

ANALYSIS OF QUANTUM 1/f EFFECTS IN FREQUENCY STANDARDS

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Abstract

There is a fundamental limitation of the stability of frequency standards, caused by the quantum 1/f effect. This effect represents a quantum fluctuation of the physical currents, scattering cross sections and process rates, caused by the reaction of Bremsstrahlung on the charged particles which have emitted it. It is the only known quantum effect which is inversely proportional to Planck's constant, and which is therefore important in the limit of low, rather than high frequencies. In frequency standards the rate of change of the dipolar moment of the oscillating material system suffers a discontinuity every time an oscillation quantum is absorbed in the loss process. This causes the emission of elf bremsstrahlung and therefore quantum 1/f fluctuations in the loss rate, which translate into 1/fQ⁴ frequency fluctuations.

A newly discovered aspect of quantum physics results in a fundamental stability limit in frequency standards such as quartz resonators, Cs atomic clocks and H maser oscillators. This aspect is the quantum 1/f effect which causes physical cross sections σ and process rates Γ to fluctuate with a 1/f spectral density $2\alpha A\Gamma^2/f$. Physically, whenever the system loses a main oscillator mode phonon (quartz) or photon (Cs or H), the electric polarization current $\mathbf{J}=\dot{\mathbf{P}}$ caused by the oscillation suffers a sudden change which generates a bremsstrahlung amplitude $(\alpha A/f)^{1/2}$ for the emission of photons of arbitrarily low frequency f , which can not be detected. This yields a fluctuation of Γ because the final quantum state of the after the emission becomes a superposition of states with slightly different energies.

According to the general quantum 1/f formula, $\Gamma^{-2}S_{\Gamma}(f)=2\alpha A/f$ with $\alpha\equiv e^2/\hbar c = 1/137$ and $A\equiv 2(\Delta\mathbf{J}/ec)^2/3\pi$. Setting $\mathbf{J}=\dot{\mathbf{P}}/dt\equiv\dot{\mathbf{P}}$ where \mathbf{P} is the vector of the dipole moment of the quartz crystal, or of the group of N oscillating atoms of Cs or H, we obtain for the fluctuations in the rate Γ of phonon/photon removal from the main resonator oscillation mode by scattering on a phonon/photon from any other mode of average frequency $\langle\omega\rangle$ of the quartz crystal or of the Cs/H clock blackbody radiation background, (or via a two-phonon-

process at a crystal defect or impurity, involving a phonon of average frequency $\langle\omega\rangle$, or, e.g., by photon absorption in the cavity walls in the case of Cs or H standards) the spectral density

$$S_{\Gamma}(f) = \Gamma^{-2}4\alpha(\Delta\dot{\mathbf{P}})^2/3\pi e^2 c^2, \quad (1)$$

where $(\Delta\dot{\mathbf{P}})^2$ is the square of the dipole moment rate change associated with the process causing the removal of a phonon (or photon for Cs and H) from the main oscillator mode. To calculate it, we write the energy W of the interacting mode $\langle\omega\rangle$ in the form

$$W = \hbar f \langle\omega\rangle = 2(Nm/2)(dx/dt)^2 = (Nm/e^2)(e dx/dt)^2 = (m/Ne^2)\epsilon^2(\dot{\mathbf{P}})^2; \quad (2)$$

The factor two includes the potential energy contribution. Here m is the reduced mass of the elementary oscillating dipoles, e their charge, ϵ a polarization constant, and N their number in the resonator. Applying a variation $\Delta n=1$ we get

$$\Delta n/n = 2|\Delta\dot{\mathbf{P}}|/|\dot{\mathbf{P}}|, \text{ or } \Delta\dot{\mathbf{P}}=\dot{\mathbf{P}}/2n. \quad (3)$$

Solving Eq. (2) for $\dot{\mathbf{P}}$ and substituting, we obtain

$$|\Delta\dot{\mathbf{P}}| = (N\hbar\langle\omega\rangle/n)^{1/2}(e/2\epsilon). \quad (4)$$

Substituting $\Delta\dot{\mathbf{P}}$ into Eq. (6), we get

$$\Gamma^{-2}S_{\Gamma}(f) = N\alpha\hbar\langle\omega\rangle/3n\pi mc^2\epsilon^2 \equiv \Lambda/f. \quad (5)$$

This result is applicable to the fluctuations in the loss rate Γ of the frequency standard.

The corresponding resonance frequency fluctuations of the standard are given by

$$\omega^{-2}S_{\omega}(f) = (1/4Q^4)(\Lambda/f) = N\alpha\hbar\langle\omega\rangle/12n\pi mc^2\epsilon^2 Q^4; \quad \omega^2 = \omega_0^2 - 2\Gamma^2, \quad (6)$$

where $Q = \omega/2\Gamma$ is the quality factor of the single-mode resonator considered, and $\langle\omega\rangle$ is the circular frequency of the main oscillation mode ω_0 in the case of cesium clocks, Hydrogen maser clocks and laser clocks. However, for quartz resonators $\langle\omega\rangle$ is not the circular frequency of the main resonator mode, but rather the practically constant frequency of the average interacting phonon (or

photon for Cs and H standards).

For the case of cesium clocks, Hydrogen maser clocks, and for the case of the laser clocks being developed, we can set $N/n = 1$, because the number of excited (oscillating) ions or atoms equals the number of quanta n which they harbor (one quantum per atom), while in the case of a quartz resonator there are an average $n_{\omega} = kT/\hbar\omega$ quanta (phonons) present in any mode of frequency ω . Therefore, for these (non-quartz) systems, we particularize Eq. (10) by setting $N/n=1$ and we write

$$\begin{aligned} \Gamma^{-2} S_{\Gamma}(f) &= \Lambda/f = \alpha \hbar \omega_0 / 3\pi m c^2 f \\ &= 10^{-27} \text{erg.s. } \omega_0 / 10 \cdot 137 \cdot 0.9 \cdot 10^{-27} \text{ g. } 9 \cdot 10^{20} \text{ cm}^2/\text{s}^2 \\ &= 10^{-24} \omega_0 \end{aligned} \quad (7)$$

Using the approximate values of the resonance frequency in each standard, we obtain the following quantum 1/f flicker floors:

$$\begin{aligned} \text{For the Hydrogen Maser: } \omega_0 &= 10^{10} \text{ s}^{-1}; \\ S_{\delta\omega/\omega}(f) &= 2.5 \cdot 10^{-15} / Q^4 f; \end{aligned} \quad (8)$$

$$\begin{aligned} \text{For the Cs clock: } \omega_0 &= 6.3 \cdot 10^{10} \text{ s}^{-1}; \\ S_{\delta\omega/\omega}(f) &= 1.5 \cdot 10^{-14} / Q^4 f; \end{aligned} \quad (9)$$

$$\begin{aligned} \text{For the Rb clock: } \omega_0 &= 4 \cdot 10^{10} \text{ s}^{-1}; \\ S_{\delta\omega/\omega}(f) &= 10^{-14} / Q^4 f; \end{aligned} \quad (10)$$

$$\begin{aligned} \text{For the laser clock: } \omega_0 &= 10^{14} \text{ s}^{-1}; \\ S_{\delta\omega/\omega}(f) &= 2.5 \cdot 10^{-11} / Q^4 f; \end{aligned} \quad (11)$$

These are lower than the observed values; the quantum limit has not been obtained so far.

For the case of quartz resonators¹ we have used the interacting thermal mode of average frequency $\langle\omega\rangle$ to calculate the quantum 1/f effect. The corresponding $\Delta\dot{\mathbf{P}}$ in the main resonator mode of frequency ω_0 has to be also included in principle, but is negligible because of the very large number n of phonons (or photons for Cs present in the main resonator mode).

Eq. (11) can be written with $N \equiv VN/V$ for the case of quartz resonators in the form

$$S(f) = \beta V / f Q^4, \quad (12)$$

where, with an intermediary value $\langle\omega\rangle = 10^8/\text{s}$, with $n = kT/\hbar\langle\omega\rangle$, $T = 300\text{K}$ and $kT = 4 \cdot 10^{-21} \text{ J}$

$$\begin{aligned} \beta &= (N/V) \alpha \hbar \langle\omega\rangle / 12 n \pi \epsilon^2 m c^2 \\ &= 10^{22} (1/137) (10^{-27} 10^8)^2 / 12 k T \pi 10^{-27} 9 \cdot 10^{20} = 1. \end{aligned} \quad (13)$$

The form of Eq. (7) shows that the level of 1/f frequency noise depends not only as Q^{-4} as previously proposed by Handel, but also on the oscillation frequency or the volume of the active region. This theory qualitatively fits the data of Gagnepain who varied the Q-factor with temperature in the same resonator (but not frequency or volume) for the quartz case, and the data of Walls who considered several resonators which differ in volume and frequency, again only for the quartz case. The comparison with the experiment is not yet possible in the case of cesium and hydrogen standards, because they have not reached the quantum 1/f limit. At the present time, these standards are still limited in their stability by well-known factors which do not introduce a 1/f spectral dependence.

The theory also provides the basis for predicting how to improve the 1/f level of resonators, beyond just improving the Q-factor, which has been known for many years, and which has been related to fluctuations in the dissipation. Since the level depends on the active volume, one should use the lowest overtone and smallest diameter consistent with other circuit parameters. This is in good agreement with the experiment from all points of view.

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