grasyuk and A. N. Oraevskii, *Quantum Electronics*, P. A. Miles, Ed. (Academic Press Inc., New York, 1964), pp. 192–197.

II. GENERAL BEHAVIOR OF SOLUTIONS

One solution of Eqs. (6) is apparent. We can let A and M be zero. Then, W relaxes to the value I/γ_1. All derivatives will be zero and the system will remain in this steady state indefinitely.

It is not enough, however, to just identify a steady-state solution of a system of nonlinear-differential equations. We must also consider the possibility that the system is momentarily disturbed. Will it return to the steady state? Suppose in Eqs. (6) that A and M are small, but not zero. Their product in Eq. (6c) will be very small; hence, we neglect it and assume W remains constant at I/γ_1. The remaining two equations are linear in A and M. Eliminating M and assuming the system is tuned ($\omega = \omega_c = \omega_a$), we obtain

$$(d/dt)^2 A + (\beta + \gamma_2)(d/dt) A + (\beta\gamma_2 - b^2 W) A = 0.$$

The solution of this equation has the form $c_1 \exp(\lambda_1 t) + c_2 \exp(\lambda_2 t)$, where λ_1 and λ_2 are roots of the characteristic equation

$$\lambda^2 + (\beta + \gamma_2)\lambda + \beta\gamma_2 - b^2 W = 0.$$

When the real parts of λ_1 and λ_2 are both negative, A damps to zero, that is, the $A = M = 0$ steady-state solution is stable. This occurs if, and only if, all coefficients in the characteristic equation are positive[21]:

$$W < \beta\gamma_2/b^2 \quad \text{or} \quad I < \beta\gamma_1\gamma_2/b^2.$$

We, therefore, define a threshold value for our source of excited atoms $I_{th} = \beta\gamma_1\gamma_2/b^2$. For $I < I_{th}$, the system decays to the steady-state condition of no field present. For $I > I_{th}$, this steady state is unstable; a small initial field value will increase. When the system is not tuned, the threshold is larger. To minimize the threshold we want the cavity tuned, a large interaction matrix element b, and all three relaxation rates small.

A more interesting solution is one with the maser oscillating. For a steady state, we set the derivatives to zero in Eqs. (6) and solve Eq. (6a) for M:

$$M = (i/b)[\beta - i(\omega - \omega_c)]A. \tag{8}$$

[21] G. Birkhoff and S. MacLane, *A Survey of Modern Algebra* (The Macmillan Co., New York, 1953), p. 115.

Substituting for M in (6b), we find

$$[\gamma_2 - i(\omega - \omega_a)][\beta - i(\omega - \omega_c)]A = b^2 W A.$$

If the maser is oscillating, A is nonzero, so we divide it out and separate the real and imaginary parts of the remainder:

$$\beta\gamma_2 - (\omega - \omega_a)(\omega - \omega_c) = b^2 W, \qquad (9a)$$

$$\beta(\omega - \omega_a) + \gamma_2(\omega - \omega_c) = 0. \qquad (9b)$$

Equation (9b) determines the frequency ω at which the maser oscillates in the steady state:

$$\omega = (\beta\omega_a + \gamma_2\omega_c)/(\beta + \gamma_2). \qquad (10)$$

This frequency is an average of the cavity and atomic resonance frequencies weighted by the relative strengths of the cavity and atomic relaxation rates. It is one of the earliest discoveries in maser theory.[22]

Substituting Eq. (8) for M in Eq. (6c), we have in the steady state:

$$I = \gamma_1 W + 4\beta A^* A. \qquad (11)$$

Multiplying by $\frac{1}{2}\hbar\omega$, we obtain a representation of energy conservation:

$$\tfrac{1}{2}\hbar\omega I = \gamma_1(\tfrac{1}{2}\hbar\omega W) + 2\beta(\hbar\omega A^* A). \qquad (12)$$

The term on the left represents the power supplied by the source of excited atoms. On the right, $\frac{1}{2}\hbar\omega W$ is the steady-state energy of the atoms in the cavity, relative to a zero of energy at equal populations of the states, and γ_1 is the rate at which atomic relaxations remove this energy. Similarly, $\hbar\omega A^* A$ is the field energy stored in the cavity and $2\beta = \omega/Q$ is the rate at which the field relaxations remove this energy. Thus, Eq. (12) shows how the input power is divided between the atomic and field loss mechanisms.

Inserting the value of ω from Eq. (10), we solve Eq. (9a) for W:

$$W = (\beta\gamma_2/b^2)[1 + (\omega_c - \omega_a)^2/(\beta + \gamma_2)^2]. \qquad (13)$$

The steady-state population inversion is independent of the pumping I. It is determined solely by balancing the atomic and field losses against the gain provided by the population inversion. Thus, as I is increased in Eq. (12), the additional

[22] J. P. Gordon, H. J. Zeiger, and C. H. Townes, Phys. Rev. 99, 1264 (1955).

power goes entirely into the field losses (including output power) while the atomic relaxations absorb a constant amount. Substituting Eq. (13) in Eq. (11), we get an expression for the steady-state photon number:

$$A^* A = I/4\beta - (\gamma_1\gamma_2/4b^2)$$
$$\times [1 + (\omega_c - \omega_a)^2/(\beta + \gamma_2)^2]. \qquad (14)$$

The output power of a maser, which is proportional to $A^* A$, is thus an increasing linear function of I and a decreasing quadratic function of the detuning $(\omega_c - \omega_a)$.

Equations (8), (10), (13), and (14) constitute the complete steady-state solution for our oscillating maser. Note that while Eq. (8) gives the relative phase between A and M, the absolute phase is not defined in the steady-state solution. In terms of the real variables of Eqs. (7) and the parameter $z \equiv I/I_{th}$, the solution is

$$r^2 = (\gamma_1\gamma_2/4b^2)(z - \sec^2\phi),$$
$$\mu = (\beta/b)r\sec\phi,$$
$$W = (\beta\gamma_2/b^2)\sec^2\phi,$$
$$\tan\phi = (\omega_a - \omega_c)/(\beta + \gamma_2), \qquad (15)$$

and θ is arbitrary. This solution can exist only if $z > \sec^2\phi$, the same threshold condition that makes the nonoscillating solution unstable.

We now investigate the stability of the oscillating steady-state solution when the system is tuned. To do this, we linearize Eqs. (7) about their steady state in Eqs. (15). We abbreviate the steady-state value of r by r_1, and define

$$r = r_1 + \delta r,$$
$$\mu = (\beta/b)r_1 + \delta\mu,$$

and

$$W = (\beta\gamma_2/b^2) + \delta W.$$

The amplitude deviations satisfy

$$\delta\dot{r} = -\beta\delta r + b\delta\mu, \qquad (16a)$$

$$\delta\dot{\mu} = -\gamma_2\delta\mu + (\beta\gamma_2/b)\delta r + br_1\delta W, \qquad (16b)$$

and

$$\delta\dot{W} = -\gamma_1\delta W - 4\beta r_1\delta r - 4br_1\delta\mu, \qquad (16c)$$

where product terms in the deviations have been

neglected. The characteristic equation for this system is

$$\lambda^3 + (\beta + \gamma_1 + \gamma_2)\lambda^2 + (\beta\gamma_1 + \gamma_1\gamma_2 + 4b^2r_1^2)\lambda$$
$$+ 8\beta b^2 r_1^2 = 0.$$

All coefficients are positive, hence the condition for stability is that the product of the first and second coefficients is greater than the third.[21] We replace $4b^2r_1^2$ by $\gamma_1\gamma_2(z-1)$ to reduce this condition to

$$(\beta + \gamma_1 + 3\gamma_2)\beta > (\beta - \gamma_1 - \gamma_2)\gamma_2 z.$$

Steady-state maser oscillation is, therefore, always stable unless[20] $\beta > (\gamma_1 + \gamma_2)$ and

$$z > z_{cr} = [\beta(\beta + \gamma_1 + 3\gamma_2)/\gamma_2(\beta - \gamma_1 - \gamma_2)].$$

Similar results are obtained for a detuned system, but the characteristic equation is fourth order. These conditions for instability are rarely met in practice.

For the phases we use Eqs. 15 in Eqs. (7d) and (7e) on tune and linearize to find:

$$\dot{\phi} = -(\beta + \gamma_2)\phi \tag{16d}$$

and

$$\dot{\theta} = -\beta\phi. \tag{16e}$$

The relative phase ϕ damps to zero, so is stable. The maser frequency $\omega - \dot{\theta}$ is therefore also stable, but the phase itself does not return to any fixed value, but remains wherever it started, except that it shifts if ϕ is disturbed. This sort of neutral equilibrium is associated with a zero root of the characteristic equation for the pair (16d) and (16e), and results in a different type of low-frequency response to noise sources, as discussed in Sec. V.

So far we have found two steady-state solutions for our maser Eqs. (6) and investigated their stability. Since the equations, though nonlinear, are relatively simple, we might guess that the qualitative behavior contains no new features. That is, when below threshold the system decays to the nonoscillating steady state regardless of the initial conditions. When above threshold, the system moves to the oscillating steady state. This guess is supported by studying the signs of the derivatives in Eqs. (7). For any initial values the system eventually, if not immediately, turns toward the steady state. Numerical solutions
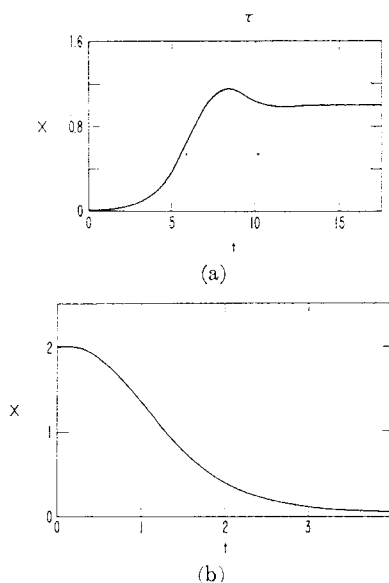

(a)

(b)

FIG. 1. (a) Buildup of maser oscillation intensity for $\beta = \gamma_1 = \gamma_2 = 1, z = 2$. Initial value of the normalized intensity x was 0.01. (b) Decay of maser oscillation intensity for $\beta = \gamma_1 = \gamma_2 = 1$, $z = 0.5$. Initial conditions corresponded to steady state oscillation at $z = 3$.

(Fig. 1) bear out this behavior. The normalized intensity x is $4b^2r^2/\gamma_1\gamma_2$.

III. PARTICULAR CASES

Although many restrictive assumptions were made in the formulation of this theory, the relative magnitudes of the relaxation rates are still arbitrary. In most physical systems, however, relaxation rates differ by orders of magnitude. In such cases, the dynamic behavior is largely controlled by the slower relaxation rates. The faster relaxations damp their variables so that they adiabatically follow the slower variables. Some of the variables can then be eliminated, yielding simpler equations of motion. Following Tang,[23] we discuss two cases.

A. Hydrogen Maser

For the hydrogen maser,[24] γ_1 and γ_2 are typically of the order of 3 sec^{-1}, while β is of the order 10^5 sec^{-1}. Therefore, $\beta \gg \gamma_1$, γ_2 and Eq. (6a) relaxes

[23] C. L. Tang, J. Appl. Phys. **34**, 2935 (1963).

[24] D. Kleppner, H. M. Goldenberg, and N. F. Ramsey, Phys. Rev. **126**, 603 (1962). The value 0.3 for γ is a misprint.

much faster than the other two, until $\dot{A}$ is small compared to βA and can be neglected. We then have (on tune)

$$\beta A = -ibM. \tag{17}$$

As M evolves at a characteristic rate γ_2, A will adiabatically follow it. The relation (17) is maintained since $\dot{A}$ will be only of order $\gamma_2 A$, small as compared to βA.

Equation (17) expresses Anderson's "reaction field" principle[25] that the field is instantaneously generated by the polarization present.

Using Eq. (17) to eliminate M, we obtain the dynamic equations for the tuned hydrogen maser:

$$\dot{A} = (-\gamma_2 + b^2 W/\beta) A$$

and

$$\dot{W} = I - \gamma_1 W - 4 A^* A, \tag{18}$$

which essentially are in agreement with Anderson[25] and Audoin.[26] For those who prefer second-order equations in one variable, we have $(r = |A|)$

$$(\ddot{r} - \dot{r}^2/r + \gamma_1 \dot{r}) + \gamma_1 \gamma_2 (1-z) r + 4b^2 r^3 = 0.$$

The term $\dot{r}^2/r$ is innocuous since $\dot{r}$ goes to zero whenever r does. It can be linearized by the change of variables $r = e^u$. The r^3 term is the real nonlinearity.

The steady-state solutions of Eqs. (18) are the same as those of the preceding section. The frequency from Eq. (10) can be approximated by

$$\omega = \omega_a + (\gamma_2/\beta)(\omega_c - \omega_a).$$

Since γ_2/β is very small, the cavity tuning has a small effect on the maser frequency, a useful feature for frequency standards applications.

A stability analysis of the oscillating steady state for Eqs. (18) leads to the second-order characteristic equation

$$\lambda^2 + \gamma_1 \lambda + 2\gamma_1 \gamma_2 (z-1) = 0,$$

with the solution

$$\lambda = -\tfrac{1}{2}\gamma_1 \pm \tfrac{1}{2}[\gamma_1(\gamma_1 + 8\gamma_2 - 8\gamma_2 z)]^{1/2}.$$

For $z < 1 + \gamma_1/8\gamma_2$ both roots are real and negative, so the maser amplitude comes in smoothly to its steady state as in Fig. 2(a). For $z > 1 + \gamma_1/8\gamma_2$,
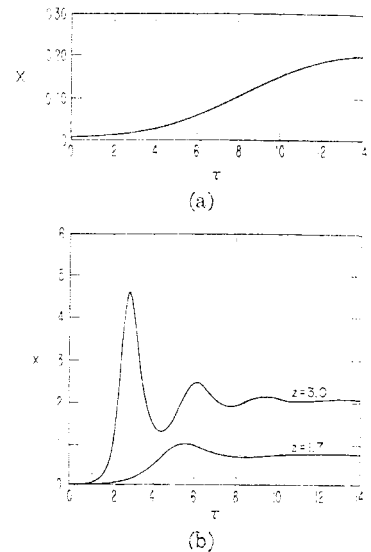

(a)

(b)

FIG. 2. (a) Buildup of maser oscillation intensity for β large, $\gamma_1 = \gamma_2 = 1$, $z = 1.2$. Initial value of x was 0.01. (b) Buildup of maser oscillation intensity for β large, $\gamma_1 = \gamma_2 = 1$, and two values of z, showing the increase in initial oscillations with increased pumping. Initial value of x was 0.01 in both cases.

the amplitude oscillates about its equilibrium value before settling down as in Fig. 2(b). These two different behaviors were discovered by Grasyuk and Oraevskii.[27] The initial overshoot (if any) of the maser is a qualitative measure of how far above threshold it is.

By defining a normalized intensity $x = 4b^2 r^2/\gamma_1\gamma_2$, population inversion $v = b^2 W/\beta\gamma_2$, and time $\tau = \gamma_1 t$, we reduce Eqs. (18) to the simpler form

$$dx/d\tau = (2\gamma_2/\gamma_1) x (v-1),$$

and

$$dv/d\tau = z - v - x. \tag{19}$$

These equations were integrated numerically to obtain Fig. 2.

For the hydrogen maser the thermal-equilibrium populations at room temperature are almost equal, so thermal relaxation tends to drive W to zero. The source term I can then be directly interpreted as the number of atoms per second in the upper state being supplied to the maser cavity by the atomic beam and state selector. Also, the assump-

[25] P. W. Anderson, J. Appl. Phys. 28, 1049 (1957).

[26] C. Audoin, Compt. Rend. 263, 489 (1966).

[27] A. S. Grasyuk and A. N. Oraevskii, Radio Eng. Electron. 9, 424 (1964).

tions of a single cavity mode and no spatial dependence are exceedingly good for the hydrogen maser. However, its behavior is somewhat complicated, owing to the presence of spin-exchange relaxation, which depends on the density of atoms inside the maser. With I interpreted as the incoming flux of atoms, γ_2 and γ_1 become dependent on I. The steady-state solution including this dependence has been given by Kleppner et al.[28]

B. Solid-State Laser

In solid-state lasers, we will usually have $\gamma_2 \gg \beta, \gamma_1$, i.e., the polarization relaxes much faster than the field or population difference. For this case, the M equation (on tune) will relax to an adiabatic condition $\gamma_2 M = ibWA$. We now use this condition to remove M from Eqs. (6a) and (6c) leaving

$$\dot{A} = -\beta A + (b^2/\gamma_2)WA$$

and

$$\dot{W} = I - \gamma_1 W - 4(b^2/\gamma_2)WA^*A, \qquad (20)$$

as the dynamic equations for the laser. If we let $n = A^*A$ and $\alpha = 2b^2/\gamma_2$, Eqs. (20) become the so-called rate equations long used for describing laser dynamics[29]:

$$\dot{n} = -2\beta n + \alpha W n$$

and

$$\dot{W} = I - \gamma_1 W - 2\alpha W n. \qquad (21)$$

Although rate equations are usually written down directly from physical arguments, we wish to emphasize that the assumption of large γ_2 is always implicit in their use. Rate equations contain no information on the phase of the field, as Eqs. (20) do, so they cannot be used to study some phenomena. On the other hand they are valid for incoherent as well as coherent radiation and can have a spontaneous emission source term explicitly added.

To make a correspondence with another common form of the rate equations,[30] we write W in

[28] D. Kleppner, H. C. Berg, S. B. Crampton, N. F. Ramsey, R. F. C. Vessot, H. E. Peters, and J. Vanier, Phys. Rev. 138, A972 (1965).

[29] H. Statz and G. deMars, *Quantum Electronics*, C. H. Townes, Ed. (Columbia University Press, New York, 1960), p. 530.

[30] For example, P. Walsh and G. Kemeny, J. Appl. Phys. 34, 956 (1963).

terms of the population of the individual states: $W = N_2 - N_1$, where $N_2 + N_1 = N = $ const. We also write $I = (p - \kappa)N$ and $\gamma_1 = p + \kappa$ to obtain

$$\dot{n} = -(\omega/Q)n + \alpha(N_2 - N_1)n,$$

$$\dot{N}_2 = pN_1 - \kappa N_2 - \alpha(N_2 - N_1)n,$$

and

$$\dot{N}_1 = -pN_1 + \kappa N_2 + \alpha(N_2 - N_1)n. \qquad (22)$$

These equations are mathematically equivalent to Eqs. (21) since we have merely changed notation. However, in the laser case the parameters p and κ may be more readily interpretable physically than I and γ_1. When p is zero, N_2 decays to zero and all the atoms are in the lower state, the usual thermal distribution at optical frequencies. The parameter κ is then the decay rate of the upper state. Similarly, p is the pumping rate removing atoms from the ground state and putting them in the excited state.

We can still use the results of Sec. II for the solution of Eqs. (22) by merely changing notation. For example, the steady-state solutions on tune corresponding to Eqs. (13) and (14) are

$$W = 2\beta/\alpha$$

and

$$n = p\left(\frac{N}{4\beta} - \frac{1}{2\alpha}\right) - \kappa\left(\frac{N}{4\beta} + \frac{1}{2\alpha}\right).$$

For fixed N, the power output still increases linearly with the pumping p. However, the threshold condition for oscillation $n > 0$ now appears to have two parts, one for the number of atoms present $N > 2\beta/\alpha$, and one for the pumping rate $p > \kappa(\alpha N + 2\beta)/(\alpha N - 2\beta)$. In the hydrogen maser we had only a condition on the pumping I, since the number of active atoms $N = I/\gamma_1$ was not considered an independent variable.

If we define $x = 2\alpha n/\gamma_1, v = (\alpha/2\beta)W, z = \alpha I/2\beta\gamma_1$, and $\tau = \gamma_1 t$ [the same definitions used to produce Eqs. (19)], Eqs. (21) become

$$dx/d\tau = (2\beta/\gamma_1)x(v - 1)$$

and

$$dv/d\tau = z - v - vx. \qquad (23)$$

These equations differ from Eqs. (19) only in the name of a constant (β instead of γ_2) and the term vx instead of x in the second equation. They have the same steady-state solutions $x = 0$, $v = z$, or

$x = z - 1$, $v = 1$, and the same qualitative behavior as in Fig. 2. However, the characteristic equation for stability of the oscillating steady state is

$$\lambda^2 + \gamma_1 z \lambda + 2\beta\gamma_1(z - 1) = 0.$$

The laser amplitude approaches steady state without oscillating not only close to threshold as the maser does, but also for very large z.

In the ruby laser, we also have $\beta \gg \gamma_1$. Equations (23) then exhibit an interesting behavior called spiking.[31] Suppose, initially, $v \approx z > 1$ and x is small. Then, $dx/d\tau$ is positive and x increases. Since β/γ_1 is large, x increases very rapidly while v changes very little. Eventually, x becomes large enough (of the order β/γ_1, much larger than its steady-state value) to make $dv/d\tau$ as large (negative) as $dx/d\tau$. Then v drops rapidly below unity. This reverses the sign of $dx/d\tau$, so that now x decays just as rapidly as it built up. When x again becomes small, $dv/d\tau$ is positive, and the pump z slowly builds v up again until it exceeds unity and the process repeats. The time between spikes is much greater than the duration of an individual spike. Succeeding spikes are not as high; the whole process eventually damps down to the steady state[32] (see Fig. 3). Sokolov and Zubarev[33] have carried the numerical integration
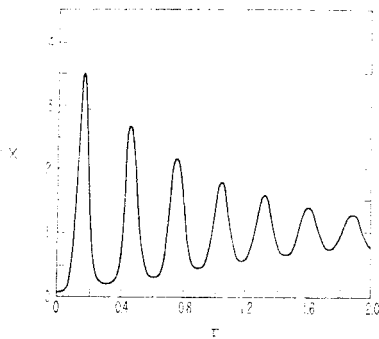

FIG. 3. Buildup of laser oscillation intensity showing damped spiking. Parameters for Eq. (23) are $\beta/\gamma_1 = 256$ and $z = 2$. Initial conditions were $x = 0.07$ and $v = 1$.

[31] D. A. Kleinman, Bell System Tech. J. **43**, 1505 (1964).

[32] That there are no undamped spiking solutions of the simple rate equations has been proved by G. Makhov, J. Appl. Phys. **33**, 202 (1962), and D. M. Sinnett, *ibid.*, **33**, p. 1578.

[33] A. K. Sokolov and T. N. Zubarev, Fiz. Tverd. Tela **6**, 2590 (1964) [English transl. Soviet Phys.—Solid State **6**, 2065 (1965)].

over more than 50 peaks and also report an experimental observation in qualitative agreement.

The so-called giant pulse of a Q-switched ruby laser can also be described by Eqs. (23). In this mode of operation, β is kept large enough to keep the system below threshold until the pump (pulsed) has built up a substantial inversion. Then the value of β is suddenly reduced. The system finds itself far above threshold and emits one of the spikes just described. The pumping is usually not maintained long enough to produce more than one spike.

To describe many laser phenomena, our equations must be generalized. The simplest and most common generalization is to three- or even four-state atoms. In the case of a ruby laser, the pump excites atoms from the ground state to a third state which then decays to the upper lasing state. If this decay is rapid as compared to the dynamics of the laser action, a two-state description is still adequate, since no appreciable number of atoms accumulates in the third state. However, in some laser systems, the lower lasing state is not the ground state and both lasing states can decay to the ground state (at different rates). In such cases, the atomic relaxation and threshold conditions can be quite different from those presented here.

Other laser phenomena, such as random spiking and mode locking, require the inclusion of several modes of the optical cavity in the equations. The spatial dependence of these modes and of the population inversion must often be considered, too. Also, atoms in solids and plasmas usually do not all have the same resonance frequency (inhomogeneous broadening). The latter is especially true of gas lasers where the Doppler linewidth exceeds the relaxation linewidth γ_2.

IV. COHERENT INPUT

We now consider the behavior of a maser when an external field source is introduced which drives the cavity at frequency ω_1. Such a source can be simulated by adding a term $\beta r_0 \exp[i(\omega - \omega_1)t]$ to Eq. (6a). In the absence of atoms, this equation then has the solution

$$A = \beta r_0 \exp[i(\omega - \omega_1)t]/[\beta + i(\omega_c - \omega_1)],$$

i.e., there is a signal in the cavity at frequency ω_1

with amplitude r_0 on tune. In the Hamiltonian (1) the external source can be represented by adding the terms $i\hbar\beta r_0[a^\dagger \exp(-i\omega_1 t) - a \exp(i\omega_1 t)]$.

If the external signal is strong as compared to the field generated by the atoms, we can ignore M in the A equation. Then A becomes a prescribed signal in the M and W equations. The latter are just the Bloch equations whose solutions have been adequately treated elsewhere.[17]

Consider now a very weak external signal with the maser below threshold. Any oscillation present will be due to the external signal, so we can take $\omega_1 = \omega$. As in Sec. II, we make the small signal linearizing approximation that $W = \beta\gamma_2 z/b^2$ is constant. We then have

$$\dot{A} = -\beta A + i(\omega - \omega_c) A - ibM + \beta r_0,$$

and

$$\dot{M} = -\gamma_2 M + i(\omega - \omega_a)M + i(\beta\gamma_2 z/b)A. \quad (24)$$

On tune these have the steady-state solution

$$M = i(\beta/b)zA,$$

and

$$A = r_0/(1-z),$$

stable for $z < 1$. The maser acts as an amplifier of the external signal with an amplitude gain $(1-z)^{-1}$. For a large gain, the pumping must be held close to threshold. The gain does not become infinite as z approaches one, since the approximation of constant W breaks down first.

By considering the cavity on tune $\omega_c = \omega_a$, but the external signal not, we can determine the bandwidth of the maser amplifier from the steady-state solution of Eqs. (24). For simplicity, we look only at the two special cases of Sec. III. For $\gamma_2 \gg \beta$, $(\omega - \omega_a)$, we have $A = Gr_0$ with

$$|G|^2 = \beta^2[\beta^2(1-z)^2 + (\omega - \omega_a)^2]^{-1}.$$

The amplifier has a simple Lorentzian passband with a width $2\beta(1-z)$ of the order of the cavity bandwidth, but reduced for z close to one (high gain). For $\beta \gg \gamma_2$, $(\omega - \omega_a)$ the gain is

$$|G|^2 = [\gamma_2^2 + (\omega - \omega_a)^2][\gamma_2^2(1-z)^2 + (\omega - \omega_a)^2]^{-1},$$

which can be rewritten as unit gain plus a Lorentzian of width $2\gamma_2(1-z)$. In general, the maser-amplifier gain–bandwidth product is determined by the smaller of the atomic and cavity linewidths

and is independent of the gain in the linear region (W unaffected). Expressions for the steady-state maser gain and bandwidth were first found by Basov and Prokhorov.[34]

Finally, we consider a weak signal introduced while a tuned maser is oscillating. We use Eqs. (7) and assume that the driving term is small enough to make little change in the steady-state solutions (15). However, it may profoundly affect the phase of the maser, which was previously arbitrary. Using the steady-state values of r, μ, and W from Eq. (15) with ϕ small, Eqs. (7d) and (7e) plus driving term become

$$\dot{\phi} = -(\beta + \gamma_2)\phi + \beta(r_0/r)\sin(\Delta\omega t - \theta)$$

and

$$\dot{\theta} = -\beta\phi + \beta(r_0/r)\sin(\Delta\omega t - \theta),$$

where $\Delta\omega = \omega - \omega_1$. For $\beta + \gamma_2 \gg \Delta\omega - \dot{\theta}$, the relative phase relaxes to the adiabatic value

$$\phi = \beta(\beta + \gamma_2)^{-1}(r_0/r)\sin(\Delta\omega t - \theta).$$

The phase θ then obeys

$$\dot{\theta} = \beta\gamma_2(\beta + \gamma_2)^{-1}(r_0/r)\sin(\Delta\omega t - \theta). \quad (25)$$

If θ remains small, we ignore it in the argument of the sine and integrate:

$$\theta = \theta_0 - \frac{\beta\gamma_2}{\beta + \gamma_2}\frac{r_0}{r}\frac{\cos\Delta\omega t}{\Delta\omega}.$$

The external signal simply adds (in quadrature) to the maser signal. The result is a phase or frequency modulation at the difference frequency.

Equation (25) has a nonoscillating solution $\theta = \Delta\omega t - \theta_0$, where

$$\sin\theta_0 = (\beta + \gamma_2)(\beta\gamma_2)^{-1}(r/r_0)\Delta\omega. \quad (26)$$

The maser frequency is $\omega - \dot{\theta} = \omega - \Delta\omega = \omega_1$, the same as that of the external signal. In this condition, we say that the maser is phase locked to the external signal. Since the sine function is bounded, we see from Eq. (26) that this solution can exist only if

$$|\Delta\omega| < \beta\gamma_2(\beta + \gamma_2)^{-1}(r_0/r). \quad (27)$$

This condition for phase locking was found for the

[34] N. G. Basov and A. M. Prokhorov, Zh. Eksp. Teor. Fiz. **30**, 560 (1956) [English transl. Soviet Phys.—JETP **3**, 426 (1956)].

maser case $\beta \gg \gamma_2$ in an early paper by Oraevskii.[19] The corresponding result for the laser case $\gamma_2 \gg \beta$ has been derived more recently.[35] Equation (25) can be integrated in terms of elementary functions and the full dynamics of the phase followed for either the locked or unlocked condition. The behavior is identical with that found by Adler[36] for an electronic oscillator.[37]

V. STOCHASTIC INPUT

In Sec. I we mentioned that dissipative processes are always accompanied by noise sources. We now illustrate the effect of such sources by considering thermal noise associated with the cavity losses. We use the so-called Langevin method[38,39] in which a stochastic driving force is added to the equation of motion along with the damping term. Assuming the stochastic force is small, the equations are linearized and solved for the stochastic part of the maser variables. We can then ensemble average to find the mean fluctuations, or their spectrum.

We should, perhaps, mention that the noise theory presented here is a classical one. Senitzky,[15] Lax,[16,40] and Haken[41] have used a quantum mechanical noise theory in which the stochastic driving force is an operator added to the operator equation of motion. By including such noise operators for the atoms as well as the field, they are then able to find the effects of spontaneous emission noise as well as cavity thermal noise. We have chosen to consider only the thermal noise here since it can be adequately treated by classical noise theory.

We add to Eq. (6a) the term $\beta f(t)$, where f is a complex-valued stochastic function. For

thermal noise, we expect f to have a white-noise spectrum $S_f(\omega) = 2\bar{n}/\beta$, or be uncorrelated with itself:

$$\langle f^*(t) f(t+\tau) \rangle = (2\bar{n}/\beta)\delta(\tau).$$

Here the angle brackets denote an ensemble average (or time average for a stationary process) and $\bar{n} = [\exp(\hbar\omega/kT) - 1]^{-1}$ is the mean number of photons in the cavity at thermal equilibrium.

If no atoms are present, we set $\omega = \omega_c$ and have simply

$$\dot{A} = -\beta A + \beta f.$$

To obtain the spectral density for A, the Fourier transform technique is convenient. Denoting Fourier transforms by tildes, we find

$$i\omega \tilde{A} = -\beta \tilde{A} + \beta \tilde{f}$$

or

$$\tilde{A} = [\beta/(\beta + i\omega)] \tilde{f}.$$

The coefficient of $\tilde{f}$ is the transfer or system function for the differential equation. The spectral density of A is just the absolute square of the system function times the spectral density of f[42]:

$$S_A(\omega) = 2\beta\bar{n}/(\beta^2 + \omega^2).$$

This is the Lorentzian expression for the cavity passband. Remember that ω_c has been factored from A, so the Fourier frequency ω is measured relative to the cavity frequency ω_c. The total number of photons present is the integral of the spectral density:

$$\int \frac{S_A(\omega) d\omega}{2\pi} = \bar{n}.$$

This integral verifies our choice of constants in the spectral density of f.

Consider now atoms present, but the system below threshold ($z < 1$). How do the atoms affect the thermal noise of the cavity? Assuming the signals small enough so that W is constant, we have from Eqs. (6a) and (6b) on tune:

$$\dot{A} = -\beta A - ibM + \beta f$$

and

$$\dot{M} = -\gamma_2 M + i(\beta\gamma_2/b)zA.$$

[25] R. H. Pantell, Proc. IEEE **53**, 474 (1965).

[26] R. Adler, Proc. IRE **34**, 351 (1946).

[27] When phase locking was first observed with lasers [H. L. Stover and W. H. Steier, Appl. Phys. Letters **8**, 91 (1966)], the authors knew of no theory for it, so they borrowed Adler's, guessing the corresponding parameters. Our theory fully verifies their guess.

[28] N. Wax, *Noise and Stochastic Processes* (Dover Publications, Inc., New York, 1954).

[39] M. Lax, Rev. Mod. Phys. **38**, 541 (1966).

[40] M. Lax, *Physics of Quantum Electronics*, P. L. Kelley, B. Lax, and P. E. Tannenwald, Eds. (McGraw-Hill Book Co., New York, 1966), pp. 735–747.

[41] H. Haken, Z. Physik **190**, 327 (1966).

[42] W. B. Davenport, Jr. and W. L. Root, *An Introduction to the Theory of Random Signals and Noise* (McGraw-Hill Book Co., New York, 1958), Sec. 9-3.

We Fourier transform these equations and solve for $\tilde{A} = g(\omega)\tilde{f}$:

$$g(\omega) = \beta(\gamma_2 + i\omega)/[(\beta + i\omega)(\gamma_2 + i\omega) - \beta\gamma_2 z].$$

The spectral density of A is then $S_A(\omega) = |g(\omega)|^2 S_f(\omega)$. When ω is small, $g(\omega) \approx (1-z)^{-1}$. Frequencies very close to the atom frequency are amplified. Frequencies outside the passband of the cavity are attenuated. In fact, $g(\omega)$ is the same function as the G used in Sec. IV to describe the response to a coherent input signal, and the special cases given there are applicable.

The integral of $|g(\omega)|^2$ can be carried out to give the total number of photons present:

$$n = [(\gamma_2 + \beta - \beta z)/(1-z)(\beta + \gamma_2)]\bar{n}.$$

For $\gamma_2 \gg \beta$, the number of thermal photons is amplified: $n = \bar{n}(1-z)^{-1}$. Note that the energy of a tuned coherent signal is amplified by $(1-z)^{-2}$, so having the thermal energy distributed throughout the cavity linewidth reduces the over-all amplification. For $\beta \gg \gamma_2$, we have

$$n = [1 + (\gamma_2/\beta)z/(1-z)]\bar{n}.$$

In this case, only the thermal energy within the atomic linewidth is amplified.

Finally, we consider the tuned maser to be oscillating. We separate $f = f_1 + if_2$ into real and imaginary parts and use Eqs. (7). Linearizing about the steady state (15), we obtain two sets of equations, one for the amplitudes and one for the phases. The amplitude equations are the same as Eqs. (16) except the term βf_1 is added to the δr equation. The spectral density of the amplitude

FIG. 4. Spectral density of amplitude fluctuations caused by thermal noise in a hydrogen maser with $\beta = 10^5$, $\gamma_1 = \gamma_2 = 3$, and $z = 4$. Unity on the vertical scale corresponds to the thermal value kT for the power spectral density.

obtained from these equations is algebraically complex. A representative case appropriate to the hydrogen maser is plotted in Fig. 4. At high frequencies, we have just the cavity line shape. The peak at $\omega = 7.4$ corresponds to the frequency of the relaxation oscillations undergone by the amplitude while approaching the steady state. The position of this peak and the noise level at low frequencies depend on z.

For the phase variables (assumed small), we have the equations

$$\dot{\theta} = -\beta\phi + \beta f_2/r,$$

and

$$\dot{\phi} = -(\beta + \gamma_2)\phi + \beta f_2/r. \qquad (28)$$

If we proceed as before, we would obtain for the spectral density of the phase:

$$S_\theta(\omega) = \beta(\gamma_2^2 + \omega^2)\bar{n}/\omega^2[(\beta + \gamma_2)^2 + \omega^2]n, \qquad (29)$$

where the spectral density for f_2 has been taken as half of that for f. Now Eq. (29) diverges for small ω, so that

$$\langle\theta^2\rangle = \int \frac{S_\theta(\omega)d\omega}{2\pi}$$

does not exist. Small ω corresponds to a large time solution of Eq. (28). We can solve for θ at large times directly by assuming ϕ has relaxed to its steady state $\beta f_2/(\beta + \gamma_2)r$. We substitute in the θ equation and integrate with $\theta = 0$ at $t = 0$:

$$\theta = \beta\gamma_2(\beta + \gamma_2)^{-1}r^{-1}\int_0^t f_2(t')dt'.$$

Then the ensemble average of θ^2 is:

$$\langle\theta^2\rangle = \beta^2\gamma_2^2(\beta + \gamma_2)^{-2}r^{-2}\int_0^t dt'\int_0^t dt''\langle f_2(t')f_2(t'')\rangle;$$

or using the delta function correlation for f_2,

$$\langle\theta^2\rangle = \beta\gamma_2^2(\beta + \gamma_2)^{-2}(\bar{n}/n)t. \qquad (30)$$

Hence, the phase fluctuations are not a stationary process. This particular type of behavior is well known from the theory of Brownian motion[33] as well as the theory of electronic oscillators, and is often referred to as a random walk of phase. We can deduce from Eq. (30) the fluctuations in frequency of a maser that would be inferred from measurements of its phase over a time t by using
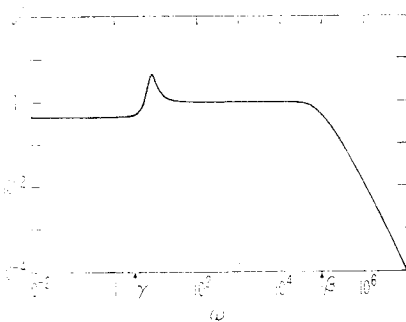
$\langle(\Delta\omega)^2\rangle = \langle\theta^2\rangle/t^2$. For the maser case of large β, we obtain the relative frequency stability

$$\langle(\Delta\omega)^2\rangle^{1/2}/\omega = (\gamma_2/\omega)\,(\bar{n}/n\beta t)^{1/2}$$
$$= (1/\sqrt{2}Q_l)\,(kT/Pt)^{1/2},$$

where $Q_l = \omega/2\gamma_2$ is the "Q" of the atoms and $P = 2\beta\hbar\omega n$ is the total power output absorbed by the cavity losses. This agrees with Blaquiere[12] and (within a factor 2π) with Ramsey's result for the hydrogen maser.[24,43]

Alternatively, if we assume that the stochastic variable θ has a Gaussian distribution, we have

$$\langle e^{i\theta}\rangle = \exp[-\tfrac{1}{2}\langle\theta^2\rangle] \equiv \exp(-\Delta\nu t),$$

where $\Delta\nu$ is the "optical" linewidth[12] of the maser oscillator that would be measured by an ideal spectrum analyzer after an arbitrarily long time. This linewidth is, from Eq. (30),

$$\Delta\nu = [\tfrac{1}{2}\beta\gamma_2{}^2/(\beta+\gamma_2)^2]\,(\bar{n}/n).$$

which agrees with the results of others[40,41] for the thermal contribution. When $\bar{n}\gg1$, as it is for the hydrogen maser, this is the dominant contribution to the linewidth. When $\bar{n}\ll1$, as it is for lasers, the contribution from spontaneous emission noise is more important.

VI. SUMMARY

In the preceding, we have presented an elementary derivation of the dynamical equations for a simple maser model starting from a quantum-mechanical Hamiltonian. We have indicated how various types of solutions can be found and their physical significance. Many other aspects, such as spontaneous emission noise, the maser amplitude near threshold when driven by an external signal, and off-tune behavior, can be worked out using the techniques illustrated. Generalizations to many levels, many modes, spatial dependence, and inhomogeneous broadening are straightforward, but more involved.

APPENDIX

We wish to relate the second-order semiclassical equations of others with our first-order Eq. (6).

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[43] The missing factor of 2π has been found by L. Cutler, Ph.D. thesis, Stanford University, 1966.

For definiteness, we start with the equations of Davis[18]:

$$\ddot{E} + (\omega_c/Q)\,\dot{E} + \omega_c{}^2E = -\epsilon^{-1}\ddot{P}, \tag{A1}$$

$$\ddot{P} + (2/T_2)\,\dot{P} + \omega_a{}^2P = -k^2W'E, \tag{A2}$$

$$\dot{W}' + (W'-W_e')/T_1 = \dot{P}E. \tag{A3}$$

We have written E in Eq. (A1) where Davis uses the mode amplitude v, since we use only one mode and assume that the spatial dependence has been factored out. We also added a prime to Davis' W to distinguish it from ours. Equation (A1) comes from Maxwell's equations. Equations (A2) and (A3) are derived from the Schrödinger equation for a two-level atom. The relaxation terms are introduced phenomenologically.

The electric field is related to the creation and annihilation operators by

$$E = (\hbar\omega_c/2\epsilon V)^{1/2}\langle a+a^\dagger\rangle.$$

From Davis' definition, we can identify P and W' as

$$P = (\mu/V)\sum_j \langle\sigma_j{}^x\rangle$$

and

$$W' = (\hbar\omega_c/2V)\sum_j \langle\sigma_j{}^z\rangle.$$

where μ is the dipole moment of the atom and V is the volume of the system. Recalling the definitions (4) of our macroscopic variables and that $\sigma^x = \sigma^+ + \sigma^-$, we can relate Davis' variables to ours:

$$E = (\hbar\omega_c/2\epsilon V)^{1/2}(Ae^{-i\omega t} + A^*e^{i\omega t}). \tag{A4}$$

$$P = (\mu/V)\,(Me^{-i\omega t} + M^*e^{i\omega t}). \tag{A5}$$

$$W' = (\hbar\omega_a/2V)\,W. \tag{A6}$$

Before substituting these relations back into Davis' equations, we must compute some derivatives. For example, we find that

$$\dot{P} = (\mu/V)\,(-i\omega Me^{-i\omega t} + i\omega M^*e^{i\omega t}$$
$$+ \dot{M}e^{-i\omega t} + \dot{M}^*e^{i\omega t}).$$

We originally defined M with the idea that it would be a slowly varying function of time as compared to $e^{i\omega t}$, that is, $\dot{M}\ll\omega M$. Accordingly, we neglect the last two terms in the expression for $\dot{P}$ and substitute the first two, along with Eqs. (A4)

and (A6) into (A3):

$$\left(\frac{\hbar\omega_a}{2T}\right)\left(W+\frac{W}{T_1}\right)-\frac{W_e'}{T_1}$$

$$=\left(\frac{\mu}{V}\right)(-i\omega Me^{-i\omega t}+i\omega M^*e^{i\omega t})$$

$$\times\left(\frac{\hbar\omega_c}{2\epsilon V}\right)^{1/2}(Ae^{-i\omega t}+A^*e^{i\omega t}).$$

We define $I=2VW_e'/\hbar\omega_a T_1$ and $\gamma_1=T_1^{-1}$ on the left, use $\omega\approx\omega_a$, and expand the right side:

$$\dot{W}+\gamma_1W-I=2i(\mu/\hbar)(\hbar\omega_c/2\epsilon V)^{1/2}$$

$$\times(M^*A-MA^*-MAe^{-2i\omega t}+M^*A^*e^{2i\omega t}). \quad (A7)$$

Now, the last two terms are rapidly oscillating as compared to the others. If we imagine averaging this equation over a few cycles of $e^{2i\omega t}$, the last two terms drop out, while the others are hardly affected. Equation (A7) then becomes identifiable with Eq. (6c) if we take for the coupling constant

$$b=-(\mu/\hbar)(\hbar\omega_c/2\epsilon V)^{1/2}. \quad (A8)$$

Dropping the last two terms of Eq. (A7) is equivalent to the rotating field approximation, for if we had not made the rotating field approximation in the Hamiltonian (1), Eq. (6c) would have come out like Eq. (A7).

To find Eq. (6a) from Eq. (A1), we must compute second derivatives. On the right, we can use $\ddot{P}\approx-\omega^2P$, neglecting the derivatives of M. On the left, the term ω_c^2E almost cancels the leading term of the second derivative, so a better approximation must be used:

$$\ddot{E}\approx(\hbar\omega_c/2\epsilon V)^{1/2}$$

$$\times(-\omega^2Ae^{-i\omega t}-\omega^2A^*e^{i\omega t}-2i\omega\dot{A}e^{-i\omega t}+2i\omega\dot{A}^*e^{i\omega t}).$$

Here, we have dropped only the terms in the second derivative of A. We can use just the leading term in the first derivative of E since we assume Q

is large. This assumption is necessary to make the phenomenological introduction of the relaxation terms in the second-order equations equivalent to their introduction in first order equations.

With these approximations and $\omega_c/Q=2\beta$, Eq. (A1) becomes

$$(\hbar\omega/2\epsilon V)^{1/2}[(-\omega^2Ae^{-i\omega t}-\omega^2A^*e^{i\omega t}-2i\omega\dot{A}e^{-i\omega t}$$

$$+2i\omega\dot{A}^*e^{i\omega t})+2\beta(-i\omega Ae^{-i\omega t}+i\omega A^*e^{i\omega t})$$

$$+\omega_c^2(Ae^{-i\omega t}+A^*e^{i\omega t})]=-\epsilon^{-1}(\mu/V)$$

$$\times(-\omega^2Me^{-i\omega t}-\omega^2M^*e^{i\omega t}).$$

If A and M are slowly varying, no term containing $e^{i\omega t}$ can cancel against one containing $e^{-i\omega t}$, so we equate the coefficients of these two types of terms separately. For the $e^{-i\omega t}$ terms, we have

$$(\hbar\omega_c/2\epsilon V)^{1/2}[-\omega^2A-2i\omega\dot{A}-2i\beta\omega A+\omega_c^2A]$$

$$=\epsilon^{-1}(\mu/V)\omega^2M. \quad (A9)$$

For the $e^{i\omega t}$ terms, we have the complex-conjugate equation. Since $\omega_c\approx\omega$, we can ignore their distinction in the constants and also use $\omega_c^2-\omega^2=(\omega_c+\omega)(\omega_c-\omega)\approx2\omega(\omega_c-\omega)$ on the right. Then Eq. (A9) becomes

$$2\omega(\omega_c-\omega)A-2i\omega(\dot{A}+\beta A)$$

$$=(\mu/\hbar)(\hbar\omega/2\epsilon V)^{1/2}2\omega M.$$

Dividing by $-2i\omega$ and inserting Eq. (A8), we wind up with Eq. (6a),

$$\dot{A}=-\beta A+i(\omega-\omega_c)A-ibM.$$

Equation (6b) follows from Eq. (A2) in a similar fashion.

The reduction of the semiclassical Eqs. (A1)–(A3) to Eqs. (6) required the rotating-field approximation, large Q's, closeness to resonance, and some labor. The automatic appearance of first-order equations with only the one constant b appearing in the coupling is an advantage of the derivation from the quantum-mechanical Hamiltonian.