SPECTRAL DENSITY ANALYSIS:

FREQUENCY DOMAIN SPECIFICATION AND MEASUREMENT OF SIGNAL STABILITY*

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Summary

Stability in the frequency domain is commonly specified in terms of spectral densities. The spectral density concept is simple, elegant, and very useful, but care must be exercised in its use. There are several different but closely related spectral densities, which are relevant to the specification and measurement of stability of the frequency, phase, period, amplitude, and power of signals. Concise, tutorial descriptions of useful spectral densities are given in this survey. These include the spectral densities of fluctuations of (a) phase, (b) frequency, (c) fractional frequency, (d) amplitude, (e) time interval, (f) angular frequency, and (g) voltage. Also included are the spectral densities of radio frequency power and its two normalized components, Script $\mathcal{L}(f)$ and Script $\mathcal{M}(f)$, the phase modulation and amplitude modulation portions, respectively. Some of the simple, often-needed relationships among these various spectral densities are given. The use of one-sided spectral densities is recommended. The relationship to two-sided spectral densities is explained. The concepts of cross-spectral densities, spectral densities of time-dependent spectral densities, and smoothed spectral densities are discussed.

Key words (for information retrieval): Amplitude Fluctuations, Cross-Spectral Density, Frequency Domain, Frequency Noise, Modulation Noise, Noise Specification and Measurement, Oscillator Noise, Phase Noise, Radio Frequency Power Spectral Density, Script $\mathcal{L}(f)$, Script $\mathcal{M}(f)$, Script $\mathcal{N}$, Sidebands, Signal Stability, Spectral Density.

Introduction

The economic importance of highly stable signals, signal sources, and signal-processors is increasing, e.g., in communications systems, navigation systems, and metrology. It is accompanied by an increasing need for carefully-defined and widely-disseminated terminology and language for specification and measurement of signal stability. A significant contribution was made in 1964 by the IEEE-NASA Symposium on the Definition and Measurement of Short-Term Frequency Stability. Its Proceedings[1] were followed in February 1966 by the very useful IEEE Special Issue on Frequency Stability.[2] In 1970 the authoritative paper, "Characterization of Frequency Stability", was published by the Subcommittee on Frequency Stability of the IEEE.[3] It is the most definitive discussion to date of the characterization and measurement of frequency stability.

Recently Shoaf et al. have prepared a tutorial, how-to-do-it technical report (NBS TN632) on specification and measurement of frequency stability.[4] The present paper is complementary to NBS TN632 and relies especially upon the material in it and in references 1 - 3.

Stability in the frequency domain is commonly specified in terms of spectral densities. The spectral density concept is simple, elegant, and very useful,[5] but care must be exercised in its use. There are several different, but closely related, spectral densities which are relevant to the specification and measurement of stability of the frequency, phase, period, amplitude and power of signals. In this paper, we present a tutorial discussion of spectral densities. For background and additional explanation of the terms, language, and methods, we encourage reference to NBS TN632[4] and to references 1 - 3. Other very important measures of stability exist;[2] we do not discuss them in this paper.

Some Relevant Spectral Densities

Twelve spectral densities which are especially useful in the specification and measurement of signal stability are listed below. Symbols otherwise undefined will be defined in a later section, where we also will define and derive the relationships of the various quantities. In our choice of concepts and symbols, we try to optimize conformity with traditional usage in the field and simultaneously to minimize what we see to be the hazards of vagueness, lack of completeness, and inconsistency. By fluctuations we mean noise, instability, and modulation. The twelve definitions follow.

$S_{\delta\phi}(f)$	Spectral density of phase fluctuations $\delta\phi$. The dimensionality is radians squared per hertz. The range of Fourier frequency f is from zero to infinity.	(1)
$S_{\delta\nu}(f)$	Spectral density of frequency fluctuations $\delta\nu$. The dimensionality is hertz squared per hertz. The range of f is from zero to infinity. We use the relation $2\pi(\delta\nu) = d(\delta\phi)/dt$, where t is the running time variable.	(2)
$S_y(f)$	Spectral density of fractional frequency fluctuations y. The dimensionality is per hertz. The range of f is from zero to infinity. By definition $y \equiv \delta\nu/\nu_o$. The symbol y is defined and used in reference 3.	(3)
$S_{\delta\varepsilon}(f)$	Spectral density of amplitude fluctuations $\delta\varepsilon$ of a signal. The dimensionality is amplitude (e.g., volts) squared per hertz. The range of f is from zero to infinity.	(4)

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$S_{\delta\tau}(f)$ Spectral density of time interval fluctuations $\delta\tau$. The dimensionality is seconds squared per hertz. The range of f is from zero to infinity. We use the relation $\delta\tau = \delta\phi/(2\pi\nu_o)$. (5)

$S_x(f)$ Spectral density of time interval fluctuations. Same as $S_{\delta\tau}(f)$. Both x and $\delta\tau$ are equal to $\delta\phi/(2\pi\nu_o)$. See Eq. (35). The symbol x is defined and used in reference 3. (6)

$S_{\delta\Omega}(f)$ Spectral density of angular frequency fluctuations $\delta\Omega$. The dimensionality is $(rad/s)^2/Hz$. The range of f is from zero to infinity. By definition, $\Omega \equiv 2\pi\nu$. (7)

$S_{\delta v}(f)$ Spectral density of voltage fluctuations δv. The dimensionality is volts squared per hertz. The range of f is from zero to infinity. Many commercial spectrum analyzers and wave analyzers exist which measure and display spectral density (or square root of spectral density) of voltage fluctuations. A metrologist often will choose to transduce the fluctuations of a quantity of interest into analogous voltage fluctuations. Then the spectral density (corresponding to fluctuations of frequency, phase, time interval, amplitude, or . . .) is measured with voltage spectrum analysis equipment. (8)

$S_{\sqrt{RFP}}(\nu)$ Spectral density of (square root of) the radio frequency power P. The power of a signal is dispersed over the frequency spectrum due to noise, instability, and modulation. The dimensionality is watts per hertz. The range of the Fourier variable ν is from zero to infinity. This concept is similar to the concept of spectral density of voltage fluctuations, $S_{\delta v}(f)$. Typically the latter, $S_{\delta v}(f)$, is more convenient for characterizing a baseband signal where voltage rather than power is relevant. The former, $S_{\sqrt{RFP}}(\nu)$, typically is more convenient for characterizing the dispersion of the signal power in the vicinity of the nominal carrier frequency ν_o. To relate the two spectral densities, it is necessary to specify the impedance associated with the signal. The choice of ν or f for the Fourier frequency variable is somewhat arbitrary. We prefer ν for carrier-related measures and f for modulation-related measures. See later section on relationships among spectral densities. (9)

$\mathcal{L}(f)$ Normalized frequency domain measure of phase fluctuation sidebands.[6] We have defined Script $\mathcal{L}(f)$ to be the ratio of the power in one phase modulation sideband, referred to the input carrier frequency, on a spectral density basis, to the total signal power, at Fourier frequency difference f from the signal's average frequency ν_o, for a single specified signal or device. The dimensionality is per hertz. Because here f is a frequency difference, the range of f is from minus ν_o to plus infinity. (10)

$\mathcal{M}(f)$ Normalized frequency domain measure of fractional amplitude fluctuation sidebands of a signal. We define Script $\mathcal{M}(f)$ to be the ratio of the spectral density of one amplitude modulation sideband to the total signal power, at Fourier frequency difference f from the signal's average frequency ν_o, for a single specified signal or device. The dimensionality is per hertz. Because here f is a frequency difference, the range of f is from minus ν_o to plus infinity. Script $\mathcal{L}(f)$ and Script $\mathcal{M}(f)$ are similar functions; the former is a measure of phase modulation (PM) sidebands, the latter is a corresponding measure of amplitude modulation (AM) sidebands. We introduce the symbol Script $\mathcal{M}(f)$ in order to have useful terminology for the important concept of normalized AM sideband power. (11)

$S_{\delta g}(f)$ Spectral density of fluctuations of any specified time-dependent quantity g(t). The dimensionality is the same as the dimensionality of the ratio g^2/f. The range of f is from zero to infinity. The total variance of $\delta g(t)$ is the integral of $S_{\delta g}(f)$ over all f. The spectral density is the distribution of the total variance over frequency. (12)

Discussion of Spectral Densities

One-Sided Versus Two-Sided Spectral Densities

Each of the above twelve spectral densities is one-sided and is on a per hertz of bandwidth density basis. This means that the total mean-square fluctuation (the total variance) of frequency, for example, is given mathematically by the integral of the spectral density over the total defined range of Fourier frequency f

$$\text{Total Variance} = \int_0^\infty S_{\delta\nu}(f)df, \qquad (13)$$

As another example, since Script $\mathcal{L}(f)$ is a normalized density, its integral over the defined range of difference frequency f is equal to unity, i.e.,

$$\int_{-\nu_o}^{+\infty} \mathcal{L}(f)\, df = 1. \tag{14}$$

The definite integral between two frequencies of the spectral density of the fluctuations of a quantity is the variance of that quantity for the frequency band defined by the two limit frequencies. We will consider this in more detail later.

Occasionally, unnecessary confusion arises concerning one-sided versus two-sided spectral densities. Two-sided spectral densities are defined such that the frequency range of integration is from minus infinity to plus infinity. For specification of signal fluctuations as treated in this paper, our one-sided spectral density is twice as large as the corresponding two-sided spectral density. For example, the total variance is

$$\int_{-\infty}^{+\infty} [S^{\text{Two-Sided}}]\, df = 2 \int_{0}^{+\infty} [S^{\text{Two-Sided}}]\, df = \int_{0}^{+\infty} [S^{\text{One-Sided}}]\, df, \tag{15}$$

where f is the Fourier frequency variable. Two-sided spectral densities are useful mainly in pure mathematical analysis involving Fourier transformations. We recommend and use one-sided spectral densities for experimental work. References 3 and 4 use one-sided spectral densities. The terminology for single sideband (upper or lower) signals versus double sideband (upper and lower) signals is totally distinct from the one-sided spectral density versus two-sided spectral density terminology. They are totally different concepts.

Cross-Spectral Densities

Another important concept is cross-spectral densities. For the case of two fluctuating quantities $\delta a(t)$ and $\delta b(t)$, we can choose the quantity $\delta g(t)$ such that

$$[\delta g(t)]^2 \equiv [\delta a(t)] \cdot [\delta b(t)]. \tag{16}$$

The real part of the cross-spectral density of the fluctuations of a(t) and b(t) can be represented then by

$$S_{\delta g}(f) = S_{\sqrt{(\delta a)\cdot(\delta b)}}(f) \tag{17}$$

using the general representation for a spectral density as given in (12). The normalized cross-spectral density is obtained by dividing both sides of Eq. (17) by $\sqrt{[S_{\delta a}(f)]\cdot[S_{\delta b}(f)]}$. The information contained in the cross-spectral density is not trivial when δa and δb are correlated. Several manufacturers provide cross-spectral density analyzers. We note there is a greater possible complexity in the cross-spectral density concept than in the (auto) spectral density concept. For example, the cross-spectral density can be negative as well as positive, and an imaginary component can be defined and measured also. Although we will not discuss cross-spectral densities further in this paper, we recommend their use for special problems of measurement.[5]

Time-Dependent Spectral Densities

The spectral density concept as used by experimentalists allows the spectral density to be time dependent. This time dependence is an observable, and we may sometimes desire to specify the spectral density of its fluctuations, as a statistical measure of the time dependence. Hence, we arrive at the concept of the spectral density of the fluctuations of a spectral density. Although not much use has been made yet of spectral densities of the fluctuations of spectral densities, in part due to the large quantity of data required for a precise measurement, the concept can be exploited where confidence measures are desired for spectral density statistics. It is a quantitative, statistical measure for characterizing the stability of noise statistics. Unless otherwise stated, it is usually understood that the spectral density of the fluctuations of a spectral density is assumed to be white, that is, it is assumed that the fluctuations of the spectral density are random and uncorrelated.

It is commonly found that measured fluctuations have a spectral density of the fluctuations of their spectral density which is not random and uncorrelated. Some authors attempt to describe such observations by saying that the fluctuations were measured and were found to be non-stationary. Such a conclusion is logically absurd in statistics, for in statistics the definition of stationarity[7] (a measure on ensembles rather than on a portion of an ensemble) makes it independent of the observations of the fluctuations of any particular entity. Note that any particular entity is simultaneously a member of stationary ensembles as well as of non-stationary ensembles. While stationarity is not a physical observable, it can be postulated for a hypothetical ensemble, thereby giving one bit of information about the hypothetical ensemble.

Some of the fundamental theoretical aspects of time-dependent spectral densities are discussed in references 8 - 10. Some practical aspects of the time dependence of the RF power spectral densities of typical high quality frequency sources are treated in reference [11] where the practical concepts of the fast linewidth and the fast RF power spectral density are used. See also reference 12 for an analysis of measurements of time-dependent RF power spectral densities of low-noise oscillators using the fast linewidth concept. Considerable work is needed in this area in order to develop operational terminology, languages, and measurement methods which will be useful and commonly accepted.

The concept of the spectral density of the fluctuations of a (time dependent) spectral density is one of several concepts which allow quantitative, as well as qualitative, characterization and measurement of the instability of the statistical measures of a signal. We recommend it for that purpose.

Measurement of Spectral Densities

Smoothed Spectral Densities

In the measurement of spectral densities, there is an important operational aspect which occasionally leads to some unnecessary confusion. A practical measurement actually gives us the (weighted) average of the spectral density over a range of frequency. We measure a smoothed spectral density — it is smoothed by being averaged in a frequency window. To characterize the measurement, we may specify an effective lower frequency, f_1, an effective higher frequency, f_2, and information about the shape of the window function (bandpass response). Alternatively, we may prefer to specify a center frequency $f_o = (f_2 + f_1)/2$, the effective bandwidth $B = (f_2 - f_1)$, and information about the shape of the window function. The first type of specification typically is more useful when the higher frequency is large compared with the lower frequency. The alternative typically is more useful in the narrowband case when the bandwidth is small compared to the center frequency.

As possible terminology, we suggest the use of Script $\mathcal{N}$ squared as a frequency domain measure to represent this weighted average variance. This allows us to reserve the terminology of spectral density [symbol: $S(f)$] for the differential element of variance between f and $f + df$, and to use $\mathcal{N}^2$ for the smoothed spectral density (averaged over a range of f).

Rectangular Bandpass Example

To illustrate the relationships, consider the hypothetical case of a window having a rectangular bandpass. We can write for the measurable variance, Script $\mathcal{N}$ squared,

$$\mathcal{N}^2 = \frac{\int_{f_1}^{f_2} \left[1\right] S(f)\, df}{\int_{f_1}^{f_2} df} = \frac{1}{B} \int_{f_1}^{f_2} \left[1\right] S(f)\, df, \qquad (18)$$

where the limits of integration reflect the rectangular response of the filter (the response is unity between f_1 and f_2 and is zero for all other f). Equation (18) can be recast in the f_o, B terminology as

$$\mathcal{N}^2 = \frac{\int_{f_o - \frac{1}{2}B}^{f_o + \frac{1}{2}B} \left[1\right] S(f)\, df}{\int_{f_o - \frac{1}{2}B}^{f_o + \frac{1}{2}B} df} = \frac{1}{B} \int_{f_o - \frac{1}{2}B}^{f_o + \frac{1}{2}B} \left[1\right] S(f)\, df. \qquad (19)$$

A strictly rectangular bandpass is not realizable, but it is a useful approximation for many practical situations.

Lorentzian Bandpass Example

As another useful illustration, consider a window having a Lorentzian bandpass response,[13] with a center frequency f_o and a quality factor Q. The normalized filter response for a Lorentzian shape function is

$$|H(f)|^2 = \frac{1}{\left[1 + \left(\frac{2Q}{f_o}\right)^2 \left(f - f_o\right)^2\right]} \, . \qquad (20)$$

The effective bandwidth of a Lorentzian is

$$B = \int_o^{\infty} \frac{1}{\left[1 + \left(\frac{2Q}{f_o}\right)^2 \left(f - f_o\right)^2\right]} \, df, \qquad (21)$$

$$B = \left(\frac{\pi}{2}\right)\left(\frac{f_o}{Q}\right) \, . \qquad (22)$$

Hence, for Script $\mathcal{N}^2$ we may write

$$\mathcal{N}^2 = \frac{\int_o^{\infty} \left[\frac{1}{1 + \left(\frac{2Q}{f_o}\right)^2 \left(f - f_o\right)^2}\right] S(f)\, df}{\int_o^{\infty} \left[\frac{1}{1 + \left(\frac{2Q}{f_o}\right)^2 \left(f - f_o\right)^2}\right] df} \, , \qquad (23)$$

$$\mathcal{N}^2 = \frac{1}{B}\int_0^{\infty}\left[\frac{1}{1+\left(\frac{2Q}{f_o}\right)^2\left(f-f_o\right)^2}\right]S(f)\ df\ . \quad (24)$$

Uniformity Approximation

If S(f) is approximately uniform over the region of frequency f where the response of the bandpass is significant, then a useful approximation is

$$S(f = f_o) \approx \frac{1}{B}\int_{f_o - \frac{1}{2}B}^{f_o + \frac{1}{2}B} S(f)df, \quad (25a)$$

where B is the effective bandwidth, and f_o is the center frequency of the frequency window function. If the frequency window can be usefully approximated by a rectangular bandpass response function, then

$$\mathcal{N}^2 \approx \frac{1}{B}\int_{f_o - \frac{1}{2}B}^{f_o + \frac{1}{2}B} S(f)df. \quad (25b)$$

Combining Eqs. (25a) and (25b), and recalling the approximations made, we may write

$$S(f = f_o) \approx \mathcal{N}^2\left\{f_o,\ B,\ \text{window}\right\}. \quad (26)$$

This approximation, Eq. (26), is commonly used to describe and interpret measurements made by a wave analyzer or a spectrum analyzer. It is exact for white noise for the case of the (unrealizable) rectangular bandpass [compare Eq. (19)].

In summary, note that a wave analyzer or a spectrum analyzer actually gives us a measurement of $\mathcal{N}^2$ or its square root, $\mathcal{N}$. The corresponding spectral density may be inferred by using the approximation of Eq. (26). If for a specific case Eq. (25a), (25b), or both, is a poor approximation, then Eq. (26) may also be a poor estimate of the spectral density. For such cases, we recommend the use of Script $\mathcal{N}$ squared, and the parameters associated with the measurement of Script $\mathcal{N}$ squared should also be quoted. The critical parameters include at least f_o, B, and the shape of the window function or equivalent information.

Spectral Density Estimation — Fourier Transform Analysis

It is implicit in some of the previous discussion that we are considering measurements of spectral densities which use frequency domain techniques, e.g., filtering in frequency domain with resonant circuits. Another widely used technique is to acquire data which are quantized in the time domain and then to use Fourier transformation to obtain the frequency domain statistics. There are several possible procedures, and a common name for this technique is spectral density estimation. A rather readable discussion, with some relevant references, is presented by Richards.[14] The availability of hard-wired digital computation programs for implementing the Fast Fourier Transform (FFT), together with their steadily decreasing cost, is making this technique more and more attractive for on-line measurement (estimation) of spectral density.

Some Relationships Among Spectral Densities

In this section we further define and describe the symbols and terminology we are using, and we derive or explain some of the useful relationships among spectral densities.

We use the Greek letter nu, ν, in two somewhat different senses in this paper. In one usage it is a Fourier frequency variable (index), for example, as the argument in the RF power spectral density $S_{\sqrt{RFP}}(\nu)$. In another usage it is the frequency of a signal, for example, the instantaneous frequency $\nu(t)$ or the average (nominal) frequency ν_o of a signal. Our usage of the symbol f is similarly two-fold. The distinction is usually obvious in context in our treatment, and we sometimes state the usage explicitly. We urge caution on this point in general; it is a hazard in this paper and in other papers on signal stability.

We use the operator δ as the fluctuations operator. For example, by ϕ we mean phase, and by $\delta\phi$ we mean fluctuations of phase. The rate of change of phase with time t is defined as angular frequency Ω. To obtain cycle frequency ν, we normalize by dividing by 2π. For the fluctuations of these quantities we may write

$$\delta\nu \equiv \frac{\delta\Omega}{2\pi} \equiv \frac{1}{2\pi}\ \frac{d(\delta\phi)}{dt}\ . \quad (27)$$

It follows from transform theory that

$$\boxed{S_{\delta\nu}(f) = \left(\frac{1}{2\pi}\right)^2 S_{\delta\Omega}(f) = f^2 S_{\delta\phi}(f)\ .} \quad (28)$$

The Fourier variable for angular frequency, ω, is commonly used. It is related to the Fourier variable for cycle frequency, f, by $\omega = 2\pi f$. The value of $\delta\nu$, normalized to the average (nominal) signal frequency ν_o, is called y. This usage is similar to the usage in reference 3.

$$y \equiv \frac{\delta\nu}{\nu_o}\ . \quad (29)$$

Hence we may use

$$S_y(f) = \left(\frac{1}{\nu_o}\right)^2 S_{\delta\nu}(f). \tag{30}$$

The basic relationship between phase ϕ, frequency ν, and time interval τ is

$$\phi = 2\pi\nu\tau + \phi_o, \tag{31}$$

where ϕ_o is an appropriate constant. The fluctuations of time interval $\delta\tau$ are hence related to the fluctuations of phase $\delta\phi$ by

$$\delta\phi = (2\pi\nu_o)\delta\tau . \tag{32}$$

Hence

$$S_{\delta\phi}(f) = (2\pi\nu_o)^2 S_{\delta\tau}(f). \tag{33}$$

Combining Eqs. (28) to (33)

$$S_{\delta\tau}(f) = \left(\frac{1}{2\pi f}\right)^2 S_y(f). \tag{34}$$

As in reference 3, x is defined such that

$$\frac{dx}{dt} \equiv y, \tag{35}$$

$$S_x(f) = \left(\frac{1}{2\pi f}\right)^2 S_y(f). \tag{36}$$

Hence we see that x and $\delta\tau$ are the same quantity.

Script $\mathcal{L}(f)$

Script $\mathcal{L}(f)$ is the normalized version of the phase modulation (PM) portion of $S_{\sqrt{RFP}}(\nu)$, with its frequency parameter f referenced to the signal's average frequency ν_o as the origin such that the difference frequency f equals $\nu - \nu_o$. A complete definition is given earlier by (10). Script $\mathcal{L}(f)$ and Script $\mathcal{M}(f)$ represent concepts which commonly arise in the languages of modulation noise and of stability of signals.

Script $\mathcal{L}(f)$ can be related in a simple way to $S_{\delta\phi}(f)$ but only for the condition that the phase fluctuations occurring at rates f and faster are small compared to one radian. Otherwise Bessel function algebra must be used to relate Script $\mathcal{L}(f)$ to $S_{\delta\phi}(f)$. Fortunately, the "small angle condition" is often met in random noise problems. Specifically we find as a good approximation

$$\boxed{\mathcal{L}(f) \approx \left(\frac{1}{2\ \text{rad}^2}\right) S_{\delta\phi}(f),} \tag{37}$$

provided that

$$\int_f^{+\infty} S_{\delta\phi}(f')df' << 1\ \text{rad}^2 \tag{38}$$

where f prime is a dummy index for integration.

For the types of signals under discussion and for $|f| < \nu_o$, we often may use as a good approximation

$$\mathcal{L}(-f) \approx \mathcal{L}(f). \tag{39}$$

We note that for pure phase modulation, the RF power spectral density of the signal with sidebands is not necessarily symmetrical. An asymmetric RF power spectral density may reflect a mixture of correlated AM and PM, but it can also arise from special cases of pure PM (or FM). There is some confusion in the literature on this point. For pure AM, the RF power spectral density is strictly symmetrical. This symmetry property is used in a later section on Script $\mathcal{M}(f)$. See Eq. (55).

A simple derivation of Eq. (37) is possible. We combine the derivation with an example which illustrates the operation of a double-balanced mixer as a phase detector. Consider two sinusoidal 5-MHz signals (having negligible amplitude modulation) feeding the two input ports of a double-balanced mixer. When the two signals are slightly different in frequency, a slow, nearly sinusoidal beat with a period of several seconds at the output of the mixer is measured to have a peak-to-peak swing of A_{ptp}.

Without changing their amplitudes, the two signals are tuned to be at zero beat and in phase quadrature (that is, $\pi/2$ out of phase with each other), and the output of the mixer is a small fluctuating voltage centered on zero volts. Provided this fluctuating voltage is small compared to $A_{ptp}/2$, the phase quadrature condition is being closely maintained, and the "small angle condition" is being met.

Phase fluctuations $\delta\phi$ between the two signals of phases ϕ_2 and ϕ_1, respectively, where

$$\delta\phi \equiv \delta(\phi_2 - \phi_1) , \tag{40}$$

give rise to voltage fluctuations δA at the output of the mixer

$$\delta A \approx \frac{A_{ptp}}{2}\delta\phi , \tag{41}$$

where we use radian measure for phase angles, and we use

$$\sin\delta\phi \approx \delta\phi \tag{42}$$

for small $\delta\phi$ ($\delta\phi$ << 1 rad). We solve Eq. (41) for $\delta\phi$, square both sides, and take a time average

$$\langle(\delta\phi)^2\rangle \approx 4\frac{\langle(\delta A)^2\rangle}{(A_{ptp})^2}\,. \tag{43}$$

The angular brackets indicate averaging over time. In practice, the averaging time typically may be ten to one thousand times the inverse bandwidth of the measurement system. If we interpret the mean-square fluctuations of $\delta\phi$ and of δA, respectively, in Eq. (43) in a spectral density fashion, we may write

$$S_{\delta\phi}(f) \approx \frac{S_{\delta A}(f)}{2(A_{rms})^2}\,, \tag{44a}$$

where we use

$$(A_{ptp})^2 = 8(A_{rms})^2\,, \tag{44b}$$

which is valid for the sinusoidal beat signal.

For the types of signals under consideration, the two phase fluctuation sidebands (lower sideband and upper sideband, at $-f$ and $+f$ from ν_o, respectively) of a signal are coherent with each other by definition. As already expressed in Eq. (39), they are, to a good approximation, of equal intensity also. The operation of the mixer when it is driven at quadrature is such that the amplitudes of the two phase sidebands add linearly in the output of the mixer, resulting in four times as much power in the output as would be present if only one of the phase sidebands were allowed to contribute to the output of the mixer. Hence for $|f| < \nu_o$ we obtain

$$\frac{S_{\delta A}(|f|)}{(A_{rms})^2} \approx 4\,\frac{\left[S_{\sqrt{RFP}}(\nu_o + f)\right]_{PM}}{P_{total}}\,, \tag{45}$$

where P_{total} is the total power of the signal, and, using the definition of Script $\mathcal{L}(f)$,

$$\mathcal{L}(f) \equiv \frac{\left[S_{\sqrt{RFP}}(\nu_o + f)\right]_{PM}}{P_{total}}\,, \tag{46}$$

we find for one fluctuating signal

$$\mathcal{L}(f) \approx \frac{1}{2} S_{\delta\phi}(|f|)\,, \tag{47}$$

provided the phase quadrature condition is approximately valid. The phase quadrature condition will be met for a time interval at least τ long, provided

$$\int_{(2\pi\tau)^{-1}}^{\infty} S_{\delta\phi}(f')df' \ll 1\ \mathrm{rad}^2, \tag{48}$$

and hence Eq. (47) is useful for values of f at least as low as $(2\pi\tau)^{-1}$. Eqs. (47) and (48) correspond to Eqs. (37) and (38) respectively.

Script $\mathcal{L}(f)$ by its definition [see (10)] is a measure of the phase fluctuation sidebands referred to the input (rti) of the unit under discussion. Sometimes it is more meaningful to quote a measure which is referred to the output (rto).[11-12, 15-17] For this purpose we find it to be convenient to place a caret (^) over the quantity which is referenced to the output.[11-12] For example, $\hat{\mathcal{L}}(f)$, $\delta\hat{\phi}$, and $S_{\delta\hat{\epsilon}}(f)$ mean, respectively, Script $\mathcal{L}(f)$ rto, fluctuations of phase rto, and spectral density of $\delta\epsilon$ rto.

A Method of Measurement of Script $\mathcal{L}(f)$

Using a double-balanced mixer as a phase sensitive detector,[18] we have one of many ways to easily measure Script $\mathcal{L}(f)$ with a voltage spectrum analyzer. Eq. (49) is valid for the case where the reference signal has negligible phase fluctuations compared to the test signal.

$$\boxed{\mathcal{L}(f) = \frac{2}{n^2}\left(\frac{v_{\text{one unit}}}{A_{ptp}}\right)^2 = \frac{\hat{\mathcal{L}}(f)}{n^2}\,,} \tag{49}$$

where v is the measured value of (δA) rms on a square root spectral density basis. However, Eq. (50) is the valid equation when we have two equally noisy signals (test and reference) driving the mixer.

$$\boxed{\mathcal{L}(f) = \frac{1}{n^2}\left(\frac{v_{\text{two units}}}{A_{ptp}}\right)^2 = \frac{\hat{\mathcal{L}}(f)}{n^2}\,.} \tag{50}$$

In case the device being measured has frequency multiplication (synthesis) by the factor n, the definition of Script $\mathcal{L}(f)$ requires that the factor $(1/n)^2$ appear in Eq. (49) and (50). Note that these measurements are smoothed estimates of the spectral densities.

Figures 1 and 2 illustrate this use of double-balanced mixers driven in phase quadrature for measurement of $\mathcal{L}(f)$. The method of Figure 1, which measures the differential stability between two parallel phase-processing arms, is used for singles or for pairs of phase-processing devices other than oscillating signal sources.[18] The method of Figure 2 is used for oscillator pairs.

Script $\mathcal{M}(f)$

Script $\mathcal{M}(f)$ is the normalized version of the amplitude modulation (AM) portion of $S_{\sqrt{RFP}}(\nu)$, with its frequency parameter f referenced to the signal's average frequency ν_o taken as the origin such that the difference frequency f equals $\nu - \nu_o$. A complete definition is given earlier by (11).

Script $\mathcal{M}(f)$ can be related in a simple way to $S_{\delta\varepsilon}(f)$, the spectral density of the amplitude fluctuations $\delta\varepsilon$ of a signal, defined in (4). Specifically we find as an exact relation

$$\mathcal{M}(f) = \frac{1}{2}\left(\frac{1}{V_o}\right)^2 S_{\delta\epsilon}(f)\,, \tag{51}$$

and

$$\mathcal{M}(f) = \frac{1}{4}\left(\frac{1}{V_{rms}}\right)^2 S_{\delta\epsilon}(f), \tag{52}$$

where we use

$$\left(V_o\right)^2 = 2\left(V_{rms}\right)^2 \tag{53}$$

which is valid for a sinusoidal signal. The signal under consideration is represented by

$$V(t) = \left[V_o + \delta\epsilon(t)\right] \sin\left[2\pi\nu_o t + \phi_o + \delta\phi(t)\right] \tag{54}$$

where V(t) is the instantaneous value of the signal, V_o is the average (nominal) amplitude of the signal, $\delta\varepsilon(t)$ represents the fluctuations of the amplitude, t is the running time index, ϕ_o is a constant, and $\delta\phi(t)$ is the fluctuation of the phase of the signal. This relation is similar to Eq. 2 of reference 3.

For $|f| < \nu_o$, Script $\mathcal{M}(f)$ is symmetrical about the signal's average frequency ν_o.

$$\mathcal{M}(f) = \mathcal{M}(-f)\,. \tag{55}$$

A simple derivation of Eqs. (51) and (52) is possible. We combine the derivation with an example which illustrates the operation of a double-balanced mixer as an amplitude detector. Consider two 5-MHz signals (having phase fluctuations of much less than one radian) feeding the two input ports of a double-balanced mixer. When the two signals are slightly different in frequency, a slow, nearly sinusoidal beat with a period of several seconds at the output of the mixer is measured to have a peak-to-peak swing of A_{ptp}.

Without changing their amplitudes, the two signals are tuned to be at zero beat and in colinear phase (that is, either zero or $\pm n\pi$ radians phase angle difference), and the output of the mixer is a fluctuating voltage centered on $A_{ptp}/2$ volts. We note there is no requirement that the output fluctuations be small compared to $A_{ptp}/2$ in the measurement of Script $\mathcal{M}(f)$.

To assure linearity of the demodulation, and to make the measurement be sensitive to the AM of only one (the test signal) of the two signals, we cause the other signal (the reference signal) to drive the double-balanced mixer at a higher level than the test signal. A ratio of 10 dB is adequate except for special cases, e.g., where the utmost linearity is required.

Amplitude fluctuations $\delta\varepsilon$ of the signal under test give rise to voltage fluctuations δA at the output of the mixer

$$\delta A = \left(\frac{A_{ptp}}{2}\right)\frac{\delta\epsilon}{V_o}\,. \tag{56}$$

We solve Eq. (56) for $\delta\varepsilon$, square both sides, and take a time average

$$\langle(\delta\varepsilon)^2\rangle = \left(4V_o^{\,2}\right)\frac{\langle(\delta A)^2\rangle}{\left(A_{ptp}\right)^2}\,. \tag{57}$$

Equation (57) is similar to Eq. (43). If we interpret the mean-square fluctuations of $\delta\varepsilon$ and of δA, respectively, in Eq. (57) in a spectral density fashion, we may write

$$S_{\delta\varepsilon}(f) = \left(\frac{V_o^{\,2}}{2}\right)\frac{S_{\delta A}(f)}{\left(A_{rms}\right)^2} \tag{58}$$

where we use

$$\left(A_{ptp}\right)^2 = 8\left(A_{rms}\right)^2 \tag{59}$$

which is valid for the sinusoidal beat signal.

For the types of signals under consideration, by definition the two amplitude fluctuation sidebands (lower sideband and upper sideband, at -f and +f from ν_o, respectively) of a signal are coherent with each other. As already expressed in Eq. (55), they are of equal intensity also. The operation of the mixer when it is driven at colinear phase is such that the amplitudes of the two AM sidebands add linearly in the output of the mixer, resulting in four times as much power in the output as would be present if only one of the AM sidebands were allowed to contribute to the output of the mixer. Hence for $|f| < \nu_o$ we obtain

$$\frac{S_{\delta A}\left(|f|\right)}{\left(A_{rms}\right)^2} = 4\,\frac{\left[S_{\sqrt{RFP}}\left(\nu_o + f\right)\right]_{AM}}{P_{total}}, \tag{60}$$

and, using the definition of Script $\mathcal{M}(f)$,

$$\mathcal{M}(f) \equiv \frac{\left[S_{\sqrt{RFP}}\left(\nu_o + f\right)\right]_{AM}}{P_{total}}\,, \tag{61}$$

we find

$$\mathcal{M}(f) = \left(\frac{1}{2V_o^{\,2}}\right) S_{\delta\varepsilon}\left(|f|\right). \tag{62}$$

or, using the normalized (fractional) fluctuations of amplitude $\delta\varepsilon/V_o$, we may write

$$\boxed{\mathcal{M}(f) = \frac{1}{2} S_{\frac{\delta\varepsilon}{V_o}}\left(|f|\right) .} \tag{63}$$

We note the similarity of Eq. (63) and Eq. (47). These are two very useful equations to remember.

A Method of Measurement of Script $\mathcal{M}(f)$

Using a double-balanced mixer as an amplitude sensitive detector, we have one of many ways to easily measure Script $\mathcal{M}(f)$ with a voltage spectrum analyzer. Equation (64) is valid for the case where the reference signal contributes only a negligible amount to the output fluctuations of the mixer as compared to the amount contributed by the test signal.

$$\mathcal{M}(f) = 2\left(\frac{v_{\text{one unit}}}{A_{\text{ptp}}}\right)^2 \tag{64}$$

where v is the measured value of (δA) rms on a square root spectral density basis. The arrangements shown in Figures 1 and 2 need to be modified slightly in order to do the amplitude sensitive measurements.

The major change is that the two input signals to the mixer must be in colinear phase, that is, the output voltage of the mixer is at an extremum rather than at zero. A more complicated phase lock loop is used (although in some cases no loop is required) than the one shown in Figure 2; the signal under test must be the weaker of the two input signals (hence the 10-dB pad shown in Figure 2 must be used in the other input for AM sensitive measurements); and some type of DC-blocking may have to be used with the amplifiers when measuring the voltage noise of the demodulated AM in order to avoid overload due to the large DC component.

Suggestions for Further Reading

A measurement system for AM and FM noise is thoroughly described by Ondria. Another excellent standard reference for PM and FM noise measurements is by Cutler and Searle,[20] but they do not consider AM noise explicitly. Compared to the number of papers on PM and FM noise measurements, there are relatively few publications on AM noise measurements. For an operational approach to the description of electrical noise in general, we strongly recommend the excellent book by Bennett.[21] For references to additional relevant publications, we recommend selection from the more than one hundred items in Appendix H and the Bibliography of NBS TN632.[4]

Summing Up

In this tutorial survey we have discussed some practical aspects of the use of spectral densities for the measurement and characterization of stability of signals. We have suggested some terminology which may be useful to the practicing metrologist. Some relationships among various types of spectral densities were explained. Some common pitfalls were identified and discussed.

We thank many people whose many excellent insights, as well as some occasional confusion, have provided content for this survey and have given impetus to its preparation. We will appreciate comments and feedback.

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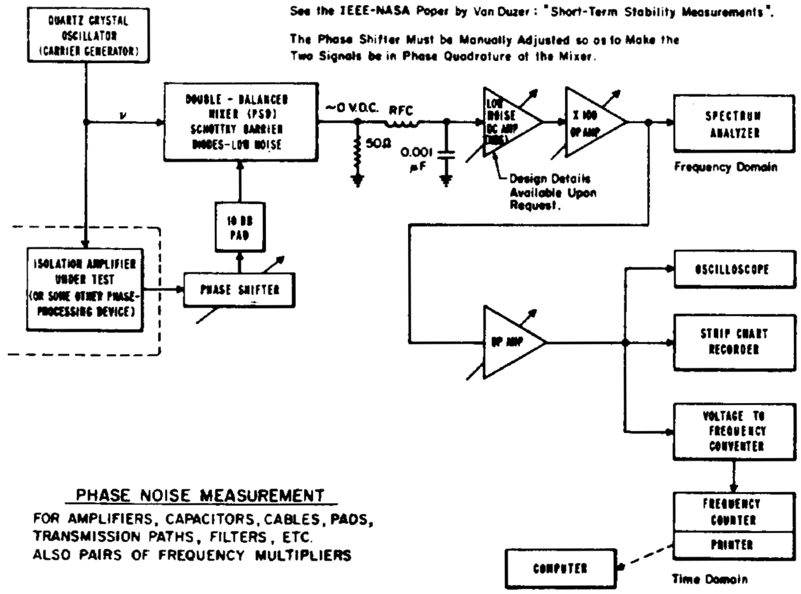

Figure 1. To measure Script $\mathcal{L}(f)$, the double-balanced mixer is driven in phase quadrature as shown. By driving the two input ports in phase-parallel instead, this system measures Script $\mathcal{M}(f)$ of the device under test. See text for additional details for measuring Script $\mathcal{M}(f)$. Note that time domain measurements can be made concurrently.

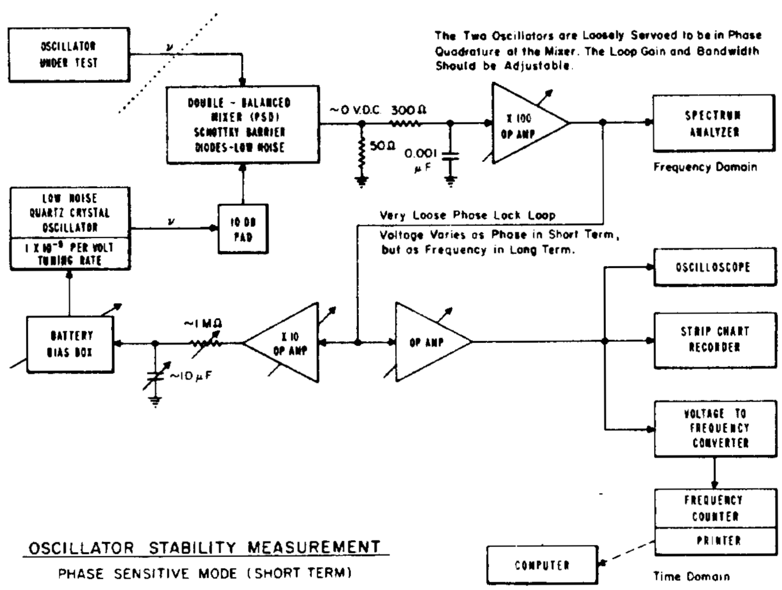

Figure 2. To measure Script $\mathcal{L}(f)$, the double-balanced mixer is driven in phase quadrature as shown. By driving the two input ports in phase-parallel instead, this system measures Script $\mathcal{M}(f)$ of the weaker of the two driving signals. See text for additional details. Note that time domain measurements can be made concurrently.