SPACETIME COORDINATE SYSTEMS

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Spacetime Coordinate Systems

Does the principle of covariance preclude the necessity for specifying a set of coordinates operationally ? There will be given a simple example which illustrates the answer to this question. This leads to the basic problem of specifying coordinate systems of physical use and interest over the vast reaches of spacetime. Examples of two operational definitions for spacetime networks associated with the surface of the spinning earth serve to make definite some of the requirements, and show the importance and use of the general Doppler effect. For space navigational purposes, the Doppler effect becomes a prime tool. Its uses and the relative effects of gravitational fields, medium refraction and dispersion, and instrument uncertainty are to be discussed with some numerical examples.

The point of view we have adopted here stems from, but is not limited to, a consideration of general relativistic effects and uses much of the formalism of relativity theory. It suggests moreover, a new general approach to time synchronization, space communication, and navigation problems, which is reminiscent of several concepts introduced by various authors. One of these is the elementary 4/3-earth notion of Schelling, Burrows, and Ferrell. A related one is the use of "homologue" space to eliminate major refraction effects in a medium. Still a third is Fock's concept of harmonic coordinates. These approaches will be explained in elaborating the conceptual basis for this paper.

Systèmes de coordonnées espace-temps

Le principe de covariance rend-il superflue la nécessité de spécifier opérationnellement un système de coordonnées ? Un simple exemple sera donné, illustrant la réponse à cette question. Cela mène au problème de base de la spécification des systèmes de coordonnées pour l'usage en physique et intéressant une région étendue d'espace-temps. Deux exemples de définitions pratiques pour des réseaux d'espace-temps associés à la surface de la terre en rotation servent à rendre explicites certaines exigences et montrent l'importance et l'utilisation de l'effet Doppler général. Pour les besoins de la navigation spaciale, l'effet Doppler acquiert une importance primordiale. Ses utilisations et les effets relatifs dus aux champs gravitationnels, à la réfraction du milieu, à la dispersion et à l'incertitude des instruments seront discutés avec quelques exemples numériques.

Le point de vue adopté ici s'appuie sur une considération d'effets de la théorie de la relativité générale et utilise largement le formalisme de cette théorie. Ce point de vue suggère en plus une nouvelle façon d'envisager la synchronisation du temps, les communications spaciales et les problèmes de navigation s'approchant ainsi de plusieurs conceptions introduites par divers auteurs. Une de celles-ci est la notion élémentaire de "4/3-terre" de Schelling, Burrows et Ferrell. Une notion analogue est celle de l'espace "homologue" utilisé pour éliminer des effets importants de réfraction dans un milieu. Enfin, il faut noter la notion de coordonnées harmoniques de Fock. Ces différentes façons de s'approcher du problème seront expliquées en élaborant le concepte de base de la présente communication.

Raum-Zeit Koordinatensysteme

Macht das Prinzip der Kovarianz es überflüssig, ein Koordinatensystem operationell besonders festzulegen ? Die Antwort zu dieser Frage wird an einem einfachen Beispiel erläutert werden. Dies führt zum fundamentalen Problem, wie Koordinatensysteme zum praktischen Gebrauch in der Physik und gültig über grosse Raum-Zeit Bereiche spezifiziert werden sollen. Zwei Beispiele von praktischen Definitionen für Raum-Zeit-Netze, verbunden mit der rotierenden Erdoberfläche, dienen dazu, gewisse Anforderungen klarzustellen und zeigen die Bedeutung und die Verwendung des allgemeinen Dopplereffektes. Für die Bedürfnisse der Raumschifffahrt wird der Dopplereffekt zum wesentlichen Werkzeug. Seine Anwendungen und die verschiedenen durch Gravitationsfelder, Brechung und Dispersion der Umgebung und instrumentelle Unschärfen bedingten Effekte werden an Hand von einigen numerischen Beispielen besprochen.

Der hier vertretene Standpunkt stützt sich auf eine Betrachtung von Effekten der allgemeinen Relativitätstheorie und verwendet wesentlich den Formalismus dieser Theorie. Von diesem Standpunkt aus ist es naheliegend, die Probleme der Zeitsynchronisierung, Raumfernverbindungen und der Raumfahrt auf neue Weise anzufassen in Anlehnung an verschiedene von verschiedenen Autoren eingeführten Begriffe. Einer dieser Begriffe ist der der elementaren "4/3-Erde" von Schelling, Burrows und Ferrell. Ein verwandter Begriff ist die Verwendung des "homologen" Raumes zur Ausscheidung von grösseren Brechungseffekten einer Umgebung. Schliesslich sei Focks Begriff der harmonischen Koordinaten erwähnt. Diese verschiedenen Methoden werden im Verlauf der Erarbeitung der wesentlichen Begriffe dieser Mitteilung erläutert werden.

ERRATA

PAGE	LINE*	READS	SHOULD READ
200	7	....Astronomy,	....Astronomy
200	11	...., however. For it....	...., however, for it....
200	24	...., but for scientific purposes. For	...., but also for scientific purposes.
200	25	several....	Several....
200	13*	....that "any....	....that any....
200	10*	all systems," then the....	all systems, the....
201	2	....P is. For the....	.....P is. The....
202	6	different than this,....	different from this,....
202	9*	$G \equiv d\tau^2 = \bar{dt}^2 - \frac{d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2}{c^2}$	$G \equiv c^2 d\tau^2 = c^2 d\bar{t}^2 - d\bar{x}^2 - d\bar{y}^2 - d\bar{z}^2$
204	10	....notices, that....	....notices that....
204	12	...."flat," presents....	...."flat" presents....
204	7*	....meridian. For the....	....meridian. The....
205	6	....cross-produce term...	....cross-product term....
205	13	....meaningless (the....	....meaningless. (The....
206	6*	cancels out the....	greatly reduces the....
207	4*	....a near perfect	.....a nearly perfect
209	13*	....events, which....	.....events which....
209	10*	....Figure 1).	....Figure 1.) ....
212	1*	....at A.	....at A. In it $\partial\varphi$ means the partial derivatives with respect to the upper limit in the integral definition of φ.
215	3	$\varphi(\bar{'}) = ...$	$\varphi(\bar{1}) = ...$

*An asterisk beside a number means count up from the bottom line.

PAGE	LINE*	READS	SHOULD READ
215	7	$x' = \frac{1}{2} c(\tau' + \tau^{o})$	$x' = \frac{1}{2} c(\tau^{\bar{r}} - \tau^{\bar{o}})$
217	3	....from event x' to	....from event x to
217	4	event x, so....	event x', so....
217	1*	$\partial\varphi(x, X)/\partial x^{\iota} = -\partial\varphi(x, X)/\partial X^{\iota}$	$\varphi(x, X) = \int_x^X K_\iota dx^\iota$ and $-\partial\varphi(\bar{\iota})/\partial x^{\iota}{}_E = K_\iota(E, \bar{\iota})$, $\partial\varphi(\bar{\iota})/\partial X^{\iota}_{\bar{\iota}} = K_\iota(\bar{\iota}, E)$. For flat spacetime and n = 1, $K_\iota(E, \bar{\iota}) = K_\iota(\bar{\iota}, E)$; otherwise this is usually not so.
218	3	$\frac{\partial\tau^{\bar{\iota}}}{\partial\tau^j} = \frac{\partial_j\varphi(\bar{\iota})}{\partial_k\varphi(\bar{\tau})U^k_{\bar{\iota}}(\bar{\tau})}$	$\frac{\partial\tau^{\bar{\iota}}}{\partial x^j{}_E} = \frac{-\partial\varphi(\bar{\iota})/\partial x^j{}_E}{(\partial\varphi(\bar{\iota})/\partial X^k_{\bar{\iota}})U^k_{\bar{\iota}}(\bar{\iota})}$
218	9	$\left\lvert \partial_j\varphi(\bar{\tau}) \right\rvert \neq 0.$	$\left\lvert \partial\varphi(\bar{\iota})/\partial x^j{}_E \right\rvert \neq 0.$
218	11	$\frac{\partial\tau^{\bar{\iota}}}{\partial x^j} = \frac{K_j(\bar{\iota})}{2\pi\nu_{\bar{\iota}}}$.	$\frac{\partial\tau^{\bar{\iota}}}{\partial x^j{}_E} = \frac{K_j(E, \bar{\iota})}{2\pi\nu_{\bar{\iota}}}$.
218	15	$\bar{g}^{\bar{\iota}\bar{j}} = \frac{\bar{g}^{\iota j} K_\iota(\bar{\iota}) K_j(\bar{j})}{4\pi^2 \nu_{\bar{\iota}} \nu_{\bar{j}}} c^2$.	$\bar{g}^{\bar{\iota}\bar{j}} = \frac{\bar{g}^{\iota j} K_\iota(E, \bar{\iota}) K_j(E, \bar{j})}{4\pi^2 \nu_{\bar{\iota}} \nu_{\bar{j}}} c^2$.
218	16	....Since $\bar{g}^{\iota j} K_\iota(\bar{\iota}) K_j(\bar{\iota}) = 0$	....Since $\bar{g}^{\iota j} K_\iota(E, \bar{\iota}) K_j(E, \bar{\iota}) = 0$
219	4*	....$\left\lvert \frac{K_j(\bar{\iota})}{2\pi\nu_{\bar{\iota}}} \right\rvert^2$.....	....$\left\lvert \frac{K_j(E, \bar{\iota})}{2\pi\nu_{\bar{\iota}}} \right\rvert^2$....

*An asterisk beside a number means count up from the bottom line

Outline

Figures

I. Introduction

(A) Space Coordinates and Time

Historically speaking, the thesis that time and its measurement play a great role in the definition and use of spatial coordinate systems is by no means new. For the need by the science of navigation for a practicable and exact method for determination of ship longitudes linked it with the science of chronometry--as is well-known. Astronomy, strengthened this bond in its eminently successful attempt to define a uniform time scale in terms of the ephemerides of heavenly bodies. This overt wedding of the two concepts ultimately took on a biologically more suspect character, however. For it turned out that the happy match was not only a convenient alliance between well-suited mates of honorable but independent ancestry. A far deeper union appeared when the theory of relativity aptly pointed out that time and space are no longer completely independent modes of perception of events, but are indissolubly united blood relatives.

Now, among other things, I wish to point out in this article some consequences of these observations for the purposes of navigation. For our age, this means navigation in space, although one example I shall mention could apply, ideally, to navigation at the earth's surface. These considerations have a bearing too on the question of how important it is to disseminate time over extensive regions of spacetime; and they will bear on the question of the use of spacetime coordinate systems, not only for engineering application, but for scientific purposes. For several fundamental questions raised by the theory of relativity--the propagation of gravitational waves, the nature of gravitational field singularities, the proper solution of the field equations--are all linked with the notion of spacetime coordinate systems.

(B) The Operational Viewpoint

Several writers have been very concerned with the notion of spacetime coordinate systems. One view held by some is that, since the relativistic principle of covariance states that "any important physical quantity must have a value which is independent of any particular choice of coordinate system and be calculable in a form which is the same for all systems", then the choice of the coordinate system is immaterial. To this we must object--along with V. Fock,[13] and J. L. Synge; arbitrariness in choice does not imply that no choice has to be made. Indeed the choice must be made in a very specific operationally definable manner.

One can see that this is so if he simply takes as his important physical quantity some measure or aspect of the coordinate system itself. For example, consider the point, P, marked $x = 2$, $y = 0$, referred to some system of axes in a plane. Now which point this is may be very important--yet from the given information, and given the particular plane in question as a physical object, one could not tell which point P is. For the point at the origin must first be specified, the direction of the x-axis (and whether it is straight or curved), and the perhaps non-uniform scale on it must all be specified before these coordinates can be used to pick out P. Astronomers who use the fixed stars as background coordinates can appreciate this point. Scientists who wish to have the scale of proper time given in terms of a particular standard atomic frequency--say Hydrogen--will appreciate this point. So let us accede and grant that coordinate systems--although ideally not partaking in the physical phenomena they are used to describe--must nevertheless be treated as physical objects which must be determinable operationally.

The choice of coordinates is very wide indeed--and it is important that this be so. For if one has this freedom, many theoretical problems and experimental determinations can be greatly simplified. This is obvious. So one should take care when he imposes a restriction on the class of coordinate systems. A classical example of such over-restriction which has led to much confusion is the sometimes seen dictum that in special relativistic treatments of phenomena no accelerated coordinates may be allowed. This even got extended to mean no accelerated motions could be considered in the framework of this theory--in spite of the obvious example of Thomas precession to the contrary notwithstanding. The restricting statement was clearly meant originally to apply only to the important comparison of inertial systems of reference--nothing else. Now we have knocked down our straw men and can get on with the game of specifying operationally some spacetime coordinate systems.

II. Some Preliminary Examples

(A) Simplifying Conditions

First I wish to simplify our problem as much as possible and consider situations where gravitational fields are not present to complicate the formulation. Many of the basic notions remain unchanged even if such fields are present, and I shall show later what must be done to include such effects. The same is true of variations in index of refraction of the interplanetary medium we might find ourselves in, to say nothing of questions of dispersion and diffraction.

So what is left? Actually quite a bit. We can consider rotating spherical coordinate systems, or uniformly accelerated ones. We can, and in the main portion of the paper will, consider a very useful kind which I shall call a range and range-rate coordinate system--named after the similar but usually earth-bound, tracking method.[1] This system is closely related, as we shall show, to coordinates known to geometers as null-coordinates (2, 3). J. L. Synge in his excellent treatise on Relativity[4] also considers the problem of detecting space-time curvature and acceleration by operational means. We too shall point out such a method.

But in this section we shall assume, since our primary purpose is different than this, that we can define, operationally, rectangular or spherical inertial reference frames. These may be defined rather tediously by the familiar radar and direction finding techniques.

(B) Inertial Coordinates

Such a frame of reference may be visualized as that of the "fixed stars", or one moving with a constant speed relative to the fixed stars. It is well-known that the proper time interval between two successive events (ticks of a clock) happening to a moving object is given by

$$\Delta\tau = \int_1^2 \sqrt{d\bar{t}^2 - \frac{d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2}{c^2}}$$

$$= \int_1^2 \sqrt{1 - \frac{v^2}{c^2}}\; d\bar{t}$$

where: $d\bar{x}$, $d\bar{y}$, $d\bar{z}$ are spatial coordinate separations of two infinitesimally near events on the trajectory of the object, $d\bar{t}$ is their coordinate time separation, c is the speed of light, and v is the coordinate speed of the object. To every event there is a set of these rectangular coordinates $\bar{t}$, $\bar{x}$, $\bar{y}$, $\bar{z}$.

If two events are connected by a continuous curve of events, the sign of the expression

$$G \equiv d\tau^2 = d\bar{t}^2 - \frac{d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2}{c^2}$$

for two neighboring points on an element of such a trajectory tells whether the element is timelike (G is positive) or spacelike (G is negative). If timelike the element can be regarded as a portion of the trajectory of an object--and $d\tau$ is the elementary time interval measured by a clock comoving with the object. If spacelike, a portion of a suitably moving rigid meter stick could be placed momentarily in coincidence with the element, and $(-c^2 d\tau^2)$ would be the square of the spatial length of the element.

If we wish we can of course use spherical inertial coordinates ($\bar{t}$, $\bar{r}$, $\bar{\theta}$, $\bar{\varphi}$) related in the usual way to the rectangular ones. Trajectories, and spatial or temporal elements can be obviously expressed in terms of these as well.

(C) Coordinates on a Spinning Spherical Surface

Now consider a rotating spherical surface as a simple example, somewhat illustrative of our later analysis. A point P fixed on its surface has a spacetime trajectory described by the equations

$$\bar{t} = \lambda$$

$$\bar{r} = R$$

$$\bar{\theta} = \theta$$

$$\bar{\varphi} = \varphi + \omega\lambda .$$

where: R is the sphere's radius, θ, φ are the fixed colatitude and longitude angles of the point, ω is the angular velocity and λ is a time parameter along the trajectory (not the particle's proper time!)

1. Coordinate System (A)

One sees that the quantities λ, θ, φ specify the time and space coordinates of events happening on the surface of the sphere. R and ω are simply known constant parameters.

Operationally, λ is just the "time" at an event as read off some clock located in the space outside but very near the surface of the sphere at the locus of the event in question. All the inertial clocks in space are fixed relative to the fixed stars and are synchronized and will run at the same rates, that is, remain in synchronization, if they are identically constructed and adjusted. A clock fastened to the spherical surface can be adjusted to give the "λ-time" i.e. read the same value $\bar{t}$ as the inertial clock in space near which it finds itself at any given instant. But although these surface clocks may have similar construction, the rate adjustment which specifies λ-time will depend on the latitude θ-- because of their motion relative to the fixed stars due to the spin. Hence these clocks will not all generate seconds--only the polar clock will. This λ-time is similar to our astronomically determined time. Readings from other clocks which do generate seconds on the moving surface (like standard atomic clocks) would diverge from the readings from the coordinate λ-time adjusted clocks. At the equator, taking the speed $v = R\omega$ equal to the earth's equatorial speed of 500 m/s, the λ-time unit would be shorter by the factor $\sqrt{1 - v^2/c^2}$ than a second; that is, these coordinate clocks at the equator would be adjusted to run faster by 1.4 parts in 10^{12} than those clocks on the earth's surface which generate seconds. Only in this fashion could they keep in synchronization with the spatial inertial clocks--which, in their fixed frame of reference, all generate standard seconds.

The latitude and longitude angles θ, φ of a point on the surface have the usual conventional meanings. This is not the place to delve into the question of practical navigational procedures for determining θ and φ. We merely have wished to emphasize the close connection between time and space coordinates in our example. This is accented if one writes the expression for the "metric"--the expression of the square of the temporal or spatial interval in terms of spacetime coordinate differentials:

$$G_A \equiv c^2 d\lambda^2 - R^2 \left[(d\theta)^2 + \sin^2\theta \, (d\varphi + \omega \, d\lambda)^2 \right]$$

From this expression one can reaffirm the latitude dependence of the proper length of the coordinate time unit; one also notices, that because of the spinning motion the spatial part of the coordinate system which ordinarily is "flat", presents a twisted appearance from the spacetime point of view.

2. Coordinate System (B)

One may attempt to eliminate the foregoing latitude dependence of coordinate time by utilizing identical atomic clocks sprinkled over the surface of the sphere. These would be synchronized initially so that all clocks on a circular meridian (as judged from the fixed star inertial system) are synchronized. But immediately it becomes clear that these clocks will not maintain synchronization as viewed from inertial clocks distributed along a celestial meridian. Indeed the atomic clocks at the equator will lose time in relation to those on the celestial meridian. From the viewpoint of an observer in space this is due to the relative motion. For an observer on the surface, this is due to the centrifugal field. In any event, a meridian of longitude on the surface, defined to be the locus of events which all happen at the same time as read on the atomic clocks fixed to the surface, <u>and</u> which all have the same value of φ, of course, will present a distorted appearance relative to the celestial meridian. For the clocks near the equator must travel farther before they read the same time as the polar ones. One may study this quantitatively by transforming the metric according to the equations

$$\bar{t} = \tau / \left\{ 1 - R^2 \omega^2 \sin^2\theta / c^2 \right\}^{\frac{1}{2}}$$

$$\bar{\theta} = \theta$$

$$\bar{\varphi} = \varphi + \omega \tau / \left\{ 1 - R^2 \omega^2 \sin^2\theta / c^2 \right\}^{\frac{1}{2}}$$

Here τ is the proper time read off the atomic clock. The metric becomes

$$G_B \equiv d\left[\frac{c\tau}{\sqrt{1 - R^2\omega^2\sin^2\theta/c^2}}\right]^2 - R^2\left[d\theta^2 + \sin^2\theta\left\{d\varphi + d\frac{\omega\tau}{\sqrt{1 - R^2\omega^2\sin^2\theta/c^2}}\right\}^2\right]$$

From this we see that the length of the coordinate time unit of these clocks (equal to the square root of the coefficient of $cd\tau$) is indeed unity, i.e. one second. But the cross-produce term involving $d\theta\, d\varphi$ shows that the angle between meridians (as previously defined) and parallels of latitude will become zero in a rather short time. In fact this time is

$$t_{\infty} = \frac{(1 - \beta^2\sin^2\theta)^{3/2}}{\omega\beta\sin\theta\cos\theta}, \quad \beta = R\omega/c$$

which amounts to 500 years for $R\omega$ equal to the earth's equatorial speed, and $\theta = 45^{\circ}$. At this time and in this region the spacetime coordinates as defined become physically meaningless (the singularity is only significant over an exceedingly narrow belt around 45° latitude. As time goes on this belt would widen gradually.)

So again we note the close connection between space and time in defining operationally a spacetime coordinate system, and the care one must use so that it will be utilizable.

(D) Effects of Gravity

Fortunately, one can avoid these questions--for an earth navigational system--for most practical purposes. But the difficulty in principle still remains. Its avoidance in practice is a result of two additional effects[5] which combine and tend to cancel out the foregoing clock rate effect due to spin (almost) entirely. These are

(1) effects of gravity

(2) non-sphericity of the earth.

A short discussion of this will not only serve to quiet the practical surface navigator's fears, but will also introduce the essential connection between metric components and gravitation discovered by Einstein. This is very important for our later discussion.

Consider a clock which is stationary on the rotating earth's surface at latitude θ. Then according to Einstein, the proper time interval between successive instants is, in terms of the coordinate time interval $d\lambda$,

$$d\tau_{\theta} = \sqrt{1 + \frac{2\delta_{\theta}}{c^2} - \frac{v_{\theta}^2}{c^2}}\; d\lambda$$

where

$$\delta_\theta = -(GM_E/R)\,(1 + f(\theta))$$

is the earth's gravitational potential per unit mass at θ, M_E is the earth's mass and R is its mean radius. $f(\theta)$ corrects for the non-sphericity.

$$v_\theta = R_\theta \omega$$

is the corresponding speed of the earth's surface at this latitude due to its spin ω. For a clock at the pole, $\theta = 0$, and $v_o = 0$, and we have

$$d\tau_o = \sqrt{1 + \frac{2\delta_o}{c^2}}\; d\lambda$$

Let the two clocks read the same difference $d\lambda$ in coordinate time--their physical state or phase differences are equal so that

$$\nu_o\, d\tau_o = \nu_\theta d\tau_\theta,$$

where ν_o and ν_θ are the respective proper frequencies of their ticking rates (relative to an atomic frequency standard). Then

$$\frac{\nu_o}{\nu_\theta} = \frac{d\tau_\theta}{d\tau_o} \cong \left\{1 + \frac{-\delta_o - (\frac{1}{2} v_\theta^2 - \delta_\theta)}{c^2}\right\}$$

The earth's surface is one of practically zero hydrostatic pressure, as if it were a spinning gravitating fluid, and Lamb shows[6] that

$$\delta_\theta - \delta_o = \tfrac{1}{2} v_\theta^2$$

for such a geoidal shape. Thus the gravitational "red shift" fortuitously cancels out the second order Doppler shift due to motion for clocks at the earth's surface.

The main point of all this is that in order to specify a spacetime coordinate system operationally, one must know not only the motions involved but also the gravitational potentials. All these things are included in the all-important expression for the metric of temporal and spatial intervals. In any spacetime coordinate system, if dx^{ι} ($\iota = 0, 1, 2, 3$)* represent the differences of coordinates between two events, then the spatial or temporal interval is calculated from

$$G \equiv g_{\iota j} \, dx^{\iota} dx^{j}$$

(summed on ι, j)

The coefficients $g_{\iota j}$ which are generally functions of the x's specify both the spacetime curvature and the coordinate curvature properties (perhaps due to motion). If G is positive, the interval is timelike, and negative, if it is spacelike.

III. Hamiltonian Optics

The examples have shown (1) the importance of the spacetime metric in defining the properties of a coordinate system and (2) the care one must exercise in order to specify the coordinate system in an operational way. Indeed we were (intentionally) somewhat vague about the specification of latitude and longitude variables--because that was not our main point, and because it is quite a complex question. As we shall soon see, even for a class of cases specially tailored for relativistic treatment, a complete definition like this is complicated.

It seems clear that to define the space and time variables of a coordinate system one must rely heavily on theories of (electromagnetic) wave motion. This has always been true, even in classical surveying, or in taking sights on stars in navigation--and it is even more important to base our operations on our theoretical knowledge of optics in view of the prime importance of radio signals in space navigation. Consequently, before we discuss our main subject (which is: null, or range and range-rate coordinates), we shall present a brief summary of and a few results from relativistic Hamiltonian optical theory. For this is the simplest form of wave theory, and the most applicable at present.

A. Refraction Index Theory

Sufficient generality will be introduced if we consider the medium through which our radio signals propagate to be specified completely in terms of a scalar refractive index n, which may depend on position and frequency, ν, of the wave relative to the medium, and a four-velocity vector field U^{ι} which also may be position and time dependent. These are presumed to be given. In case we have a near perfect vacuum, $n = 1$.

* Usually, but not always, one reserves the index value 0 for the time-like coordinate.

A wave front is specified by a surface in spacetime which has a covariant normal 4-vector, K_ι, also known as the propagation vector, frequency vector, or wave-number vector. The latter designation is to be preferred since if one makes an arbitrary displacement dx^ι, 2π times the number of waves passed in making it is by definition

$$d\varphi = K_\iota dx^\iota$$

Thus $d\varphi$ is the phase change in the displacement; we shall assume the existence of a definite phase function φ (coherent waves) so that

$$K_\iota = \partial_\iota \varphi$$

Note that in the simple flat space rectangular coordinate system case, K_o is $2\pi/c$ times the frequency measured relative to an observer at rest in this system; $K_{1,2,3}$ are the negative components of the usual 3-dimensional propagation vector, times 2π. But in any case the proper frequency of the wave relative to the medium is given by

$$\nu = \frac{K_\iota U^\iota}{2\pi} \quad .$$

Let u^ι be the 4-vector components of the motion of a point associated with constant phase and moving "with" the wave. Its direction coincides with the generators of $\varphi = 0$. Then

$$K_\iota u^\iota = 0$$

specifies that it is normal to the gradient of φ; furthermore, since all material velocity vectors are of length c in spacetime, $|U| = c$; and by the definition of $\vec{u}$, we have, with $U_\iota \equiv g_{\iota j} U^j$,

$$U_\iota u^\iota = c^2 .$$

The spatial part of the velocity vector $\vec{u}$ relative to the material medium is

$$\vec{v} = \vec{u} - \vec{U}$$

since $\vec{U}$ is timelike. The phase velocity is the vector $\vec{u}_\varphi$ which minimizes $|v|$ subject to the foregoing conditions. The corresponding length v_φ of $\vec{v}$ is called the phase speed.

We find that

$$u_{\varphi}^{\iota} = \frac{K^{\iota}\ (c^2/2\pi\nu) + (n^2 - 1)\ U^{\iota}}{n^2}$$

and

$$\frac{c^2}{v_{\varphi}^2} = n^2 = 1 - K_{\iota} K^{\iota} \left[c^2/(K_j \cdot U^j)^2\right]$$

a formula which is given by Synge[7]. Since $K_{\iota} = \partial_{\iota}\varphi$, this yields a partial differential equation (eikonal) for the phase function φ. In fact, we may write it in the form

$$\bar{\Omega}\ (x, \partial\varphi) \equiv \bar{g}^{\iota j}\ \partial_{\iota}\ \varphi\, \partial_j \varphi = 0$$

where

$$\bar{g}^{\iota j} = g^{\iota j} + (n^2-1) U^{\iota}\ U^j/c^2$$

can be regarded as a new "metric" tensor in case n does not depend on ν (no dispersion). The quantities $g^{\iota j}$ are the contravariant components of the usual metric tensor.

The use of this "metric" gives rise to the terminology "optical distance"--and its introduction is related to the early "4/3-earth" concept[8], and similar homologous space notions [9,10,11,12] whereby investigators have tried to replace the solution of propagation problems in physical spacetime with a variable refraction index, by their solution in a different spacetime with a different--perhaps a constant--index.

(B) Hamilton's Principle

One now pictures spacetime as a manifold of events, which is permeated with the aforementioned material medium. At each point-event, x^{ι}, there is visualized a "metric null cone" whose apex is at the event. (See Figure 1). Its equation is

$$\Omega \equiv g_{\iota j}\ dx^{\iota}\ dx^j = 0$$

and it delineates, locally, events (end points of the vector $\vec{dx}$) which separate the past events, associated with x^{ι} from the present ones, and the present ones from the future ones. There is also another cone of events, the "light cone", described by

$$\bar{\Omega} \equiv \bar{g}_{\iota j}\ \ dx^{\iota}\ dx^j = 0$$

which also has its apex at x^{ι}. It coincides with the metric cone only if $n = 1$. We note that

$$\bar{g}_{\iota j} = g_{\iota j}\ -(1 - 1/n^2)\ U_{\iota}\ U_j/c^2$$

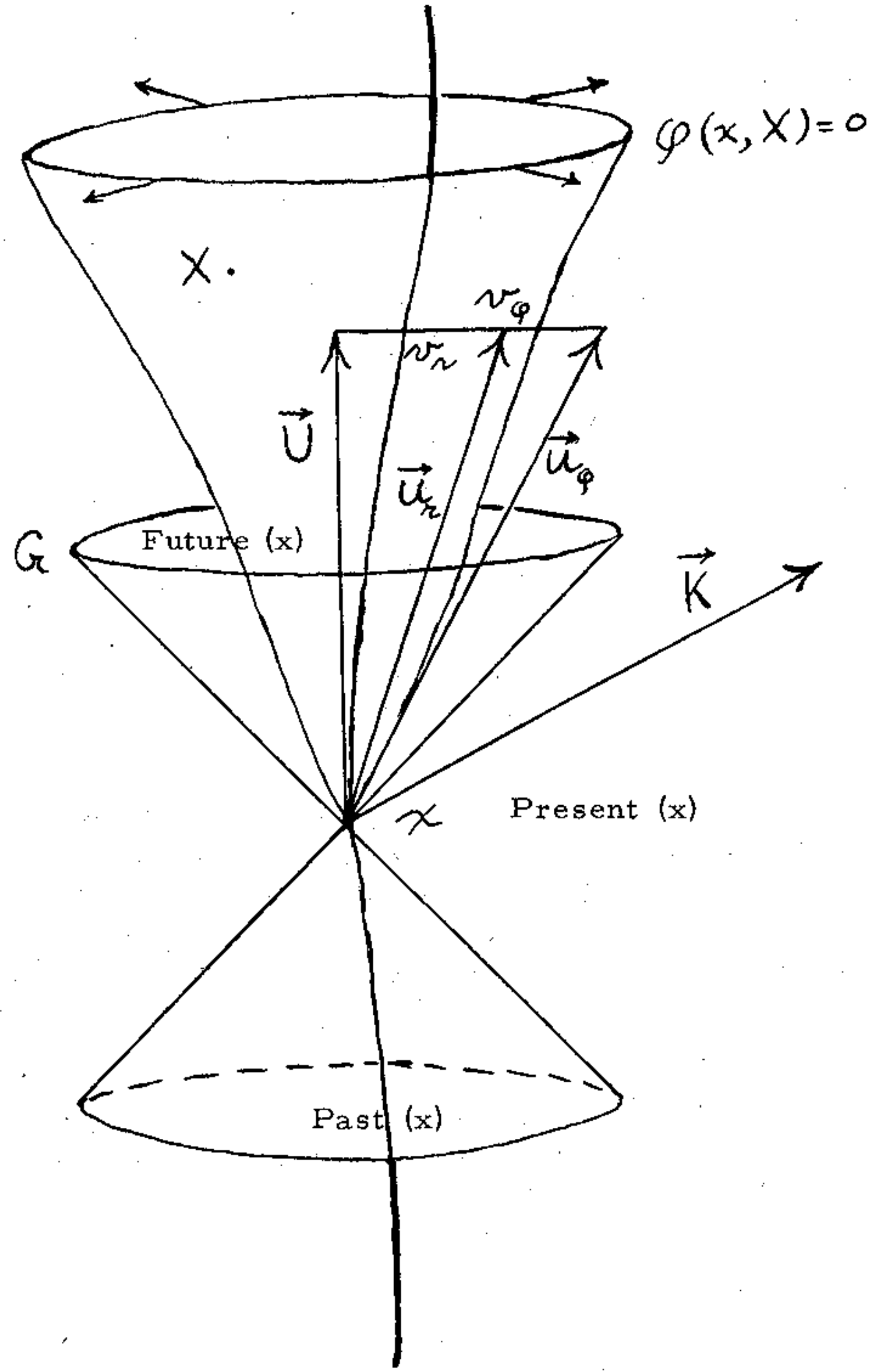

Figure 1: The metric null-cone, G, and the light phase-wave-front, φ, associated with an event x in (2 + 1)-spacetime for $n \neq 1$. Also indicated are the relations for normal dispersion between phase and group velocities and speeds, and the wave-number vector. The classes of events comprising the past, present and future of x are shown.

One frustum of this cone coincides with a phase front convergingon x^{ι}; the other coincides with a phase front emanating from x^{ι}. The "surface" $\varphi\,(x, X)=0$ coincides with this cone in the neighborhood of x.

Now the optical rays of this system are curves which do not necessarily coincide with the generators of the light cone. They are curves which extremalize the phase difference φ between two events, subject to the equation defining phase speed in terms of refraction index. Thus the principle

$$\delta\varphi = \delta\int_1^2 K_{\iota}\, dx^{\iota} = 0$$

$$\overline{\Omega} = g^{\iota j} K_j \cdot K_{\iota} + (n^2 - 1)\,(K_{\iota} U^{\iota}/c)^2 = 0$$

leads to the differential equations of the rays:

$$\frac{dx^{\iota}}{d\lambda} = \frac{\partial\overline{\Omega}}{\partial K_{\iota}}\,, \qquad \frac{dK_{\iota}}{d\lambda} = -\frac{\partial\overline{\Omega}}{\partial x^{\iota}}\,.$$

The ray velocity is

$$u_r^{\iota} \equiv \frac{dx^{\iota}}{d\lambda}\;\frac{d\lambda}{d\tau}\,;$$

its spatial component normal to U^{ι} has a length equal to the ray speed v_r relative to the medium, and so

$$u_r^{\iota}\, U_{\iota} = c^2$$

which serves to define

$$\frac{d\tau}{d\lambda} = \frac{U_{\iota}}{c^2}\;\frac{\partial\overline{\Omega}}{\partial K_{\iota}}\,.$$

The result of all this is that the phase front satisfies the eikonal or Hamilton-Jacobi equation,

$$\overline{\Omega}\,(x,\ \partial\varphi/\partial x) = 0$$

as before, and the ray velocity is

$$u_r^{\iota} = \frac{K^{\iota} c^2/2\pi\nu + (n^2 - 1 + n n'\nu)\; U^{\iota}}{n(n + n'\nu)}$$

where

$$n' \equiv \partial n/\partial\nu\,.$$

The ray speed turns out to be

$$v_r = c\Big/\frac{\partial(n\nu)}{\partial\nu}$$

which is identical with the group speed in such a dispersive medium. One sees immediately that, if the medium is not dispersive, i.e. $n' = 0$, the phase speed and velocity become identical with the ray speed and velocity--a well-known result. If the ray speed is less than the phase speed, the dispersion is normal, and if greater, the dispersion is anomalous. To maintain causality we require that $v_r < c$.

(C) The General Doppler Effect

It is appropriate to conclude our discussion of basic principles with a description of the general Doppler effect--for this effect relates measurements of velocity and range to measurements of frequencies and time--the basis for the entire sequel. Consider two observers, A and B, moving quite generally at some distance from each other. At the moment when A's spacetime velocity $\vec{U}_A$ has components U^ι_A, let him send a signal to B. This signal corresponds to a phase Φ. When this happens, let A's clock read the proper time τ^A and let B's clock read τ^B when this phase front reaches him. If

$$x^\iota_A(\tau^A) \qquad \text{is A's position}$$

and

$$x^\iota_B(\tau^B) \qquad \text{is B's position}$$

for these two events, we know that the phase front function φ defined by

$$\varphi\left\{x_A(\tau^A),\ x_B(\tau^B)\right\} = \int_A^B K_\iota\, dx^\iota = 0$$

since the same wave front passes through A and B. A moment later a different phase signal corresponding to the value $\Phi + \Delta\Phi$ is sent at time $\tau^A + d\tau^A$ and received at time $\tau^B + d\tau^B$. But since φ measures the difference in phase between two events in a wave region, we again have

$$\varphi\left\{x_A(\tau^A + d\tau^A),\ x_B(\tau^B + d\tau^B)\right\} = 0$$

Now

$$\frac{\Delta\Phi}{2\pi} = \nu_A\, d\tau^A = \nu_B\, d\tau^B$$

where ν_A and ν_B are the proper frequencies of the radiation observed at A and at B respectively. Consequently we conclude that

$$\frac{d\tau^A}{d\tau^B} = \frac{\nu_B}{\nu_A} = \frac{\partial_\iota \varphi U^\iota \big|_B}{\partial_\iota \varphi U^\iota \big|_A} = \frac{K_\iota(B) U^\iota_B}{K_\iota(A) U^\iota_A}$$

This is the general Doppler relation between the proper frequency at B and that at A.

IV. Range and Range-Rate Systems: Null Coordinates in Spacetime

We are in a position now to define operationally and discuss in detail the properties of a particular class of spacetime coordinates. They can be used to measure the components of the gravitational field plus refraction index and material velocity field, i.e. the light cone components $\overline{g}^{ij}$; they can be used to determine the range and coordinates of a particle moving arbitrarily relative to any coordinate system; they can be used to specify the coordinate velocity components of such a particle. And these measurements are all relatable to frequency and time measurements made in terms of standard devices and systems carried in suitably moving reference vehicles.

(A) An Example

As a first very simple example, we shall consider motion only in one spatial dimension, and shall assume that the refractive index $n = 1$, and that spacetime is flat (no gravity). For this situation the phase function is simply

$$\varphi = (x^o - X^o)^2 - (x' - X')^2$$

where:

$$x^o = ct, \quad X^o = cT$$

$$x' = x, \quad X' = X$$

are the coordinates of two arbitrary events.

Let us consider a reference vehicle situated at the origin of the x-axis. Its trajectory in these coordinates is

$$x^o = X^o(\tau) = c\tau$$

$$x' = X'(\tau) = 0$$

where τ is the proper time as read off a standard frequency generator and clock carried by the vehicle--it happens, for this simple case, to be the same as the "fixed star" time t. We wish to associate readings on this clock with the position of a particle having arbitrary coordinates, (x^o, x') and associate frequency measurements with the velocity components of a particle moving arbitrarily in (1 + 1)-spacetime.

Light or radio signals sent from the reference vehicle can be scattered back to it from the particle in question, and the times and frequencies of these sent-and-received signals measured. Let a signal be sent at time $\tau^{\overline{o}}$ and received back at time $\tau^{\overline{1}}$, and let the corresponding frequencies be $\nu_{\overline{o}}$ and ν_{T}. See Figure 2a.

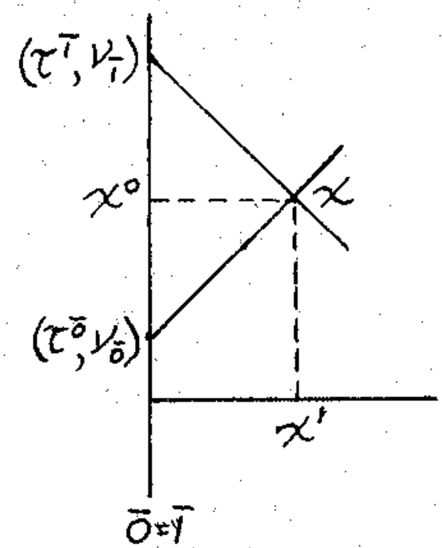

Figure 2a: An example for (1 + 1)-spacetime

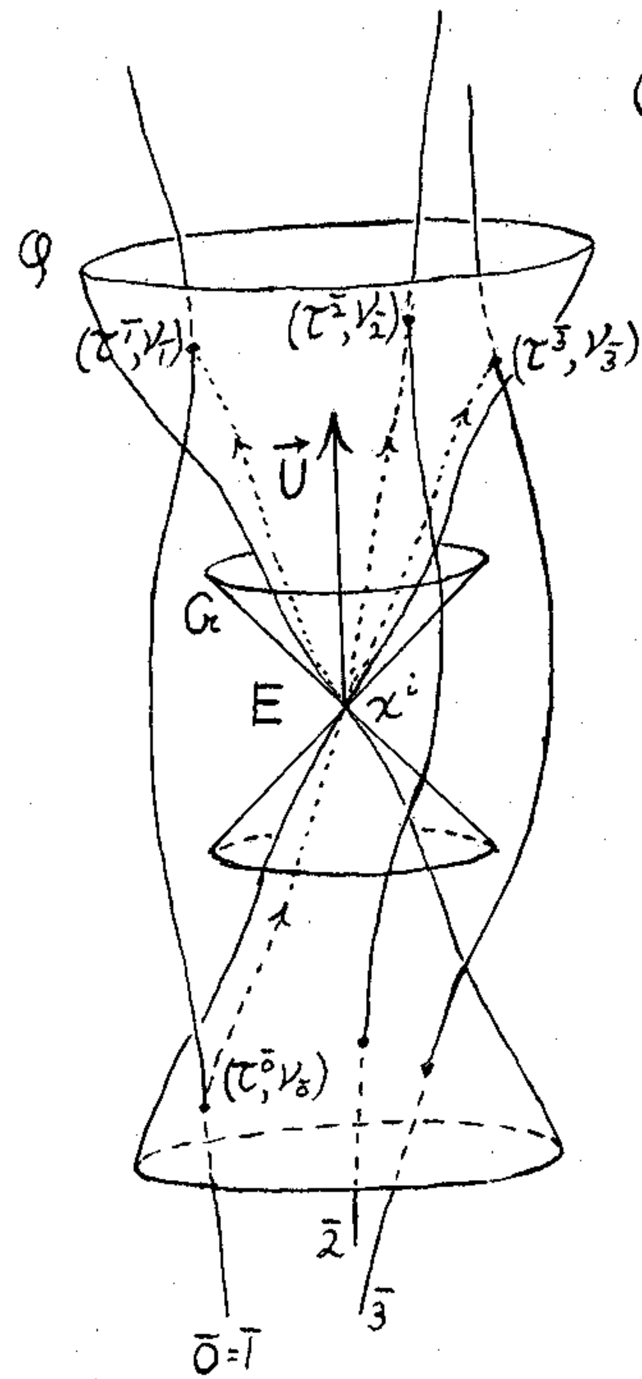

Figure 2: Null coordinates $\tau^{\bar{i}}_{\rightarrow}$ and frequencies $\nu_{\bar{i}}$ associated with a particle moving with velocity U at event E, which has the general position coordinates x^i. The cone is shown as a (2 + 1) - dimensional cone instead of a (3 + 1)-figure, for simplicity in the drawing. The third spatial dimension is suppressed.

Now since

$$\varphi(\bar{o}) = (x^o - c\tau^{\bar{o}})^2 - (x')^2 = 0$$

$$\varphi(\bar{1}) = (x^o - c\tau^{\bar{1}})^2 - (x')^2 = 0$$

we can determine the coordinates x^o and x' from the measured times $\tau^{\bar{o}}$ and $\tau^{\bar{1}}$. In fact either

$$x^o = \tfrac{1}{2}c(\tau^{\bar{o}} + \tau^{\bar{1}})$$

$$x' = \tfrac{1}{2}c(\tau^{\bar{1}} - \tau^{\bar{o}}) \qquad \text{(I)}$$

or

$$x^o = \tfrac{1}{2}c(\tau^{\bar{o}} + \tau^{\bar{1}})$$

$$x' = \tfrac{1}{2}c(\tau^{\bar{o}} - \tau^{\bar{1}}) \qquad \text{(II)}$$

The ambiguity can be resolved if one knows whether x' is positive or negative. One of these sets, say the first (I), constitutes a transformation equation from the x^i coordinates to the $\tau^{\bar{i}}$ set. Using it, one may easily calculate relations between quantities described in either coordinate system. The metric tensor has components

$$g_{ij} : \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

in the original rectangular system, and components

$$g_{\bar{i}\bar{j}} : \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$$

in the $\tau^{\bar{i}}$--or range system. Hence the metric expression is

$$G = (dx^o)^2 - (dx')^2$$

$$= c^2\, d\tau^{\bar{o}}\, d\tau^{\bar{1}}$$

in the two systems.

Similarly, consider the particle at (x^o, x') to be moving with spacetime velocity components

$$U^o = dx^o/d\tau = c\, dt/d\tau$$
$$U' = dx'/d\tau = dx/d\tau$$

(Note: $dt/d\tau = 1/\sqrt{1-(v/c)^2}$, where $v = dx/dt$ is the usual speed of the particle; also $d\tau = \sqrt{(dx^o)^2 - (dx')^2}\,/c = \sqrt{d\tau^{\bar{o}} d\tau^{\bar{1}}}$.) These can be transformed to the range-system, and yield

$$U^{\bar{o}} = c\frac{d\tau^{\bar{o}}}{d\tau} = c\sqrt{\frac{d\tau^{\bar{o}}}{d\tau^{\bar{1}}}}$$

$$U^{\bar{1}} = c\frac{d\tau^{\bar{1}}}{d\tau} = c\sqrt{\frac{d\tau^{\bar{1}}}{d\tau^{\bar{o}}}}$$

If a phase difference

$$d\varphi = \nu_{\bar{o}}\, d\tau^{\bar{o}}$$

is recorded at the sender, where $\nu_{\bar{o}}$ is the frequency of the signal as sent, the frequency observed at the particle will be ν, where

$$\nu\, d\tau = \nu_{\bar{o}}\, d\tau^{\bar{o}}$$

and at the receiver it will be $\nu_{\bar{1}}$ where

$$\nu_{\bar{1}}\, d\tau^{\bar{1}} = \nu_{\bar{o}}\, d\tau^{\bar{o}} .$$

Hence we see that

$$\nu^2 = \nu_{\bar{o}}\, \nu_{\bar{1}}$$

(a special case of a very general result soon to be noted) and

$$U^{\bar{o}} = c\sqrt{\frac{\nu_{\bar{1}}}{\nu_{\bar{o}}}}$$

$$U^{\bar{1}} = c\sqrt{\frac{\nu_{\bar{o}}}{\nu_{\bar{1}}}}$$

The velocity components in the range and range-rate coordinates are simply obtained in terms of the measured frequencies.

Other simple examples have been worked out. The motion of the standard reference vehicle has been made more general in relation to the x^i-coordinate system. An example in three spatial dimensions and one time dimension, ignoring gravity effects and refraction, and with special reference vehicle motions has been considered. But those general principles underlying the range and range-rate system which are illustrated above can be just as easily derived in the rather general case, which follows.

(B) The General Dispersionless Case

Let $\varphi(x,x')$ be the phase function defined in the foregoing sections. It measures the difference in phase of a signal sent from event x' to event x, so that if the two events experience the same phase, they are on the same wave-front, and

$$\varphi(x, x') = 0.$$

Let there be three reference vehicles moving in arbitrary but known ways. See Figure 2. These vehicles carry identical standard frequency and time generating and measuring equipment. Their motions may be described in terms of some convenient set of coordinates by

$$x^{\iota} = X^{\iota}_{\bar{j}}(\tau), \quad (\bar{j} = \bar{0}, \bar{1}, \bar{2}, \bar{3})$$

where τ, in each case, is the proper time recorded on the corresponding vehicle's clock. Since there are just three vehicles, we require in addition that

$$X^{\iota}_{\bar{0}}(\tau) \equiv X^{\iota}_{\bar{1}}(\tau).$$

Let x^{ι}_E be general spacetime coordinates in this system of a particle when some event E happens to it. Then the wave-front, or light "cone" with apex at E intersects the trajectories of the reference vehicles at various proper times. We are assuming no dispersion ($n' = \partial n/\partial \nu = 0$) so that the ray (or group, or signal) velocities coincide with the phase velocity--and the signal trajectories coincide with the light cone generator curves. These are shown dotted in Figure 2.

If a signal is sent from vehicle $\bar{0}$ ($=\bar{1}$) at its proper time reading $\tau^{\bar{0}}$ this signal is scattered from the particle at E back to $\bar{o}$ and to $\bar{2}$ and $\bar{3}$, being received at times $\tau^{\bar{1}}$, $\tau^{\bar{2}}$, and $\tau^{\bar{3}}$ respectively. If the carrier frequency of the original signal is $\nu_{\bar{o}}$, it will be Doppler shifted during the scattering because of the relative motions of the reference vehicles and the particle velocity U^{ι}_E. We suppose its frequencies as received to be $\nu_{\bar{1}}$, $\nu_{\bar{2}}$, and $\nu_{\bar{3}}$.

1. Range

Now because the event E and the signal transmission and reception events lie on the same wave-front (light cone), we can write

$$\varphi(\bar{\iota}) \equiv \varphi(x_E, X_{\bar{\iota}}(\tau^{\bar{\iota}})) = 0, \quad (\bar{\iota} = \bar{0}, \bar{1}, \bar{2}, \bar{3})$$

These are four equations to determine the general coordinate values x^{ι}_E of E, in terms of the measured "range coordinates" $\tau^{\bar{\iota}}$. From its definition, we know that

$$\partial\varphi(x, X)/\partial x^{\iota} = -\partial\varphi(x,X)/\partial X^{\iota}.$$

So by differentiation we find the transformation coefficients between the x-coordinates and the range coordinates to have the values

$$\frac{\partial \tau^{\bar{\iota}}}{\partial \tau^{j}} = \frac{\partial_j \varphi(\bar{\iota})}{\partial_k \varphi(\bar{\iota})\, U^{k}_{\bar{\iota}}(\bar{\iota})}$$

The quantities $U^{k}_{\tau}(\bar{\iota})$ are the velocity components in the x-system, of the ιth reference vehicle at its time τ^{ι}:

$$U^{k}_{\bar{\iota}}(\bar{\iota}) = \left.\frac{d X^{k}_{\bar{\iota}}(\tau)}{d\tau}\right|_{\tau = \tau^{\bar{\iota}}}$$

In order for the inverse transformation to exist we see that the determinant

$$\left|\partial_j \varphi(\bar{\iota})\right| \neq 0 \; .$$

Now we remember that $K_{\iota} = \partial_{\iota}\varphi$ and $2\pi\nu = K_{\iota} U^{\iota}$. Hence

$$\frac{\partial \tau^{\bar{\iota}}}{\partial x^{j}} = \frac{K_j(\bar{\iota})}{2\pi \nu_{\bar{\iota}}} \; .$$

The contravariant light cone tensor components are particularly interesting to calculate in the range-system. We find, according to the general tensor transformation law, that

$$\bar{g}^{\bar{\iota}\bar{j}} = \frac{\bar{g}^{\iota j} K_{\iota}(\bar{\iota}) K_{j}(\bar{j})}{4\pi^2 \nu_{\bar{\iota}}\, \nu_{\bar{j}}}\, c^2 \; .$$

c^2 is simply a convenient normalization constant. Since $\bar{g}^{\iota j} K_{\iota}(\bar{\iota}) K_{j}(\bar{\iota}) = 0$ is the basic eikonal equation for φ, we see that

$$\bar{g}^{\bar{\iota}\bar{\iota}} = 0$$

When $n = 1$, this means that $g^{\bar{\iota}\bar{\iota}} = 0$ and the metric for range coordinates has a vanishing diagonal and six non-vanishing components, in general. This identifies this class of coordinates with Synge's class of "null" coordinates. It should be noted that the components of the inverse matrix to $\bar{g}^{\iota j}$ can readily be calculated, but the expressions are too lengthy to include here.

2. Range-rate

Suppose that in time $d\tau^{\bar{o}}$ we have $\nu_{\bar{o}}\, d\tau^{\bar{o}}$ waves emitted from the sender of vehicle $(\overline{0} - \overline{1})$. These are scattered off the particle at event E and sent back to the sender and to the other two reference vehicles. If the frequency ν_E is observed at E during time $d\tau^{E}$, and measured as $\nu_{\bar{1}}$, $\nu_{\bar{2}}$, $\nu_{\bar{3}}$ at the other receivers in times $d\tau_{\bar{1}}$, $d\tau_{\bar{2}}$, $d\tau_{\bar{3}}$ respectively, then we have

$$\nu_{\bar{o}}\, d\tau^{\bar{o}} = \nu_E\, d\tau^E = \nu_{\bar{1}}\, d\tau^{\bar{1}} = \nu_{\bar{2}}\, d\tau^{\bar{2}} = \nu_{\bar{3}}\, d\tau^{\bar{3}} .$$

Now we know that the spacetime velocity vector of the particle at E is of length c, in spacetime, so that

$$U_E^{\bar{i}}\ U_E^{\bar{j}}\, g_{\bar{i}\bar{j}} = c^2$$

But

$$U_E^{\bar{i}} = c\,\frac{d\tau^{\bar{i}}}{d\tau^E} = c\,\frac{\nu_E}{\nu_{\bar{i}}}$$

so that

$$\frac{1}{\nu_E^2} = \frac{g_{\bar{i}\bar{j}}}{\nu_{\bar{i}}\,\nu_{\bar{j}}} .$$

These very basic results show that the velocity components of a particle in the range and range-rate coordinates are determinable entirely in terms of frequency measurements at the reference vehicles. Of course the metric $g_{\bar{i}\bar{j}}$ must be known--but the last formula suggests a method for measuring these six components. If one makes a determination of ν_E and the $\nu_{\bar{i}}$ for six arbitrarily moving particles in the neighborhood of event E, then he may calculate the $g_{\bar{i}\bar{j}}$ from these measurements.

By a transformation of coordinates using the components $\partial\tau^{\bar{i}}/\partial x^j$, or the inverse matrix, one can determine the components of quantities relative to the x^i-coordinate system. We should note that to do this, one must have a knowledge of the K_i-vector at the standard reference vehicles.

(C) Error Estimates

Although we certainly cannot in this brief article present an exhaustive treatment of error analysis, it is perhaps instructive to write down certain preliminary relations. Assume the errors in coordinate determinations to be uncorrelated. Then from the transformation formulas, we may deduce

$$\overline{\left| d\tau^{\bar{i}} \right|^2} = \sum_j \left| \frac{K_j(\bar{i})}{2\pi\,\nu_{\bar{i}}} \right|^2 \ \left| \delta x_E^j \right|^2$$

as the relation between the mean square errors in coordinate determination. Next the observed frequency errors and those encountered in determining $g_{\bar{i}\bar{j}}$, are related by

$$\frac{\delta \nu_E}{\nu_E} = \frac{1}{2} \nu_E^2 \frac{g_{\bar{j}\bar{k}}}{\nu_{\bar{j}} \nu_{\bar{k}}} \left\{ \frac{\delta \nu_{\bar{j}}}{\nu_{\bar{j}}} + \frac{\delta \nu_{\bar{k}}}{\nu_{\bar{k}}} - \frac{\delta g_{\bar{j}\bar{k}}}{g_{\bar{j}\bar{k}}} \right\} .$$

Finally

$$\frac{\delta U_E^{\bar{i}}}{U_E^{\bar{i}}} = \frac{\delta \nu_E}{\nu_E} - \frac{\delta \nu_{\bar{i}}}{\nu_{\bar{i}}} .$$

(D) Conclusions

The analysis and results contained herein suggest that an extensive investigation be made of the possibilities of practical utilization of null coordinates. Certainly many interesting problems remain from a theoretical point of view. The complete specification of the wave number vector field in terms of frequency and time measurements is one. The extension to the case of dispersion is another. The solution of the eikonal equation in the presence of various gravitational and refraction index fields is another. Applications to specific spacetime navigation problems are highly desirable.

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